Electromagnetic Fluctuations in Charged Fluids Coupled to the Radiation Field

Master's Thesis of

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Abstract

We consider the microscopic model of a quantum plasma (charged fluid) in thermal equilibrium with the electromagnetic radiation field. After a review of the semi-classical case (featuring a classical field), we focus on the full quantum system including the quantized field. Using Maxwell-Boltzmann statistics for the particles we provide an exact statistical treatment of the field correlations in the plasma. Our calculations are based on a joint path integral formulation of the thermal Gibbs weight, combining the Feynman-Kac-Itô representation of the particles with the bosonic path integral formalism for the radiation field. Performing the partial field trace, we extract an effective magnetic interaction potential and determine the field fluctuations. We confirm these exact results by means of a second-order perturbative expansion and show that the electric field correlations reveal an algebraic decay in the long distance asymptotics. This result is in apparent contradiction to the macroscopic theory developed by Landau and Lifshitz in their Course of Theoretical Physics. They characterize the matter in terms of a dielectric function and find exponentially decaying field correlations. We perform a detailed comparison between our calculations and this theory, outlining the subtleties of such a direct confrontation and reviewing Landau's and Lifshitz's hypotheses. In particular, we investigate a characteristic of the macroscopic approach which states the cancellation of the leading order terms of the transverse and longitudinal electric field correlations. For the time being, we are not able to definitely confirm or falsify this assertion from the microscopic point of view.

This thesis, constituting the first chapter of "Die Frage des Warums", is dedicated to my brother Christoph O. Ryser

Contents

Chapter 1

Introduction

The physics of a non-relativistic quantum plasma of charged particles (electrons, nuclei and ions) in interaction with the electromagnetic radiation field is presumed to be contained in the Hamiltonian

$$H = \sum_{i=1}^{N} \frac{1}{2m_{\alpha_i}} \Big(\boldsymbol{p}_i - \frac{e_{\alpha_i}}{c} \boldsymbol{A}(\boldsymbol{r}_i) \Big)^2 + \sum_{i<j}^{N} \frac{e_{\alpha_i} e_{\alpha_j}}{|\boldsymbol{r}_i - \boldsymbol{r}_j|} + \sum_{\boldsymbol{k}\lambda} \hbar\omega_{\boldsymbol{k}} a^{\dagger}_{\boldsymbol{k}\lambda} a_{\boldsymbol{k}\lambda}. \tag{1.1}$$

This very formula represents therefore, at least from a conceptual point of view, the key to the resolution of a overwhelming multitude of physical problems. Unfortunately, the description of the macroscopic world on the mere basis of this microscopic Hamiltonian (the so-called physical picture of the world) is in most cases a slightly too ambitious project. Indeed, many interesting physical phenomena have first been understood and theoretically explained in the framework of semi-macroscopic or macroscopic theories and a rigorous link to the microscopic picture is in many cases still missing.

The macroscopic theories in question are usually based on semi-heuristic arguments, simplifications, working hypotheses and approximations which do, although very plausible, often lack a microscopic justification. Famous examples are the so-called chemical picture of the world (hypothesizing the existence of atoms and molecules) and the various mean-field theories. Of course, in the vast majority of these cases we can argue that experimental confirmations are to a certain extent proof enough for the adequacy of the underlying approximations and simplifications, but from a purely microscopic point of view, this is not completely satisfying. Consequently, the missing link between the physical and the chemical (resp. macroscopic) world represents a fascinating and highly non-trivial challenge for the community of theoretical physicists. The correct prediction of macroscopic phenomena by means of fundamental physical theories, e.g. quantum mechanics, quantum electrodynamics and quantum statistics, are celebrated highlights of modern physics. And the so far unsolved problems constitute a driving force to push these considerations further and further and to recover both anticipated and so far unknown results in terms of physically exact reasonings and mathematically rigorous proofs.

Let us now have a closer look at some major results that have been obtained in the framework of the physical picture, i.e. that are a direct consequence of (1.1). To start off, we consider a classical Coulomb gas in absence of the radiation field. The screening of the Coulomb potential (predicted by Debye and Hückel in the framework of their famous mean-field theory) as well as the consequential exponential clustering of the particle correlations have been proven in a rigorous manner. More precisely, it is known that in the so-called Debye-Hückel regime, the following bound holds for the truncated two-particle density [18]

$$|\rho_T(\alpha, \alpha', \boldsymbol{r})| \leq Ce^{-|\boldsymbol{r}|/l}, \tag{1.2}$$

where l is proportional to the Debye screening length and C is a constant. Although this is a rigorous result, we should bear in mind that it originates from the framework of classical mechanics and it is therefore legitimate to question the impact of the consideration of the particles' quantum nature. The answer is indeed astonishing because it has been shown that the intrinsic quantum fluctuations destroy the Debye-Hückel screening [17]. Whereas the monopole contribution of the Coulomb potential remains screened, the multipolar contributions survive and consequently the exponential clustering of the particle correlations disappears in favor of algebraic tails. More precisely, the two-particle density is proven to decay as

$$\rho_T(\alpha, \alpha', \boldsymbol{r}) \sim \frac{A}{|\boldsymbol{r}|^6}, \qquad A = const, \quad |\boldsymbol{r}| \to \infty. \tag{1.3}$$

Another interesting result in the framework of the quantum plasma is the study of the formation of atoms and molecules. For example, one can prove (cf [4]) the emergence of hydrogen atoms in a plasma of protons and electrons in the well specified Saha-regime. These works on stability and existence of matter may be considered as being the physical confirmation of the chemical picture. Noticing that the matter is always surrounded by the photon gas (except at zero temperature), we add now the electromagnetic radiation field and pass on to the full Hamiltonian (1.1). In addition to the mere electrostatic Coulomb potential, the presence of the radiation field implicates a magnetic interaction between the particles. However, it is not only interesting to know the impact of the field on the behavior of the matter, we are also curious about the reverse influence: the properties of black radiation are well-known, but what happens if we bring the photons into thermal equilibrium with a quantum plasma? What's the impact on the field correlations and hence the coherence properties of the light? An exact microscopic investigation of these questions by means of (1.1) has been proposed in [5]. In this article, the field is treated classically and it is shown that the effective magnetic potential mediated by the field has a dipolar decay in the long distance asymptotics. The two-particle density obeys qualitatively the same algebraic decay as in absence of radiation (1.3) and the field correlations reveal algebraic tails, too. In light of these results, our inquisitiveness is once again stimulated and the following questions arise: what would be the consequences if we quantized the radiation field and envisaged therefore a complete quantum description of the system? What would be the impact of the quantum nature of the field on the effective interaction potential between the particles and on the particle correlations? Is the decay law of the electromagnetic field correlations different from the classical one? Would we, in contrary to the semi-classical case, detect an influence of the matter on the transverse electric field correlations? In addition to these intriguing questions, there is actually another very strong motivation for the investigation of the full quantum model: if we perform exact calculations on the basis of (1.1), we are able to re-examine already existing macroscopic theories and to either corroborate or falsify them in the spirit of the first paragraph. In the present case of the field correlations, we would like to focus on a macroscopic approach by Landau and Lifshitz (LL). In their famous Course of Theoretical Physics [14], they calculate the electromagnetic field correlations in a macroscopic framework by characterizing the impact of the matter on the fields in terms of a complex refraction index $n = \sqrt{\epsilon}$. As we will discuss later on, LL's approach is based on two major assumptions and it is a very interesting task to examine whether these hypotheses are physically justified or not.

The present thesis can be divided into three parts. In chapter 2, we give a detailed microscopic description of the model and outline some general considerations about the statistical formalism inalienable for the accurate description of the thermal equilibrium state (in particular, we will work in the framework of Maxwell-Boltzmann statistics, an approximation that alleviates the mathematical formalism substantially). Chapter 3 is devoted to the field-correlations in the semi-classical formalism. In particular, we introduce the very useful path integral representation of the quantum matter in terms of the Feynman-Kac-Itô (FKI) formula. This path integral formalism leads to an elegant representation of the thermal Gibbs weight and allows us to derive the effective magnetic potential. The exact field correlations are then calculated by means of a new method which involves as an intermediate step the calculation of the second order field moments.

In the second part, we switch to the full quantum model and introduce at first the bosonic path integral (BPI) which leads together with the FKI formula to a double path integral representation of the thermal Gibbs weight (chapter 4). This allows us to extract the magnetic potential mediated by the quantum field. Then, we generalize the technique of chapter 3 and calculate the exact field fluctuations. Since the joint path integral (FKI and BPI) is a rather sophisticated mathematical tool, we seek a confirmation of the exact results by means of perturbation theory in chapter 5.

Finally, the third part is devoted to a direct comparison between the microscopic approach and the theory of Landau and Lifshitz. In order to prepare this comparison, we focus in chapter 6 on the long distance asymptotics of the electric field correlations. In chapter 7, we examine the comparability of the two approaches and outline the main subtleties. Unfortunately, we have to anticipate that we are not yet able to definitively corroborate or falsify the macroscopic approach on the basis of our results.

Chapter 2

The Model

2.1 Generalities

The system we will investigate in this paper consists of a charged quantum plasma which is coupled to the radiation field. There are N non relativistic quantum charges (electrons, nuclei, ions) of $\mathcal{L}$ different species $\alpha = 1, 2, \cdots, \mathcal{L}$ and we denote by m_{α_i} and e_{α_i} their respective mass and charge ($i = 1, \ldots, N$). Since Coulomb systems expel excess charges to the boundary [4], we have to assure overall neutrality by imposing

$$\sum_{\alpha} \rho_\alpha e_\alpha = 0. \tag{2.1}$$

We consider the system in its state of thermal equilibrium at temperature T. In order to establish the partition function and the thermal Gibbs weight, we confine the system as follows: first, we put the N particles in a cube of side length L and volume $\Lambda = L^3$. The radiation field is then enclosed in a larger box of side length $R >> L$ and volume V which contains on the other hand the particle box, i.e. $\Lambda \subset V$. Since we are solving the Schroedinger equation (on Λ) and the Maxwell equations (on V) involving differential operators we have to specify the boundary conditions for both equations. In the case of the domain limiting the plasma we choose Dirichlet boundary conditions, i.e. we let the wave functions vanish on the boundary. Later on we will represent the quantum point charges by spatially extended filaments and hence Dirichlet conditions will ensure that the filaments stay confined within Λ. Regarding the cube enclosing the radiation field, we chose periodic boundary conditions which allow for propagating electromagnetic waves as well as for the very convenient Fourier decomposition of the field that we shall introduce later on. Another very important choice is the Gauge for the electromagnetic field and we will work throughout the whole thesis in the Coulomb Gauge $\nabla \cdot \boldsymbol{A} = 0$. Although we loose hereby explicit Lorentz covariance, there is a certain number of advantages. Let us briefly illuminate them: in point of fact, if we express the electric and magnetic field in terms of the scalar and vector potentials V and $\boldsymbol{A}$ as

$$\begin{aligned} \boldsymbol{E} &= -\nabla V - \frac{1}{c}\partial_t \boldsymbol{A} \\ \boldsymbol{B} &= \nabla \wedge \boldsymbol{A}, \end{aligned} \tag{2.2}$$

they satisfy automatically the homogeneous Maxwell equations

$$\begin{aligned} \nabla \cdot \boldsymbol{B} &= 0 \\ \nabla \wedge \boldsymbol{E} &= -\frac{1}{c}\partial_t \boldsymbol{B}. \end{aligned} \tag{2.3}$$

(Note that we work throughout the whole thesis in Gaussian units.) Then the inhomogeneous Maxwell equations in terms of the charge density ρ and the current density $\boldsymbol{j}$ read (using the Coulomb Gauge)

$$\begin{aligned} \nabla^2 V &= -4\pi\rho \\ \frac{1}{c^2}\frac{\partial^2 \boldsymbol{A}}{\partial t^2} - \nabla^2 \boldsymbol{A} &= \frac{4\pi}{c}\boldsymbol{j} - \frac{1}{c^2}\frac{\partial \nabla V}{\partial t}. \end{aligned} \tag{2.4}$$

The former equation is simply the Poisson equation for electrostatics and the scalar potential V is therefore the genuine Coulomb potential. According to equation (2.2) we can now decompose the electric field into its longitudinal ($\nabla \wedge \boldsymbol{E}_l = 0$) and transverse ($\nabla \cdot \boldsymbol{E}_t = 0$) components:

$$\boldsymbol{E} = \boldsymbol{E}_l + \boldsymbol{E}_t, \tag{2.5}$$

where $\boldsymbol{E}_l = -\nabla V$ and $\boldsymbol{E}_t = -\frac{1}{c}\partial_t \boldsymbol{A}$. The crucial aspect of this decomposition is that, according to (2.4), the two components can be treated independently: $\boldsymbol{E}_l$ corresponding to the instantaneous Coulomb interaction between the charges and $\boldsymbol{E}_t$ corresponding to the radiative part of the field (mediated by its two transverse degrees of freedom, the photon).

Thanks to the periodic boundary conditions, we can decompose the potential vector $\boldsymbol{A}$ into its Fourier components as follows:

$$\boldsymbol{A}(\boldsymbol{r}) = \left(\frac{4\pi\hbar c^2}{R^3}\right)^{\frac{1}{2}} \sum_{\boldsymbol{k}\lambda} g(\boldsymbol{k}) \frac{\boldsymbol{e}_{\boldsymbol{k}\lambda}}{\sqrt{2\omega_{\boldsymbol{k}}}} (a^\dagger_{\boldsymbol{k}\lambda} e^{-i\boldsymbol{k}\cdot\boldsymbol{r}} + a_{\boldsymbol{k}\lambda} e^{i\boldsymbol{k}\cdot\boldsymbol{r}}), \tag{2.6}$$

where $\boldsymbol{k} = (\frac{2\pi n_x}{R}, \frac{2\pi n_y}{R}, \frac{2\pi n_z}{R}) \backslash \{\boldsymbol{0}\}$ are the wave vectors and the mode frequency is given by $\omega_{\boldsymbol{k}} = c|\boldsymbol{k}|$. $a^\dagger_{\boldsymbol{k}\lambda}$ and $a_{\boldsymbol{k}\lambda}$ are the creation and annihilation operators for photons of the mode $(\boldsymbol{k}\lambda)$ and $\lambda = 1, 2$ labels the two transverse degrees of freedom. The two unit polarization vectors $\boldsymbol{e}_{\boldsymbol{k}\lambda}$ are orthogonal to $\boldsymbol{k}$ due to the Coulomb Gauge and we chose them real and orthogonal to each other

$$\begin{aligned} \boldsymbol{k} \cdot \boldsymbol{e}_{\boldsymbol{k}\lambda} &= 0 \\ \boldsymbol{e}_{\boldsymbol{k}\lambda} \cdot \boldsymbol{e}_{\boldsymbol{k}\lambda'} &= \delta_{\lambda\lambda'}. \end{aligned} \tag{2.7}$$

The function $g(\boldsymbol{k})$ in (2.6) is a form factor that regularizes the ultraviolet divergences, a problem that arises both in the classical and the quantum theory. This problem is due to the point-like nature of the charges and the singularity may only be smoothed out by means of the renormalization theory. The form factor $g(\boldsymbol{k})$ is a centrally symmetric and smooth function that goes to unity when $|\boldsymbol{k}| \to 0$: $g(0) = 1$. Let us note that our main interest lies in the long distance behavior of the system in real space, i.e. we shall be concerned with small wave numbers (cf [11] for a justification).

Let us finally make a comment about the transverse electric field $\boldsymbol{E}_t$ which is the time derivative of the $\boldsymbol{A}$-field and cannot be derived from the time-independent expression (2.6). However, in both the classical as well as the quantum theory of the electromagnetic field, the time-independent observable $-1/4\pi\boldsymbol{E}_t(\boldsymbol{r})$ is the canonically conjugated variable to $\boldsymbol{A}(\boldsymbol{r})$ and is given by [8]

$$\boldsymbol{E}_t(\boldsymbol{r}) = \left(\frac{4\pi\hbar c^2}{R^3}\right)^{\frac{1}{2}} \sum_{\boldsymbol{k}\lambda} g(\boldsymbol{k}) \frac{\boldsymbol{e}_{\boldsymbol{k}\lambda}}{\sqrt{2\omega_{\boldsymbol{k}}}} \left(\frac{-i\omega_{\boldsymbol{k}}}{c} a^{\dagger}_{\boldsymbol{k}\lambda} e^{-i\boldsymbol{k}\cdot\boldsymbol{r}} + \frac{i\omega_{\boldsymbol{k}}}{c} a_{\boldsymbol{k}\lambda} e^{i\boldsymbol{k}\cdot\boldsymbol{r}}\right). \tag{2.8}$$

This result can also be derived from the vector potential $\boldsymbol{A}(\boldsymbol{x}, t)$ in the Heisenberg picture. But for this we need to know the Hamiltonian as well as the time evolution operator of our system.

2.2 The Hamiltonian

The N-particle Hamiltonian for the finite volume system in Λ and V reads:

$$H^N_{L,R} = H^N_{kin} + H^N_C + H^N_{pot} + H^{rad}_0, \tag{2.9}$$

where we omit the indices L and R for the different contributions on the right-hand side and

- $H^N_{kin} = \sum_{i=1}^N \frac{\left(\boldsymbol{p}_i - \frac{e_{\alpha_i}}{c}\boldsymbol{A}(\boldsymbol{r}_i)\right)^2}{2m_{\alpha_i}}$ takes account of the kinetic energy of the particles that carry now the modified momentum $(\boldsymbol{p}_i - \frac{e_{\alpha_i}}{c}\boldsymbol{A}(\boldsymbol{r}_i))$
- $H^N_C = \sum_{i<j}^N V_C(\boldsymbol{r}_i, \boldsymbol{r}_j) + \sum_{i=1}^N V_{sr}(\boldsymbol{r}_j - \boldsymbol{r}_i)$, where $V_C(\boldsymbol{r}_i, \boldsymbol{r}_j) = \frac{e_{\alpha_i} e_{\alpha_j}}{|\boldsymbol{r}_j - \boldsymbol{r}_i|}$ is the Coulomb interaction between the charges and $V_{sr}(\boldsymbol{r}_j - \boldsymbol{r}_i)$ is a regularization potential

- $H_{pot}^N = \sum_{i=1}^N V_{walls}(\alpha_i, \boldsymbol{r}_i)$ is an external potential that confines the particles within the cube Λ. In fact, we shall choose an infinitely steep potential at the walls in order to enforce the Dirichlet boundary conditions
- $H_0^{rad} = \sum_{\boldsymbol{k}\lambda} \hbar\omega_{\boldsymbol{k}}\, a_{\boldsymbol{k}\lambda}^\dagger a_{\boldsymbol{k}\lambda}$ is the Hamiltonian of the free quantum electromagnetic radiation field

Let us make some comments on (2.9). First of all, we note that the appearance of the coefficient e/c in the modified momentum $(\boldsymbol{p}_i - \frac{e_{\alpha_i}}{c}\boldsymbol{A}(\boldsymbol{r}_i))$ is the main motivation for the choice of Gaussian units. Indeed, this form of the coupling is very practical for perturbative calculations since it gives us control over the interaction, i.e. by $e/c \to 0$ we can uncouple field and matter. The singularities arising in the Coulomb interaction for $\boldsymbol{r}_i = \boldsymbol{r}_j$ are normally regularized by the appropriate quantum statistics, either Fermi-Dirac or Bose-Einstein. However, as we will precise later on, we shall work with Maxwell-Boltzmann statistics for the many-particle system and hence we have to introduce an additional potential V_{sr} that smooths out the Coulomb singularities and assures thermodynamic stability. Following [4], we may either choose a hard core potential with the de Broglie wave lengths

$$\lambda_{\alpha_i} = \hbar\sqrt{\frac{\beta}{m_{\alpha_i}}} \tag{2.10}$$

as radius or a add

$$V_{sr}(\boldsymbol{r}) = |\boldsymbol{r}|^{-1}\exp(-|\boldsymbol{r}|/\lambda) \tag{2.11}$$

to the genuine Coulomb potential. The total Hamiltonian $H_{L,R}^N$ is defined on the Hilbert space $\mathcal{H}_{tot} = \mathcal{H}_\nu^{\otimes N} \otimes \mathcal{F}_{photons}$, where $\mathcal{H}$ is the single-particle Hilbert space $L^2(\Lambda, d^3\boldsymbol{r})$ and $\mathcal{F}_{photons}$ is the Fock space of the photons. Note that $\mathcal{H}_\nu^{\otimes N}$ is the properly symmetrized many-body Hilbert space with $\nu = +1$ in the case of Bosons and $\nu = -1$ in the case of Fermions. However, the use of the appropriate quantum statistics complicates the succeeding calculations. Referring to the results of [5] and others we suppose therefore that neglecting the quantum statistics will not yield substantial changes in the final results. Hence, we will work with Maxwell-Boltzmann statistics and simply take the many-body Hilbert space $\mathcal{H}^{\otimes N}$, i.e. we only retain the identity permutation in the symmetrizer of the states.

2.3 Time Evolution of the Fields

Now that we know the total Hamiltonian of our system, we can analyze its time evolution. Since the transverse part of the electric field is given by $\boldsymbol{E}_t = -\frac{1}{c}\partial_t\boldsymbol{A}$, we are in particular interested in the time evolution of the potential vector (2.6). The Schroedinger equation for the unitary time evolution operator reads

$$i\hbar\frac{d}{dt}U(t) = H^N_{L,R}U(t), \qquad U(0) = \mathbf{1}_{\mathcal{H}_{tot}} \tag{2.12}$$

and since the Hamiltonian $H^N_{L,R}$ is time-independent the solution of (2.12) is simply

$$U(t) = e^{-\frac{it}{\hbar}H^N_{L,R}}. \tag{2.13}$$

Hence the time evolution of $A(\boldsymbol{x})$ in the Heisenberg picture is derived from the Schroedinger operator according to

$$\boldsymbol{A}(\boldsymbol{x}, t) = U^\dagger(t)\boldsymbol{A}(\boldsymbol{x})U(t) = e^{\frac{it}{\hbar}H^N_{L,R}}\boldsymbol{A}(\boldsymbol{x})e^{-\frac{it}{\hbar}H^N_{L,R}}. \tag{2.14}$$

In the case of the free field, the time evolution of $\boldsymbol{A}(\boldsymbol{x}, t)$ can be expressed in terms of the evolution of the creation and annihilation operators [8]

$$\begin{aligned} a_{\boldsymbol{k}\lambda} &\to a_{\boldsymbol{k}\lambda}(t) = e^{i\omega_k t}a_{\boldsymbol{k}\lambda} \\ a^\dagger_{\boldsymbol{k}\lambda} &\to a^\dagger_{\boldsymbol{k}\lambda}(t) = e^{-i\omega_k t}a^\dagger_{\boldsymbol{k}\lambda}. \end{aligned} \tag{2.15}$$

In the present case however, the time evolution is much more complicated than this. Fortunately, our calculations will be based on the time-independent operators, i.e. we only need to know the field expressions at $t = 0$. The magnetic field $\boldsymbol{B}$ is trivially calculated from (2.6) and the transverse field (2.8) can be found by means of the Heisenberg equation. In point of fact, we have

$$\begin{aligned} \boldsymbol{E}_t(\boldsymbol{x}, t = 0) &= -\frac{1}{c}\left.\frac{d\boldsymbol{A}(\boldsymbol{x}, t)}{dt}\right|_{t=0} \\ &= -\left.\frac{1}{i\hbar c}U^\dagger(t)[\boldsymbol{A}(x), H^N_{L,R}]U(t)\right|_{t=0} \\ &= -\frac{1}{i\hbar c}\left\{[\boldsymbol{A}(x), H^{rad}_0] + [\boldsymbol{A}(x), H^N_{kin}]\right\} \\ &= -\frac{1}{i\hbar c}[\boldsymbol{A}(x), H^{rad}_0], \end{aligned} \tag{2.16}$$

where the second equality is the Heisenberg equation for time-independent Hamiltonians and the third one is due to the trivial commutation relations $[\boldsymbol{A}(\boldsymbol{x}), H^N_{pot}] = [\boldsymbol{A}(\boldsymbol{x}), H^N_C] = 0$. The last equality is based on the less trivial relation $[\boldsymbol{A}(\boldsymbol{x}), H^N_{kin}] = 0$ which can be proven using $[A^i(\boldsymbol{x}), A^j(\boldsymbol{y})] = 0$, cf [8]. The evaluation of the commutator $[\boldsymbol{A}(x), H^{rad}_0]$ yields finally (2.8).

2.4 Statistical Aspects

As stated above, we are interested in the thermal equilibrium of the system and hence we have to precise the statistical mechanical framework. Since the photons are massless particles, their chemical potential μ is zero and their correct statistical description refers to the grand-canonical ensemble via the Fock space $\mathcal{F}_{photons}$. Regarding the quantum charges, we can either work in the canonical ensemble via the N-particle space $\mathcal{H}_\nu^{\otimes N}$ or we can construct the corresponding particle Fock space $\mathcal{F}_{particles}$. In the former case, the appropriate total Hamiltonian is $H_{L,R}^N$ given in (2.9) whereas in the latter case we have to switch to the following Hamiltonian

$$H_{L,R} = \sum_{N=0}^{\infty} \left(H_{kin}^N + H_C^N + H_{pot}^N\right) + H_0^{rad} \tag{2.17}$$

on the joint Fock space $\mathcal{F}_{particles} \otimes \mathcal{F}_{photons}$. The canonical partition function reads

$$Z_{L,R}^N(T) = Tr\, e^{-\beta H_{L,R}^N} \tag{2.18}$$

where $Tr = Tr_{mat} Tr_{rad}$ is the total trace over the particles' (Tr_{mat}) and the field's (Tr_{rad}) degrees of freedom. The canonical average value at temperature T of an observable A is given by

$$< A >_{L,R} = \frac{Tr\, e^{-\beta H_{L,R}^N}}{Z_{L,R}^N(T)}. \tag{2.19}$$

As usual the grand-canonical description is more realistic since it allows for variable particle numbers. However, once we have calculated the canonical partition function $Z_{L,R}^N(T)$ it is merely a matter of resummation to obtain the grand-canonical extension

$$\Xi_{L,R}(\mu, T) = \sum_{\{N_\alpha\}}^{\infty} \prod_{\alpha=1}^{\mathcal{L}} \frac{z_\alpha^{N_\alpha}}{N_\alpha!} Z_{L,R}^{\{N_\alpha\}}(T). \tag{2.20}$$

We shall therefore focus our interest on the canonical partition function.

A quantity we will often use in the following calculations is the total partition function $Z_{L,R}^N(T)$ normalized by the partition function of the free radiation field $Z_0^{rad}(T)$

$$Z(T, N) = \frac{1}{Z_0^{rad}} Z_{L,R}^N(T) = \frac{1}{Z_0^{rad}} Tr\, e^{-\beta H_{L,R}^N}, \tag{2.21}$$

where

$$Z_0^{rad} \equiv Z_0^{rad}(T) = Tr_{rad} e^{-\beta H_0^{rad}}. \tag{2.22}$$

The thermal Gibbs weight is simply given by

$$\rho_{L,R}^{tot} = \frac{1}{Z_{L,R}^N} e^{-\beta H_{L,R}^N}. \tag{2.23}$$

Later on, we shall be interested in the effective particle interactions, i.e. we will integrate out the field's degrees of freedom and analyze expressions solely depending on the matter's degrees of freedom. In this aim we introduce here the reduced (i.e. effective) thermal Gibbs weight

$$\rho_{L,R} = \frac{1}{Z_0^{rad}} Tr_{rad}\, e^{-\beta H^N_{L,R}}. \tag{2.24}$$

In terms of $\rho_{L,R}$, the average of an observable $< O^{mat} >_{L,R}$ concerning only the particles reads simply

$$< O^{mat} >_{L,R} = \frac{Tr_{mat}\, \rho_{L,R}}{Tr_{mat}(O_{mat}\, \rho_{L,R})}. \tag{2.25}$$

As the indices L and R indicate it in $Z^N_{L,R}(T)$ and $\rho_{L,R}$, we are still considering a finite system confined in the boxes Λ and V. Since we focus our attention in the present paper on correlations in the large distance asymptotics, we are urged to consider an infinitely extended system, i.e. we have to perform the thermodynamic limit. In a first instance, we will take the limit $R \to \infty$ and hence obtain a system of N confined particles immersed in an infinitely extended radiation field. The existence of the limit $L \to \infty$ (keeping the particle densities $\{\rho_\alpha\}$ constant) is far from being evident and a rigorous proof is still missing in the case of the Hamiltonian (2.9). However, we shall tacitly assume its existence and suppose that in the limit $R \to \infty$ and $L \to \infty$ the thermodynamic average values $< A >$ coincide with the finite system averages $< A >_{L,R}$. For the sake of simplicity we will therefore omit the indices L and R from the beginning.

Chapter 3

Correlations in the Classical Fields

3.1 Recap of the Model with Classical Fields

In chapter 2 we have introduced the full quantum system, i.e. the quantum plasma coupled to the quantized radiation field. An interesting aspect in such situations are the correlations in both the matter and the fields. A thorough analysis of the former type can be found in [4] or [5]. The field correlations on the other hand are of special interest because they characterize the coherence properties of the radiation. However, their explicit calculation is a rather complicated task in the full quantum system and before we attack it in chapter 4, we would like to analyze foremost the case of the classical radiation field. Another motivation for this procedure is the possibility for classical limit checks: by taking the limit $\hbar \to 0$ in the quantum results we will be able to judge their correctness by comparison with the results of the present section.

First of all, we have to adapt some definitions of chapter 2 to the classical field. Basically, the Hamiltonian $H_{L,R}^N$ given in (2.9) is unaltered expcept that we have to replace the creation and annihilation operators $a_{\boldsymbol{k}\lambda}^\dagger$ and $a_{\boldsymbol{k}\lambda}$ by complex c-numbers $\alpha_{\boldsymbol{k}\lambda}^*$ and $\alpha_{\boldsymbol{k}\lambda}$. Consequently, the total Hamiltonian $H_{L,R}^N$ is now an operator acting on the reduced Hilbert space $\mathcal{H}_\nu^{\otimes N}$ (or $\mathcal{H}^{\otimes N}$ in the case of Maxwell-Boltzmann statistics) and the vector potential reads

$$\boldsymbol{A}(\boldsymbol{r}) = \Big(\frac{4\pi\hbar c^2}{R^3}\Big)^{\frac{1}{2}} \sum_{\boldsymbol{k}\lambda} g(\boldsymbol{k}) \frac{\boldsymbol{e}_{\boldsymbol{k}\lambda}}{\sqrt{2\omega_{\boldsymbol{k}}}} (\alpha_{\boldsymbol{k}\lambda}^* e^{-i\boldsymbol{k}\cdot\boldsymbol{r}} + \alpha_{\boldsymbol{k}\lambda} e^{i\boldsymbol{k}\cdot\boldsymbol{r}}). \tag{3.1}$$

Since $\boldsymbol{A}$ is now a purely classical quantity, one may choose to absorb the $\hbar$ in the Fourier coefficients of the field, $\alpha_{\boldsymbol{k}\lambda} \mapsto \alpha_{\boldsymbol{k}\lambda}/\sqrt{\hbar}$. In any case, the final results will be independent of this $\hbar$ as it is to expected for a classical quantity (of course there is still the $\hbar$ coming from the quantum nature of the matter, it is however absorbed in the de Broglie wave length λ. Another consequence of the classical nature of the field concerns the thermal Gibbs weight. By rewriting the total Hamiltonian as $H^N_{L,R} = H_A + H^{rad}_0$ with

$$\begin{aligned} H_A &= H_{kin} + H_C + H_{pot} \\ H^{rad}_0 &= \sum_{\boldsymbol{k}\lambda} \hbar\omega_{\boldsymbol{k}}\, \alpha^*_{\boldsymbol{k}\lambda}\alpha_{\boldsymbol{k}\lambda}, \end{aligned} \tag{3.2}$$

we see that $[H_A, H^{rad}_0] = 0$ and hence the Gibbs weight factorizes as follows:

$$e^{-\beta H^N_{L,R}} = e^{-\beta H^{rad}_0} e^{-\beta H_A}. \tag{3.3}$$

The free radiation partition function (2.22) becomes now a simple Gaussian integral

$$Z^{rad}_0 = Tr_{rad}\, e^{-\beta H^{rad}_0} = \int d^2\alpha_{\boldsymbol{k}\lambda} e^{-\beta H^{rad}_0} = \prod_{\boldsymbol{k}\lambda} \Big(\frac{\pi}{\beta\hbar\omega_{\boldsymbol{k}}}\Big), \tag{3.4}$$

where $d^2\alpha_{\boldsymbol{k}\lambda} = dIm(\alpha_{\boldsymbol{k}\lambda})dRe(\alpha_{\boldsymbol{k}\lambda})$ and the reduced thermal weight (2.24) can be rewritten in a more compact form as

$$\rho_{L,R} = \big\langle e^{-\beta H_A} \big\rangle_{rad}, \tag{3.5}$$

where we define for an observable $A = A(\{\alpha_{\boldsymbol{k}\lambda}\})$ depending solely on the field's degrees of freedom

$$\begin{aligned} \big\langle A \big\rangle_{rad} &= \frac{1}{Z^{rad}_0} \prod_{\boldsymbol{k}\lambda} \int d^2\alpha_{\boldsymbol{k}\lambda}\, e^{-\beta \sum_{\boldsymbol{k}\lambda} \hbar\omega_{\boldsymbol{k}}|\alpha_{\boldsymbol{k}\lambda}|^2} A(\{\alpha_{\boldsymbol{k}\lambda}\}) \\ &= \prod_{\boldsymbol{k}\lambda} \int \frac{d^2\alpha_{\boldsymbol{k}\lambda}}{\pi} \Big[\beta\hbar\omega_{\boldsymbol{k}} e^{-\beta\hbar\omega_{\boldsymbol{k}}|\alpha_{\boldsymbol{k}\lambda}|^2}\Big] A(\{\alpha_{\boldsymbol{k}\lambda}\}). \end{aligned} \tag{3.6}$$

As a matter of fact, the free radiation weight $e^{-\beta H^{rad}_0}$ is Gaussian and the arising integrals in (3.5) and (3.6) are Gaussian integrals. This will considerably facilitate explicit calculations later on. Furthermore, let us note that the Hamiltonian $H_A = H_A(\{\alpha_{\boldsymbol{k}\lambda}\})$ is now the energy of the particles in a given realization of the Gaussian random field and the elimination of the field's degrees of freedom in (3.5) yields therefore an effective statistical weight. We will precise this point later on in this section.

Let us now proceed to the calculation of the truncated equal-time field correlations in thermal equilibrium $< A^\mu(\boldsymbol{x})A^\nu(\boldsymbol{y}) >_T$, $< E^\mu(\boldsymbol{x})E^\nu(\boldsymbol{y}) >_T$ and $< B^\mu(\boldsymbol{x})B^\nu(\boldsymbol{y}) >_T$. The canonical average is a time-independent phase space average and we set $t = 0$ in order to facilitate the following calculations. Hence we will work with the fields

$$\begin{aligned} &\boldsymbol{A}(\boldsymbol{r}) = \boldsymbol{A}(\boldsymbol{r}, t = 0), \\ &\boldsymbol{B}(\boldsymbol{r}) = \boldsymbol{B}(\boldsymbol{r}, t = 0) = \nabla \wedge \boldsymbol{A}(\boldsymbol{r}), \\ &\boldsymbol{E}_t(\boldsymbol{r}) = \boldsymbol{E}_t(\boldsymbol{r}, t = 0) = -\tfrac{1}{c}\tfrac{\partial}{\partial t}\boldsymbol{A}(\boldsymbol{r}, t)\Big|_{t=0}, \end{aligned} \tag{3.7}$$

where $\boldsymbol{E}_t$ is the transverse part of the electric field (2.8) which is mediated by the photons. There is, of course, a longitudinal contribution $\boldsymbol{E}_l$ arising from the Coulomb interaction between the charges. This term has already been calculated in [9] and we shall focus our attention on the transverse part. Let us start with the calculation of the spatial correlations of the vector potential $< A^\mu(\boldsymbol{x}) A^\nu(\boldsymbol{y}) >_T$. One possibility is to proceed according to [5] by coupling the original Hamiltonian $H^N_{L,R}$ to an external current $\boldsymbol{\mathcal{J}}_{ext}$

$$H^N_{L,R}(\boldsymbol{\mathcal{J}}_{ext}) = H^N_{L,R} - i \int d\boldsymbol{r} \boldsymbol{\mathcal{J}}_{ext}(\boldsymbol{r}) \cdot \boldsymbol{A}(\boldsymbol{r}) \tag{3.8}$$

such that the correlations can be found by functional differentiation

$$< A^\mu(\boldsymbol{x}) A^\nu(\boldsymbol{y}) >_T = -\frac{1}{\beta^2} \frac{\delta^2}{\delta \mathcal{J}^\mu_{ext}(\boldsymbol{x}) \delta \mathcal{J}^\nu_{ext}(\boldsymbol{y})} \, ln\, Tr e^{-\beta H^N_{L,R}(\boldsymbol{\mathcal{J}}_{ext})} \Big|_{\boldsymbol{\mathcal{J}}_{ext}=0}. \tag{3.9}$$

However, this method is not suited for the quantum field because $[H^N_{L,R}, \boldsymbol{A}] \neq 0$ and hence formula (3.9) is not applicable any more. It is the aim of this section to calculate the correlations by an alternative method which is then easily generalized to the full quantum case in chapter 4. Let us at first consider the explicit expression which reads after substitution of (3.1)

$$\begin{aligned} < A^\mu(\boldsymbol{x}) A^\nu(\boldsymbol{y}) >_T = & \left(\frac{4\pi\hbar c^2}{R^3}\right) \sum_{\boldsymbol{k}\lambda} \sum_{\boldsymbol{k}'\lambda'} \frac{g(\boldsymbol{k}) g(\boldsymbol{k}')}{2\sqrt{\omega_{\boldsymbol{k}} \omega_{\boldsymbol{k}'}}} \boldsymbol{e}^\mu_{\boldsymbol{k}\lambda} \boldsymbol{e}^\nu_{\boldsymbol{k}'\lambda'} \times \\ & \times \Big[e^{-i\boldsymbol{kx}} e^{-i\boldsymbol{k}'\boldsymbol{y}} < \alpha^*_{\boldsymbol{k}\lambda} \alpha^*_{\boldsymbol{k}'\lambda'} > + e^{-i\boldsymbol{kx}} e^{+i\boldsymbol{k}'\boldsymbol{y}} < \alpha^*_{\boldsymbol{k}\lambda} \alpha_{\boldsymbol{k}'\lambda'} > + \\ & + e^{+i\boldsymbol{kx}} e^{-i\boldsymbol{k}'\boldsymbol{y}} < \alpha_{\boldsymbol{k}\lambda} \alpha^*_{\boldsymbol{k}'\lambda'} > + e^{+i\boldsymbol{kx}} e^{+i\boldsymbol{k}'\boldsymbol{y}} < \alpha_{\boldsymbol{k}\lambda} \alpha_{\boldsymbol{k}'\lambda'} > \Big]. \end{aligned} \tag{3.10}$$

We observe that this expression involves merely four different types of canonical averages:

$$< \alpha_{\boldsymbol{k}\lambda} \alpha_{\boldsymbol{k}'\lambda'} >, \quad < \alpha^*_{\boldsymbol{k}\lambda} \alpha^*_{\boldsymbol{k}'\lambda'} >, \quad < \alpha^*_{\boldsymbol{k}\lambda} \alpha_{\boldsymbol{k}'\lambda'} > \quad \text{and} \quad < \alpha_{\boldsymbol{k}\lambda} \alpha^*_{\boldsymbol{k}'\lambda'} > . \tag{3.11}$$

Since the Fourier coefficients are merely c-numbers, the third and the fourth average in (3.11) differ obviously by a sole exchange of indices. However, in the quantum case the c-numbers will be replaced by non-commutative creation and annihilation operators and in order to outline the resemblance between the two calculations we keep for instance all four terms. Once we will have calculated these quantities, it will merely be a matter of resummation to recover the correlation (3.10). Since $< \alpha_{\boldsymbol{k}\lambda} \alpha_{\boldsymbol{k}'\lambda'} >_T$ etc. are just second order moments in the Fourier coefficients, we can calculate them as usual by means of a generating function. In our case, the generating function is acutally the reduced thermal weight (3.5) and we shall obtain the various averages by taking appropriate second order derivatives of $\rho_{L,R}$. These intentions in mind, it is very instructive to calculate foremost the partition function $Z(T, N)$.

3.2 The Partition Function $Z(T, N)$

The calculation of $Z(T, N)$ has already been given in [5] and we outline here merely the introduction of the Feynman-Kac-Itô (FKI) formula as well as some other aspects that will be useful for further considerations. The partition function (2.21) reads explicitely:

$$Z(T,N) = \frac{1}{Z_0^{rad}} \prod_{i=1}^{N} \int d\boldsymbol{r}_i \prod_{\boldsymbol{k}\lambda} \int d^2\alpha_{\boldsymbol{k}\lambda}\, e^{-\beta H_0^{rad}} < \boldsymbol{r}_1, \cdots, \boldsymbol{r}_N | e^{-\beta H_A} | \boldsymbol{r}_1, \cdots, \boldsymbol{r}_N > . \tag{3.12}$$

Before proceeding to explicit calculations, we would like to make the following remarks: whereas the trace over the particles' degrees of freedom is far from being trivial, the trace over the radiation field is a simple Gaussian integral and is therefore feasible. However, the exponent $-\beta H_A$ in (3.12) is a quadratic form in the Fourier coefficients $\alpha_{\boldsymbol{k}\lambda}$ and the operatorial nature of H_A complicates the radiation trace substantially. The latter problem can be avoided by means of the Feynman-Kac-Itô formula [2] which yields a path integral representation of the diagonal matrix element in (3.12). The key feature of this reformulation is that we switch from the phase space of the quantum mechanical point charges to a purely classical phase space. In terms of this new phase space, the ith particle is now characterized by a so-called filament (or loop) $\mathcal{F}_i = (\alpha_i, \boldsymbol{r}_i, \boldsymbol{\xi}_i)$ where α_i denotes the particle's species, $\boldsymbol{r}_i$ its (classical) position and $\boldsymbol{\xi}_i$ is a dimensionless random vector. This random vector will, multiplied by the thermal de Broglie wavelength

$$\lambda_{\alpha_i} = \hbar \sqrt{\frac{\beta}{m_{\alpha_i}}}, \tag{3.13}$$

suggest a certain spatial extension of the particle. In point of fact, $\boldsymbol{\xi}_i$ is a closed Brownian path (the so-called Brownian bridge)

$$\boldsymbol{\xi}_i(s), \quad 0 \leq s \leq 1, \quad \boldsymbol{\xi}_i(0) = \boldsymbol{\xi}_i(1) = 0 \tag{3.14}$$

and its corresponding conditional Wiener measure is denoted by $\mathcal{D}(\boldsymbol{\xi})$. This is a Gaussian measure normalized to unity and we may formally write its integration weight as

$$\exp\left(-\frac{1}{2} \int_0^1 ds \left| \frac{d\boldsymbol{\xi}(s)}{ds} \right|^2 \right). \tag{3.15}$$

Its mean value is zero and the covariance is given by [1]

$$C^{\mu\nu} = \int \mathcal{D}(\boldsymbol{\xi}) \xi^{\mu}(s_1) \xi^{\nu}(s_2) = \delta^{\mu\nu}(min(s_1, s_2) - s_1 s_2). \tag{3.16}$$

In (3.14), s can be interpreted as a sort of imaginary time and we emphasize that the notation of (3.15) is purely formal because the time-derivative of a Brownian path with respect to time diverges at each point of the path. Consequently, we always have to refer to a discretized version of the path integral expressions if we would like to make explicit calculations. An utmost figurative interpretation of the spatial extension $\lambda_{\alpha_i}\boldsymbol{\xi}_i$ is the following: we have converted the quantum phase space of the two non-commutative canonical observables $\hat{\boldsymbol{p}}$ and $\hat{\boldsymbol{r}}$ into a classical phase space of the three commuting quantities $\mathcal{F}_i$. However, in order to achieve this mathematical simplifaction, we have to convert the operatorial incertitude into an intrinsic quantum fluctuation, i.e. the fluctuating extension $\lambda_{\alpha_i}\boldsymbol{\xi}_i$. Hence we can imagine the particles as being a collection of charged random loops. Although representing the quantum nature of the particles, these imaginary time fluctuations are not observable and we have to integrate over the Brownian bridges in order to obtain observable quantities.

These preliminary comments in mind, we can now give the representation of the diagonal matrix element in (3.12) in the path integral formalism (known as the Feynman-Kac-Itô formula):

$$\begin{aligned} < \{\boldsymbol{r}_i\}|e^{-\beta H_A}|\{\boldsymbol{r}_i\}> &= < \{\boldsymbol{r}_i\}|e^{-\beta(H_A+H_{kin}+H_{pot})}|\{\boldsymbol{r}_i\}> \\ &= \prod_{i=1}^{N}\left(\frac{1}{2\pi\lambda_{\alpha_i}^2}\right)^{\frac{3}{2}}\int \mathcal{D}(\boldsymbol{\xi}_i)e^{-\beta U_M(\mathcal{F}_1,\ldots,\mathcal{F}_N)-\beta U_A(\mathcal{F}_1,\ldots,\mathcal{F}_N,\boldsymbol{A})}, \end{aligned} \tag{3.17}$$

where

$$U_M(\mathcal{F}_1,\ldots,\mathcal{F}_N) = \sum_{i<j}^{N} e_{\alpha_i}e_{\alpha_j}V_C(\mathcal{F}_i,\mathcal{F}_j) + \sum_{i<j}^{N} e_{\alpha_i}e_{\alpha_j}V_{sr}(\mathcal{F}_i,\mathcal{F}_j) + \sum_{i=1}^{N} V_{walls}(\mathcal{F}_i) \tag{3.18}$$

and

$$U_A(\mathcal{F}_1,\ldots,\mathcal{F}_N,\boldsymbol{A}) = -i\sum_{i=1}^{N}\frac{e_{\alpha_i}}{c^2\sqrt{\beta m_{\alpha_i}}}\int_0^1 d\boldsymbol{\xi}_i(s)\cdot\boldsymbol{A}(\boldsymbol{r}_i+\lambda_{\alpha_i}\boldsymbol{\xi}_i(s)). \tag{3.19}$$

In (3.18), $V_C(\mathcal{F}_i,\mathcal{F}_j)$ is an equal-time Coulomb interaction potential between two loops

$$V_C(\mathcal{F}_i,\mathcal{F}_j) = \int_0^1 ds\frac{1}{|\boldsymbol{r}_j+\lambda_{\alpha_j}\boldsymbol{\xi}_j(s)-\boldsymbol{r}_i-\lambda_{\alpha_i}\boldsymbol{\xi}_i(s)|} \tag{3.20}$$

and $V_{sr}(\mathcal{F}_i,\mathcal{F}_j)$ and $V_{walls}(\mathcal{F}_i)$ are similarly defined. Care needs to be taken with the stochastic line integral in (3.19). If one comes down to explicit calculations there are several possibilities for the discretization of the integral. We shall stick to the middle-point rule for subsequent calculations. Other rules are possible and we refer to [2] for details. Taking into account the Feynman-Kac-Itô formula (3.17) as well as the definition (3.6), we can rewrite the partition function (3.12) as

$$Z(T,N) = \prod_{i=1}^{N} \left(\frac{1}{2\pi\lambda_{\alpha_i}} \right)^{\frac{3}{2}} \int \mathcal{D}(\boldsymbol{\xi}_i) \int d\boldsymbol{r}_i \; e^{-\beta U_M(\mathcal{F}_1,\ldots,\mathcal{F}_N)} \; < e^{-\beta U_A(\mathcal{F}_1,\ldots,\mathcal{F}_N,\boldsymbol{A})} >_{rad} . \tag{3.21}$$

Let us make some comments on the structure of this formula. Actually, in terms of the new classical phase space of filaments, $\prod_{i=1}^{N} \left(\frac{1}{2\pi\lambda_{\alpha_i}} \right)^{\frac{3}{2}} \int \mathcal{D}(\boldsymbol{\xi}_i) \int d^3\boldsymbol{r}_i$ is the trace over the particles' degrees of freedom and $e^{-\beta U_M} < e^{-\beta U_A} >_{rad}$ represents the thermal weight $\rho_{L,R}$ defined in (3.5). The first factor $e^{-\beta U_M}$ represents the direct interaction between the particles and $< e^{-\beta U_A} >_{rad}$ is an effective interaction mediated by the electromagnetic field. The effective interaction energy U_A vanishes as soon as we switch off the field-matter interaction as can be seen in (3.19) by taking the limit $e_{\alpha_i} \to 0$. In comparison to (3.12), the reformulation of the statistical weight in (3.21) has substantially alleviated our task: the remaining object is purely classical without operator-dependence and the exponent $-\beta U_A$ is now a linear form in the Fourier coefficients $\{\alpha_{\boldsymbol{k}\lambda}\}$.

Let us now calculate the effective particle interaction by integrating out the field's degrees of freedom in $< e^{-\beta U_A} >_{rad}$. In point of fact, due to the factor i in U_A, the remaining integration over the field modes is a well-known Gaussian integral, i.e. the Fourier transform of a Gaussian. In analogy with the local interaction Hamiltonian between an exterior current $\boldsymbol{J}$ and the electromagnetic field, we rewrite

$$U_A(\mathcal{F}_1, \ldots, \mathcal{F}_N, \boldsymbol{A}) = -i \int d\boldsymbol{x} \, \boldsymbol{A}(\boldsymbol{x}) \cdot \boldsymbol{\mathcal{J}}(\boldsymbol{x}) \tag{3.22}$$

where

$$\boldsymbol{\mathcal{J}}(\boldsymbol{x}) = \sum_{i=1}^{N} \boldsymbol{j}(\mathcal{F}_i, \boldsymbol{x}), \qquad \boldsymbol{j}(\mathcal{F}_i, \boldsymbol{x}) = \frac{e_{\alpha_i}}{c\sqrt{\beta m_{\alpha_i}}} \int_0^1 d\boldsymbol{\xi}_i(s) \delta(\boldsymbol{x} - \boldsymbol{r}_i - \lambda_{\alpha_i} \boldsymbol{\xi}_i(s)) \tag{3.23}$$

It is erroneous to interprete the quantities $\boldsymbol{j}(\mathcal{F}_i, \boldsymbol{x})$ as physical currents. Indeed, s is not the physical time and the appearant velocities $d\boldsymbol{\xi}(s)/ds$ are ill-defined derivatives (that diverge at every point of the Brownian path). Therefore it is by pure analogy that we may call them "imaginary" time currents. Let's define now the c-numbers $\{u_{\boldsymbol{k}\lambda}\}$ by

$$u_{\boldsymbol{k}\lambda} = \beta \left(\frac{4\pi\hbar c^2}{R^3} \right)^{1/2} \frac{g(\boldsymbol{k})}{\sqrt{2\omega_k}} \boldsymbol{\mathcal{J}}(\boldsymbol{k}) \cdot \boldsymbol{e}_{\boldsymbol{k}\lambda}, \qquad \boldsymbol{\mathcal{J}} = \int d\boldsymbol{x} \, e^{-i\boldsymbol{k}\cdot\boldsymbol{x}} \boldsymbol{\mathcal{J}}(\boldsymbol{x}) \tag{3.24}$$

which allow us to rewrite the reduced thermal weight (we omit the part due to the direct interaction for brevity) simply as

$$< e^{-\beta U_A} >_{rad} = \left\langle e^{i\beta \int d\boldsymbol{x} \boldsymbol{A}(\boldsymbol{x}) \cdot \boldsymbol{\mathcal{J}}(\boldsymbol{x})} \right\rangle_{rad} = \left\langle \prod_{\boldsymbol{k}\lambda} e^{i(u^*_{kl}\alpha_{kl} + u_{\boldsymbol{k}\lambda}\alpha^*_{\boldsymbol{k}\lambda})} \right\rangle_{rad} . \tag{3.25}$$

The right-hand side of (3.25) can be calculated by means of the well-known Gaussian integral $\int \frac{d^2\alpha}{\pi} e^{-b|\alpha|^2+i(u^*\alpha+u\alpha^*)} = b^{-1}e^{-b^{-1}|u|^2}, b>0$, and yields

$$\left\langle e^{i\beta\int d\boldsymbol{x}\boldsymbol{A}(\boldsymbol{x})\cdot\boldsymbol{\mathcal{J}}(\boldsymbol{x})}\right\rangle_{rad} = \prod_{\boldsymbol{k}\lambda}\int\frac{d^2\alpha_{\boldsymbol{k}\lambda}}{\pi}\beta\hbar\omega_{\boldsymbol{k}}\exp\left(-\beta\hbar\omega_{\boldsymbol{k}}|\alpha_{\boldsymbol{k}\lambda}|^2+i(u^*_{\boldsymbol{k}\lambda}\alpha_{\boldsymbol{k}\lambda}+u_{\boldsymbol{k}\lambda}\alpha^*_{\boldsymbol{k}\lambda})\right)$$
$$= \prod_{\boldsymbol{k}\lambda} e^{-\frac{|u_{\boldsymbol{k}\lambda}|^2}{\beta\hbar\omega_{\boldsymbol{k}}}}. \tag{3.26}$$

Finally, by taking the sum over the polarizations λ

$$\sum_\lambda e^\mu_{\boldsymbol{k}\lambda}e^\nu_{\boldsymbol{k}\lambda} = \delta^{\mu\nu} - \frac{k^\mu k^\nu}{k^2} \tag{3.27}$$

and the infinite volume limit $R\to\infty$ for the volume V enclosing the field

$$\sum_{\boldsymbol{k}} \to \frac{R^3}{(2\pi)^3}\int d\boldsymbol{k} \tag{3.28}$$

we find the final expression for the reduced thermal weight

$$\exp\left(-\frac{\beta}{2R^3}\sum_{\boldsymbol{k}\lambda}\frac{4\pi g^2(\boldsymbol{k})}{k^2}\left|\boldsymbol{\mathcal{J}}(\boldsymbol{k})\cdot\boldsymbol{e}_{\boldsymbol{k}\lambda}\right|^2\right) \to \exp\left(-\frac{\beta}{2}\int\frac{d\boldsymbol{k}}{(2\pi)^3}((\mathcal{J}^\mu(\boldsymbol{k}))^*G^{\mu\nu}(\boldsymbol{k})\mathcal{J}^\nu(\boldsymbol{k})\right), \tag{3.29}$$

where $G^{\mu\nu}$ is the covariance of the free radiation field

$$G^{\mu\nu}(\boldsymbol{k}) = \frac{4\pi g^2(\boldsymbol{k})}{k^2}\delta^{\mu\nu}_{tr}(\boldsymbol{k}), \qquad \text{and} \qquad \delta^{\mu\nu}_{tr}(\boldsymbol{k}) = \delta^{\mu\nu} - \frac{k^\mu k^\nu}{k^2} \tag{3.30}$$

is the transverse Kronecker symbol. The exponent of (3.29) is a function of the filaments only and hence we can write an effective interaction potential $W_m(i,j)$ between two filaments $\mathcal{F}_i$ and $\mathcal{F}_j$:

$$W_m(i,j) = \frac{1}{\beta c^2\sqrt{m_{\alpha_i}m_{\alpha_j}}}\int\frac{d\boldsymbol{k}}{(2\pi)^3}e^{i\boldsymbol{k}\cdot(\boldsymbol{r}_i-\boldsymbol{r}_j)}G^{\mu\nu}(\boldsymbol{k})\int_0^1 d\xi^\mu_i(s_1)e^{i\lambda_{\alpha_i}\boldsymbol{k}\cdot\boldsymbol{\xi}_i(s_1)}\int_0^1 d\boldsymbol{\xi}^\nu_j(s_2)e^{-i\lambda_{\alpha_j}\boldsymbol{k}\cdot\boldsymbol{\xi}_j(s_2)} \tag{3.31}$$

such that reduced thermal weight (3.25) becomes

$$\left\langle e^{i\beta\int d\boldsymbol{x}\boldsymbol{A}(\boldsymbol{x})\cdot\boldsymbol{\mathcal{J}}(\boldsymbol{x})}\right\rangle_{rad} = e^{-\frac{\beta}{2}\sum_{i,j}^N e_{\alpha_i}e_{\alpha_j}W_m(i,j)} = e^{-\frac{\beta}{2}\sum_i^N e^2_{\alpha_i}W_m(i,i)}e^{-\beta\sum_{i<j}^N e_{\alpha_i}e_{\alpha_j}W_m(i,j)}. \tag{3.32}$$

This interaction arises from the complex interplay between the particle currents and the radiation field and is therefore cleary of a diamagnetic nature. As discussed in [6], this effective interaction has the same form as the interaction potential between two classical current-loops. We shall therefore nominate W_m the loop-loop effective magnetic potential. We note that the first exponential of (3.32) contains the magnetic self-energies of the loops. Finally, we can now use (3.21) and (3.32) to write the final expression of the partition function:

$$Z(T,N) = \prod_{i=1}^{N} \left(\frac{1}{2\pi\lambda_{\alpha_i}} \right)^{\frac{3}{2}} \int \mathcal{D}(\boldsymbol{\xi}_i) \int d\boldsymbol{r}_i \; e^{-\beta U_M(\mathcal{F}_1,\ldots,\mathcal{F}_N)} \; e^{-\frac{\beta}{2}\sum_{i,j}^{N} e_{\alpha_i} e_{\alpha_j} W_m(i,j)}. \quad (3.33)$$

3.3 The Second Order Field Moments

Let us now calculate the four truncated averages of (3.11) using $< e^{-\beta U_A} >_{rad}$ as a sort of generating function. In point of fact, we can rewrite $< \alpha_{\boldsymbol{k}_1\lambda_1} \alpha^*_{\boldsymbol{k}_2\lambda_2} >_T$ in terms of the path integral formulation

$$\begin{aligned} < \alpha_{\boldsymbol{k}_1\lambda_1} \alpha^*_{\boldsymbol{k}_2\lambda_2} > &= \frac{1}{Z^N_{L,R}} Tr(\alpha_{\boldsymbol{k}_1\lambda_1} \alpha^*_{\boldsymbol{k}_2\lambda_2} e^{-\beta H^N_{L,R}}) \\ &= \frac{1}{Z(T,N)} \prod_{i=1}^{N} \left(\frac{1}{2\pi\lambda_{\alpha_i}} \right)^{\frac{3}{2}} \int \mathcal{D}(\boldsymbol{\xi}_i) \int d\boldsymbol{r}_i \; e^{-\beta U_M(\mathcal{F}_1,\ldots,\mathcal{F}_N)} \times \\ &\quad \times \left\langle \alpha_{\boldsymbol{k}_1\lambda_1} \alpha^*_{\boldsymbol{k}_2\lambda_2} e^{-\beta U_A(\mathcal{F}_1,\ldots,\mathcal{F}_N,\boldsymbol{A})} \right\rangle_{rad} \end{aligned} \quad (3.34)$$

and similarly for the other three types of averages. According to equation (3.25) we can rewrite the reduced thermal weight of (3.34) as

$$\left\langle \alpha_{\boldsymbol{k}_1\lambda_1} \alpha^*_{\boldsymbol{k}_2\lambda_2} e^{i\beta \int d\boldsymbol{x} \boldsymbol{A}(\boldsymbol{x})\cdot \boldsymbol{\mathcal{J}}(\boldsymbol{x})} \right\rangle_{rad} = \left\langle \alpha_{\boldsymbol{k}_1\lambda_1} \alpha^*_{\boldsymbol{k}_2\lambda_2} \prod_{\boldsymbol{k}\lambda} e^{i(u^*_{kl}\alpha_{kl} + u_{\boldsymbol{k}\lambda}\alpha^*_{\boldsymbol{k}\lambda})} \right\rangle_{rad}. \quad (3.35)$$

At this point we can apply a standard trick of classical statistical mechanics and take advantage of the calculations made in section 3.2. We introduce a set of c-numbers $\{\mu_{\boldsymbol{k}\lambda}\}$ and generate (3.35) in form of a second order derivative of the reduced thermal weight:

$$\begin{aligned} &-\frac{\partial^2}{\partial\mu^*_{\boldsymbol{k}_1\lambda_1}\partial\mu_{\boldsymbol{k}_2\lambda_2}} \left\langle \prod\nolimits_{\boldsymbol{k}\lambda} e^{i(u^*_{\boldsymbol{k}\lambda}+\mu^*_{\boldsymbol{k}\lambda})\alpha_{\boldsymbol{k}\lambda} + (u_{\boldsymbol{k}\lambda}+\mu_{\boldsymbol{k}\lambda})\alpha^*_{\boldsymbol{k}\lambda})} \right\rangle_{rad} \Bigg|_{\{\mu_{\boldsymbol{k}\lambda}\}\equiv 0} \\ &= -\frac{\partial^2}{\partial\mu^*_{\boldsymbol{k}_1\lambda_1}\partial\mu_{\boldsymbol{k}_2\lambda_2}} \prod\nolimits_{\boldsymbol{k}\lambda} e^{-\frac{|u_{\boldsymbol{k}\lambda}+\mu_{\boldsymbol{k}\lambda}|^2}{\hbar\beta\omega_{\boldsymbol{k}}}} \Bigg|_{\{\mu_{\boldsymbol{k}\lambda}\}\equiv 0} \\ &= \left(\frac{\delta^{\boldsymbol{k}_2\lambda_2}_{\boldsymbol{k}_1\lambda_1}}{\beta\hbar\omega_{\boldsymbol{k}_1}} - \frac{u_{\boldsymbol{k}_1\lambda_1}u^*_{\boldsymbol{k}_2\lambda_2}}{(\beta\hbar)^2\omega_{\boldsymbol{k}_1}\omega_{\boldsymbol{k}_2}} \right) e^{-\frac{\beta}{2}\sum_{i,j}^{N} e_{\alpha_i}e_{\alpha_j}W_m(i,j)}. \end{aligned} \quad (3.36)$$

Equation (3.34) together with the definition (3.24) yields eventually the final expression:

$$< \alpha_{\boldsymbol{k}_1\lambda_1}\alpha^*_{\boldsymbol{k}_2\lambda_2} > = \frac{1}{Z(T,N)}\prod_{i=1}^{N}\left(\frac{1}{2\pi\lambda_{\alpha_i}}\right)^{\frac{3}{2}}\int\mathcal{D}(\boldsymbol{\xi}_i)\int d\boldsymbol{r}_i\; e^{-\beta U_M(\mathcal{F}_1,\ldots,\mathcal{F}_N)}e^{-\frac{\beta}{2}\sum_{i,j}^{N}e_{\alpha_i}e_{\alpha_j}W_m(i,j)}\times$$
$$\times\left[\frac{\delta^{\boldsymbol{k}_2\lambda_2}_{\boldsymbol{k}_1\lambda_1}}{\beta\hbar\omega_{\boldsymbol{k}_1}} - \left(\frac{2\pi c^2}{\hbar R^3}\right)\frac{g(\boldsymbol{k}_1)g(\boldsymbol{k}_2)}{(\omega_{\boldsymbol{k}_1}\omega_{\boldsymbol{k}_2})^{3/2}}\mathcal{J}^\nu(\boldsymbol{k}_1)(\mathcal{J}^\mu(\boldsymbol{k}_2))^*\boldsymbol{e}^\nu_{\boldsymbol{k}_1\lambda_1}\boldsymbol{e}^\mu_{\boldsymbol{k}_2\lambda_2}\right]. \tag{3.37}$$

In the exact same manner we recover the remaining two averages which however lack the diagonal contribution proportional to $\delta^{\boldsymbol{k}_2\lambda_2}_{\boldsymbol{k}_1\lambda_1}$

$$< \alpha^*_{\boldsymbol{k}_1\lambda_1}\alpha^*_{\boldsymbol{k}_2\lambda_2} > = \frac{1}{Z(T,N)}\prod_{i=1}^{N}\left(\frac{1}{2\pi\lambda_{\alpha_i}}\right)^{\frac{3}{2}}\int\mathcal{D}(\boldsymbol{\xi}_i)\int d\boldsymbol{r}_i\; e^{-\beta U_M(\mathcal{F}_1,\ldots,\mathcal{F}_N)}e^{-\frac{\beta}{2}\sum_{i,j}^{N}e_{\alpha_i}e_{\alpha_j}W_m(i,j)}\times$$
$$\times\left[-\left(\frac{2\pi c^2}{\hbar R^3}\right)\frac{g(\boldsymbol{k}_1)g(\boldsymbol{k}_2)}{(\omega_{\boldsymbol{k}_1}\omega_{\boldsymbol{k}_2})^{3/2}}(\mathcal{J}^\nu(\boldsymbol{k}_1))^*(\mathcal{J}^\mu(\boldsymbol{k}_2))^*\boldsymbol{e}^\nu_{\boldsymbol{k}_1\lambda_1}\boldsymbol{e}^\mu_{\boldsymbol{k}_2\lambda_2}\right] \tag{3.38}$$

and

$$< \alpha_{\boldsymbol{k}_1\lambda_1}\alpha_{\boldsymbol{k}_2\lambda_2} > = \frac{1}{Z(T,N)}\prod_{i=1}^{N}\left(\frac{1}{2\pi\lambda_{\alpha_i}}\right)^{\frac{3}{2}}\int\mathcal{D}(\boldsymbol{\xi}_i)\int d\boldsymbol{r}_i\; e^{-\beta U_M(\mathcal{F}_1,\ldots,\mathcal{F}_N)}e^{-\frac{\beta}{2}\sum_{i,j}^{N}e_{\alpha_i}e_{\alpha_j}W_m(i,j)}\times$$
$$\times\left[-\left(\frac{2\pi c^2}{\hbar R^3}\right)\frac{g(\boldsymbol{k}_1)g(\boldsymbol{k}_2)}{(\omega_{\boldsymbol{k}_1}\omega_{\boldsymbol{k}_2})^{3/2}}\mathcal{J}^\nu(\boldsymbol{k}_1)\mathcal{J}^\mu(\boldsymbol{k}_2)\boldsymbol{e}^\nu_{\boldsymbol{k}_1\lambda_1}\boldsymbol{e}^\mu_{\boldsymbol{k}_2\lambda_2}\right]. \tag{3.39}$$

We see that in absence of the field-matter interaction ($\boldsymbol{\mathcal{J}} \equiv 0$), $< \alpha_{\boldsymbol{k}_1\lambda_1}\alpha_{\boldsymbol{k}_2\lambda_2} >_T$ and $< \alpha^*_{\boldsymbol{k}_1\lambda_1}\alpha^*_{\boldsymbol{k}_2\lambda_2} >_T$ are zero. The same is true for$< \alpha_{\boldsymbol{k}_1\lambda_1}\alpha^*_{\boldsymbol{k}_2\lambda_2} >_T$ except for the diagonal contributions ($\boldsymbol{k}_1\lambda_1 = \boldsymbol{k}_2\lambda_2$) that survive due to the matter-independent term proporional to $\delta^{\boldsymbol{k}_2\lambda_2}_{\boldsymbol{k}_1\lambda_1}$ in (3.37). This was to be expected because we know that in the free radiation field the modes are independent and furthermore, on purely mathematical grounds, we see that only a product of equal numbers of stars and non-stars of the same mode can give a non-zero Gaussian integral. This result will become physically meaningful in the quantum theory of radiation because only observables that conserve the number of particles are allowed to have non-zero averages. We will discuss this property and some restricted generalizations to interacting systems in appendix C.

3.4 The Field Correlations

We are now in the position to calculate explicitely the field correlations $< A^\mu(\boldsymbol{x})A^\nu(\boldsymbol{y}) >_T$, $< E^\mu(\boldsymbol{x})E^\nu(\boldsymbol{y}) >_T$ and $< B^\mu(\boldsymbol{x})B^\nu(\boldsymbol{y}) >_T$ by means of the averages (3.37), (3.38) and (3.39). As we have emphasized in the end of section 3.3, these averages contain both matter-independent as well as matter-dependent contributions. Hence we decompose the field correlations into a free field correlation $< \boldsymbol{A}^\mu(\boldsymbol{x})\boldsymbol{A}^\nu(\boldsymbol{y}) >_T^0$ and the correlation $< A^\mu(\boldsymbol{x})A^\nu(\boldsymbol{y}) >_T^{mat}$ arising from the matter-field interaction:

$$< A^\mu(\boldsymbol{x})A^\nu(\boldsymbol{y}) >_T = < A^\mu(\boldsymbol{x})A^\nu(\boldsymbol{y}) >_T^0 + < A^\mu(\boldsymbol{x})A^\nu(\boldsymbol{y}) >_T^{mat} . \tag{3.40}$$

First, we calculate the free field contribution by summing up the $\delta_{\boldsymbol{k}_1\lambda_1}^{\boldsymbol{k}_2\lambda_2}$-terms in (3.37). Using (3.10) together with the equality

$$Z(T,N) = \prod_{i=1}^{N}\left(\frac{1}{2\pi\lambda_{\alpha_i}}\right)^{\frac{3}{2}}\int \mathcal{D}(\boldsymbol{\xi}_i)\int d^3\boldsymbol{r}_i\; e^{-\beta U_M(\mathcal{F}_1,\ldots,\mathcal{F}_N)}e^{-\frac{\beta}{2}\sum_{i,j}^{N} e_{\alpha_i}e_{\alpha_j}W_m(i,j)} \tag{3.41}$$

we find

$$\begin{aligned} < A^\mu(\boldsymbol{x})A^\nu(\boldsymbol{y}) >_T^0 &= \left(\frac{4\pi\hbar c^2}{R^3}\right)\sum_{\boldsymbol{k}\lambda}\frac{g(\boldsymbol{k})^2}{2\omega_{\boldsymbol{k}}}e^{i\boldsymbol{k}\cdot(\boldsymbol{x}-\boldsymbol{y})}\boldsymbol{e}^\mu_{\boldsymbol{k}\lambda}\boldsymbol{e}^\nu_{\boldsymbol{k}\lambda}\left[\frac{2}{\beta\hbar\omega_{\boldsymbol{k}}}\right] \\ &= \frac{1}{\beta R^3}\sum_{\boldsymbol{k}}\frac{4\pi g(\boldsymbol{k})^2}{k^2}\delta^{\mu\nu}_{tr}(\boldsymbol{k})e^{i\boldsymbol{k}\cdot(\boldsymbol{x}-\boldsymbol{y})} \\ &\to \frac{1}{\beta}\int\frac{d\boldsymbol{k}}{(2\pi)^3}G^{\mu\nu}(\boldsymbol{k})e^{i\boldsymbol{k}\cdot(\boldsymbol{x}-\boldsymbol{y})}. \end{aligned} \tag{3.42}$$

In the end we have performed the infinite volume limit $R\to\infty$ according to (3.28). Let us now calculate the contribution due to the matter-field interaction. For the sake of simplicity, we consider foremost the fourth term of (3.10):

$$\begin{aligned} &\left(\tfrac{4\pi\hbar c^2}{R^3}\right)\sum_{\boldsymbol{k}\lambda}\sum_{\boldsymbol{k}'\lambda'}\tfrac{g(\boldsymbol{k})g(\boldsymbol{k}')}{2\sqrt{\omega_{\boldsymbol{k}}\omega_{\boldsymbol{k}'}}}\boldsymbol{e}^\mu_{\boldsymbol{k}\lambda}\boldsymbol{e}^\nu_{\boldsymbol{k}'\lambda'}\left[e^{i\boldsymbol{k}\boldsymbol{x}}e^{i\boldsymbol{k}'\boldsymbol{y}} < \alpha_{\boldsymbol{k}\lambda}\alpha_{\boldsymbol{k}'\lambda'} >\right] = \\ &= -\tfrac{1}{4}\tfrac{1}{Z(T,N)}\prod_{i=1}^N\left(\tfrac{1}{2\pi\lambda_{\alpha_i}}\right)^{\frac{3}{2}}\int\mathcal{D}(\boldsymbol{\xi}_i)\int d\boldsymbol{r}_i\; e^{-\beta U_M(\mathcal{F}_1,\ldots,\mathcal{F}_N)}e^{-\frac{\beta}{2}\sum_{i,j}^N e_{\alpha_i}e_{\alpha_j}W_m(i,j)}\times \\ &\times\int\tfrac{d\boldsymbol{k}}{(2\pi)^3}\int\tfrac{d\boldsymbol{k}}{(2\pi)^3}e^{i\boldsymbol{k}\boldsymbol{x}}e^{i\boldsymbol{k}'\boldsymbol{y}}G^{\mu\sigma}(\boldsymbol{k})G^{\nu\tau}(\boldsymbol{k}')(\mathcal{J}^\sigma(\boldsymbol{k}))(\mathcal{J}^\tau(\boldsymbol{k}')). \end{aligned} \tag{3.43}$$

Since $\boldsymbol{\mathcal{J}}(-\boldsymbol{k}) = (\boldsymbol{\mathcal{J}}(\boldsymbol{k}))^*$ and $G^{\mu\nu}(-\boldsymbol{k}) = G^{\mu\nu}(\boldsymbol{k})$, we can easily see that the other three matter-depending contributions in (3.10) yield the exact same expression as (3.43). Since

$$\begin{aligned} < \mathcal{J}^\mu(\boldsymbol{k})\mathcal{J}^\nu(\boldsymbol{k}') > = \frac{1}{Z(T,N)}\prod_{i=1}^N\left(\frac{1}{2\pi\lambda_{\alpha_i}}\right)^{\frac{3}{2}}\int\mathcal{D}(\boldsymbol{\xi}_i)\int d\boldsymbol{r}_i\; e^{-\beta U_M(\mathcal{F}_1,\ldots,\mathcal{F}_N)}e^{-\frac{\beta}{2}\sum_{i,j}^N e_{\alpha_i}e_{\alpha_j}W_m(i,j)}\times \\ \times(\mathcal{J}^\sigma(\boldsymbol{k}))(\mathcal{J}^\tau(\boldsymbol{k}')) \end{aligned} \tag{3.44}$$

we can now sum up the four contributions to find

$$< A^\mu(\boldsymbol{x})A^\nu(\boldsymbol{y}) >_T^{mat} = -\int\frac{d\boldsymbol{k}}{(2\pi)^3}\int\frac{d\boldsymbol{k}}{(2\pi)^3}e^{i\boldsymbol{k}\boldsymbol{x}}e^{i\boldsymbol{k}'\boldsymbol{y}}G^{\mu\sigma}(\boldsymbol{k})G^{\nu\tau}(\boldsymbol{k}') < \mathcal{J}^\sigma(\boldsymbol{k})\mathcal{J}^\tau(\boldsymbol{k}') > . \tag{3.45}$$

In the case of the electric field, we use (2.8) to find

$$< E^\mu(\boldsymbol{x}) E^\nu(\boldsymbol{y}) >_T = \left(\frac{4\pi\hbar c^2}{R^3}\right) \sum_{\boldsymbol{k}\lambda} \sum_{\boldsymbol{k}'\lambda'} \frac{g(\boldsymbol{k}) g(\boldsymbol{k}')}{2\sqrt{\omega_{\boldsymbol{k}}\omega_{\boldsymbol{k}'}}} \boldsymbol{e}^\mu_{\boldsymbol{k}\lambda} \boldsymbol{e}^\nu_{\boldsymbol{k}'\lambda'} \times \tag{3.46}$$
$$\times \Big[-\left(\frac{\omega_{\boldsymbol{k}}\omega_{\boldsymbol{k}'}}{c^2}\right) e^{-i\boldsymbol{k}\boldsymbol{x}} e^{-i\boldsymbol{k}'\boldsymbol{y}} < \alpha^*_{\boldsymbol{k}\lambda} \alpha^*_{\boldsymbol{k}'\lambda'} > + \left(\frac{\omega_{\boldsymbol{k}}\omega_{\boldsymbol{k}'}}{c^2}\right) e^{-i\boldsymbol{k}\boldsymbol{x}} e^{+i\boldsymbol{k}'\boldsymbol{y}} < \alpha^*_{\boldsymbol{k}\lambda} \alpha_{\boldsymbol{k}'\lambda'} > +$$
$$+ \left(\frac{\omega_{\boldsymbol{k}}\omega_{\boldsymbol{k}'}}{c^2}\right) e^{+i\boldsymbol{k}\boldsymbol{x}} e^{-i\boldsymbol{k}'\boldsymbol{y}} < \alpha_{\boldsymbol{k}\lambda} \alpha^*_{\boldsymbol{k}'\lambda'} > - \left(\frac{\omega_{\boldsymbol{k}}\omega_{\boldsymbol{k}'}}{c^2}\right) e^{+i\boldsymbol{k}\boldsymbol{x}} e^{+i\boldsymbol{k}'\boldsymbol{y}} < \alpha_{\boldsymbol{k}\lambda} \alpha_{\boldsymbol{k}'\lambda'} > \Big].$$

Using again $\boldsymbol{\mathcal{J}}(-\boldsymbol{k}) = (\boldsymbol{\mathcal{J}}(\boldsymbol{k}))^*$ and $G^{\mu\nu}(-\boldsymbol{k}) = G^{\mu\nu}(\boldsymbol{k})$, we can show that the matter-depending contributions of the four terms (modulo their sign) are equal. Then, according to the two plus and two minus signs in (3.46), they cancel each other and we are left with the matter-independent contributions coming from the diagonal terms $\boldsymbol{k}\lambda = \boldsymbol{k}'\lambda'$:

$$< E^\mu(\boldsymbol{x}) E^\nu(\boldsymbol{y}) >_T = < E^\mu(\boldsymbol{x}) E^\nu(\boldsymbol{y}) >^0_T = \frac{1}{\beta} \int \frac{d\boldsymbol{k}}{(2\pi)^3} G^{\mu\nu}(\boldsymbol{k})\, k^2 e^{i\boldsymbol{k}(\boldsymbol{x}-\boldsymbol{y})}. \tag{3.47}$$

This is indeed a very interesting result: the electric field correlations are not influenced by the presence of the quantum plasma. Although the charges interact with the $\boldsymbol{E}$-field, there is no influence on the field correlations. Hence we have found a generalization of the Bohr-van-Leuween theorem: indeed, this theorem states a decoupling in the case of a fully classical system and we can now extend its validity to quantum matter as well. However, we shall see that this behaviour is merely due to the classical nature of the field. As soon as we will consider the full quantum system, we will find that the correlations are indeed coupled to the quantum charges.
The $\boldsymbol{B}$-field correlations are easily found by means of (2.2) and (3.40)

$$< B^\mu(\boldsymbol{x}) B^\nu(\boldsymbol{y}) >_T = \epsilon^{\mu m n} \epsilon^{\nu i j} \frac{\partial^2}{\partial x^m \partial y^i} < A^n(\boldsymbol{x}) A^j(\boldsymbol{y}) >_T . \tag{3.48}$$

At this point, we don't give the explicit expression, but will specify the asymptotic behaviour of this correlation in the next section.

3.5 Long Distance Asymptotics

The results (3.10), (3.47) and (3.48) of the previous section are exact but they do not give us much of a phyiscal insight. As usual in physics, we are interested in the nature of the decay of such correlations; we would like to know whether there is an exponential decay (screening) or whether we recover algebraic tails in the long distance asymptotics. In order to illuminate this question we let now $|\boldsymbol{x} - \boldsymbol{y}| \equiv |\boldsymbol{r}| \to \infty$ and we analyze the long distance behaviour of the correlations. Again, this has already been done in [5] and we merely state the results. In point of fact, the correlations obey an algebraic decay law and we have

$$< A^{\mu}(\boldsymbol{x})A^{\nu}(\boldsymbol{y}) >_T \sim \frac{1}{2r}\left(\delta^{\mu\nu} + \frac{r^{\mu}r^{\nu}}{r^2}\right)\left(\frac{1}{\beta} - 4\pi a\right), \quad r \to \infty, \tag{3.49}$$

$$< E_t^{\mu}(\boldsymbol{x})E_t^{\nu}(\boldsymbol{y}) >_T \sim \left(\partial_{\mu}\partial_{\nu}\frac{1}{r}\right)\frac{1}{\beta}, \quad r \to \infty, \tag{3.50}$$

and

$$< B^{\mu}(\boldsymbol{x})B^{\nu}(\boldsymbol{y}) >_T \sim \left(\partial_{\mu}\partial_{\nu}\frac{1}{r}\right)\left(\frac{1}{\beta} - 4\pi a\right), \quad r \to \infty. \tag{3.51}$$

The complicated matter-dependence of these relations is contained in the constant $a = a(\rho_{\alpha}, \beta, \hbar)$ which reads

$$a(\rho_{\alpha}, \beta, \hbar) = \frac{1}{2}\sum_{\alpha,\alpha'}\frac{e_{\alpha}e_{\alpha'}\lambda_{\alpha}\lambda_{\alpha'}}{\beta\sqrt{m_{\alpha}m_{\alpha'}}c^2}\int d\boldsymbol{r}\int D(\xi_1)\int D(\xi_2)\,\times \tag{3.52}$$
$$\times\int_0^1 d\xi^{\mu}(s)\int_0^1 d\xi'^{\nu}(s')(\hat{\boldsymbol{k}}\cdot\boldsymbol{\xi}(s))(\hat{\boldsymbol{k}}\cdot\boldsymbol{\xi}'(s'))\delta_{tr}^{\mu\nu}(\hat{\boldsymbol{k}})n_T(\alpha, \boldsymbol{\xi}, \alpha', \boldsymbol{\xi}', \boldsymbol{r}).$$

In the low density regime, we can perform density expansion of this coefficient by means of the Mayer diagrammatics (cf Appendix D.3). To the first order in the particle densities $\{\rho_{\alpha}\}$ we find in Appendix D.1

$$a(\rho_{\alpha}, \beta, \hbar) \sim \frac{1}{12}\sum_{\alpha}\frac{e_{\alpha}^2\lambda_{\alpha}^2}{\beta m_{\alpha}c^2}\rho_{\alpha}. \tag{3.53}$$

Finally and for the sake of completeness, we would like to state the leading order contribution to the correlations of the longitudinal part of the electric field. It has been calculated by [9] and is valid for both classical and quantum plasmas in a radiation field (classical or quantum-mechanical).

$$< E_l^{\mu}(\boldsymbol{x})E_l^{\nu}(\boldsymbol{y}) >\sim -\partial_{\mu}\partial_{\nu}\frac{1}{r}\left(-\frac{2\pi}{3}\int d\boldsymbol{r}'|r'|^2 S(\boldsymbol{r}')\right). \tag{3.54}$$

$S(\boldsymbol{r})$ is the (classical or quantum-mechanical) charge-charge correlation function of the plasma which reads (for a homogenous plasma)

$$S(\boldsymbol{r}) = \sum_{\alpha\alpha'} e_{\alpha}e_{\alpha'}n_T(\alpha, \alpha', \boldsymbol{r}). \tag{3.55}$$

In (3.55), the truncated loop density correlation function is defined as

$$n_T(\alpha, \boldsymbol{r}, \alpha', \boldsymbol{r}') =< \hat{\rho}(\alpha, \boldsymbol{r}), \hat{\rho}(\alpha', \boldsymbol{r}') >= \sum_{i,j} < \delta_{\alpha\alpha_i}\delta_{\alpha'\alpha_j}\delta(\boldsymbol{\xi}, \boldsymbol{\xi}_i)\delta(\boldsymbol{\xi}', \boldsymbol{\xi}_j)\delta(\boldsymbol{r}-\boldsymbol{r}_i)\delta(\boldsymbol{r}'-\boldsymbol{r}_j) > . \tag{3.56}$$

Chapter 4

Correlations in the Quantum Fields: Exact Calculations

We are now ready to tackle the main problem, the calculation of the field correlations in the full quantum system including the quantum plasma and the quantum electromagnetic field. The system is now described by the formalism of chapter 2 without any restrictions and we will work with the full quantum Hamiltonian $H_{L,R}^{N}$ given in (2.9). Basically, we will follow the same procedure as in the classical case in chapter 3. At first, we will calculate the partition function $Z(T,N)$, whereas we perform the field trace yielding the reduced thermal weight $\rho_{L,R}$ and leave the complicated matter trace uncalculated. Then we use these preliminary calculations to deduce both the effective particle interaction as well as the second order field moments (3.11). The latter are again obtained as derivatives of $\rho_{L,R}$ which plays therefore the role of a generating function. Eventually we will sum up the different contributions of the field moments to recover the field correlations $< A^{\mu}(\boldsymbol{x})A^{\nu}(\boldsymbol{y}) >_T$, $< E^{\mu}(\boldsymbol{x})E^{\nu}(\boldsymbol{y}) >_T$ and $< B^{\mu}(\boldsymbol{x})B^{\nu}(\boldsymbol{y}) >_T$. The large distance correlations will be of a major interest, especially because they are directly comparable to the classical expressions. However, the calculations are now slightly more difficult because of the quantum nature of the field. The main difference to the classical case is the non-commutativity of $H_{\boldsymbol{A}}$ and H_0^{rad} (3.2) which prevents the factorization

$$e^{-\beta H_{L,R}^{N}} \neq e^{-\beta H_0^{rad}} e^{-\beta H_A}. \tag{4.1}$$

In the classical case we had $[H_A, H_0^{rad}] = 0$ and hence the factorization of the Gaussian factor $e^{-\beta H_0^{rad}}$ in (3.3) lead to standard Gaussian integrals in the course of the correlation calculations. In the present case however, we have to find another technique to calculate the reduced thermal weight. In this aim we will introduce the bosonic path integral (BPI), an apt tool to tackle the present problem. This technique, in combination with the FKI formula, will allow us to reformulate the partition function as a double path integral that shall substantially facilitate our calculations. The partial trace over the field degrees of freedom yielding the reduced thermal weight is hereby reduced to the evaluation of Gaussian functional integrals. These integrals are feasible although substantially more technical than their classical counterparts, the normal Gaussian integrals. After having calculated the exact field correlations, we shall investigate the classical limit $\hbar \to 0$ and compare the results with those of chapter 3. However, care needs to be taken while performing this limit: there are now two "different" species of $\hbar$, an $\hbar^{mat}$ coming from the quantum matter and an $\hbar^{rad}$ arising in the quantization of the field. In order to avoid this ambiguity, we absorb $\hbar^{mat}$ from the beginning in the thermal de Broglie wave length λ. Like this the final limit $\hbar \to 0$ will be equivalent to $\hbar^{rad} \to 0$. During the whole section we will use the same notations as in the classical calculations of chapter 3 and the operatorial nature of the objects will not be explicitly indicated.

4.1 The Partition Function $Z(T, N)$

Let us now start with the partition function $Z(T, N)$ and the reduced thermal weight $\rho_{L,R}$. As mentioned above, we will have to use the bosonic path integral (BPI), a standard technique in field theory and solid state physics. We will consecutively introduce definitions and results that are useful for our considerations, but we we suppose at the same time that the reader is already familiar with the basic concepts. We refer to [10] for a detailed analysis of the subject and to [6] for a compact review.

The key feature of the BPI is that it enables us to transform the field trace appearing in $\rho_{L,R}$ into a classical integral. For obvious reasons, we can only construct a path integral by means of a continuous base of the Hilbert space, i.e. the Fock space $\mathcal{F}_{photon}$. The famous occupation number base, constituted by the eigenvectors of the number operator $\sum_{\boldsymbol{k}\lambda} a^\dagger_{\boldsymbol{k}\lambda} a_{\boldsymbol{k}\lambda}$, is a discrete base and therefore useless for our actual purpose. Hence we switch to the coherent states for bosons which provide a, although not orthonormal, continuous base of $\mathcal{F}_{photons}$. Let us recall the basic properties of these coherent states. In the case of one photon mode (i.e. a single harmonic oscillator), we define the coherent state

$$|\alpha >= \sum_{m=0}^{\infty} \frac{(\alpha a^\dagger)^m}{m!} |0 >= e^{\alpha a^\dagger} |0 >, \tag{4.2}$$

where $|0>$ is the vacuum state defined by $a|0>= 0$ and $|\alpha>$ is an element of the continuous coherent state basis ($\alpha \in \mathbb{C}$). By construction, $|\alpha>$ is the eigenstate of the creation operator a with eigenvalue α, i.e. $a\,|\alpha >= \alpha|\alpha >$. The scalar product between two coherent states can be shown to give

$$< \alpha_2 | \alpha_1 >= e^{\alpha_2^* \alpha_1} \tag{4.3}$$

and later on we will appreciate the closure relation

$$\int \frac{d^2\alpha}{\pi} |\alpha >< \alpha| = \mathbf{1}_{Osc}. \tag{4.4}$$

Finally, as can be seen in (4.3), this base is not orthonormal; it is actually overcomplete. These definitions and properties are easily generalized to the complete Fock space $\mathcal{F}_{photon}$. We shall denote by $|\boldsymbol{\alpha} >= \prod_{\boldsymbol{k}\lambda} |\alpha_{\boldsymbol{k}\lambda} >$ a coherent state base vector of $\mathcal{F}_{photon}$ and we adopt the following notations: $\boldsymbol{\alpha} = \{\alpha_{\boldsymbol{k}\lambda}\}_{\boldsymbol{k}\lambda}$, $\boldsymbol{\alpha\alpha'} = \sum_{\boldsymbol{k}\lambda} \alpha_{\boldsymbol{k}\lambda} \alpha'_{\boldsymbol{k}\lambda}$ and $\mathcal{D}(\boldsymbol{\alpha}) = \prod_{\boldsymbol{k}\lambda} \int \frac{d^2\alpha_{\boldsymbol{k}\lambda}}{\pi}$. In terms of these definitions, the Fock space closure relation reads now

$$\int \mathcal{D}(\boldsymbol{\alpha})\, e^{-\boldsymbol{\alpha}^*\boldsymbol{\alpha}} |\boldsymbol{\alpha} >< \boldsymbol{\alpha}| = \mathbf{1}_{\mathcal{F}_{photons}}. \tag{4.5}$$

Before we proceed to the explicit expressions for the partition function $Z(T, N)$ and the reduced thermal weight $\rho_{L,R}$ we rewrite the Hamiltonian in terms of its normal ordered version

$$H_{L,R}^N =: H_{L,R}^N : + C_N, \tag{4.6}$$

where the additional constant

$$C_N = \sum_{i=1}^{N} \frac{2\pi\hbar e_{\alpha_i}^2}{m_{\alpha_i} c R^3} \sum_{\boldsymbol{k}} \frac{g^2(\boldsymbol{k})}{k} \tag{4.7}$$

arises when we put $\boldsymbol{A}^2(\boldsymbol{r}_i)$ in normal order. The normal order of the Hamiltonian is actually inalienable for the succeeding calculations as we will see in a while. The decomposition (4.7) yields now (2.21)

$$Z(T, N) = \frac{e^{-\beta C_N}}{Z_0^{rad}}\, Tr\, e^{-\beta : H_{L,R}^N :} \tag{4.8}$$

and we extract the reduced thermal weight (2.24)

$$\begin{aligned} \rho_{L,R} &= \frac{e^{-\beta C_N}}{Z_0^{rad}}\, Tr_{rad}\, e^{-\beta : H_{L,R}^N :} \\ &= \frac{e^{-\beta C_N}}{Z_0^{rad}} \int \mathcal{D}(\boldsymbol{\alpha})\, e^{-\boldsymbol{\alpha}^*\boldsymbol{\alpha}} < \boldsymbol{\alpha} | e^{-\beta : H_{L,R}^N :} | \boldsymbol{\alpha} > . \end{aligned} \tag{4.9}$$

The second equality has been obtained by means of the closure relation (4.4). Now, we take advantage of Trotter's formula

$$e^{-\beta : H_{L,R}^N :} = \lim_{n \to \infty} \left(1 - \frac{\beta}{n} : H_{L,R}^N : \right)^n \tag{4.10}$$

and insert $(n-1)$ closure relations to obtain (identifying $n \equiv 0$)

$$\rho_{L,R} = \frac{e^{-\beta C_N}}{Z_0^{rad}} \lim_{n\to\infty} \prod_{i=1}^{n} \int \mathcal{D}(\boldsymbol{\alpha}_i)\, e^{-\boldsymbol{\alpha}_i^* \boldsymbol{\alpha}_i} \times \tag{4.11}$$
$$\times < \boldsymbol{\alpha}_n| \left(1 - \frac{\beta}{n} : H_{L,R}^N :\right) |\boldsymbol{\alpha}_{n-1} > \ldots < \boldsymbol{\alpha}_1| \left(1 - \frac{\beta}{n} : H_{L,R}^N :\right) |\boldsymbol{\alpha}_0 > .$$

The matrix elements in (4.11) can be rewritten by means of (4.3)

$$< \boldsymbol{\alpha}_i| \left(1 - \frac{\beta}{n} : H_{L,R}^N :\right) |\boldsymbol{\alpha}_{i-1} >= e^{-\boldsymbol{\alpha}_i^* \boldsymbol{\alpha}_{i-1}} \left(1 - \frac{\beta}{n} H_{L,R}^N(\boldsymbol{\alpha}_i^*, \boldsymbol{\alpha}_{i-1})\right), \tag{4.12}$$

where $H_{L,R}^N(\boldsymbol{\alpha}_i^*, \boldsymbol{\alpha}_{i-1})$ has the same expression as the quantum Hamiltonian $H_{L,R}^N$ except that we replace everywhere $a_{\boldsymbol{k}\lambda}^\dagger$ by $\boldsymbol{\alpha}_i^*$ and $a_{\boldsymbol{k}\lambda}$ by $\boldsymbol{\alpha}_{i-1}$. Equation (4.12) is obviously only possible thanks to the normal order in the Hamiltonian which has allowed us to use the relations $a_{\boldsymbol{k}\lambda}|\alpha_{\boldsymbol{k}\lambda} >= \alpha_{\boldsymbol{k}\lambda}|\alpha_{\boldsymbol{k}\lambda} >$ and $< \alpha_{\boldsymbol{k}\lambda}|a_{\boldsymbol{k}\lambda}^\dagger =< \alpha_{\boldsymbol{k}\lambda}|\alpha_{\boldsymbol{k}\lambda}^*$. Hence we can now write the reduced thermal weight as

$$\rho_{L,R} = \frac{e^{-\beta C_N}}{Z_0^{rad}} \lim_{n\to\infty} \prod_{i=1}^{n} \int \mathcal{D}(\boldsymbol{\alpha}_i)\, e^{-\frac{\beta}{n}\boldsymbol{\alpha}_i^* \frac{(\boldsymbol{\alpha}_i - \boldsymbol{\alpha}_{i-1})}{\frac{\beta}{n}}} \times$$
$$\times \left(1 - \frac{\beta}{n} H_{L,R}^N(\boldsymbol{\alpha}_n^*, \boldsymbol{\alpha}_{n-1})\right) \ldots \left(1 - \frac{\beta}{n} H_{L,R}^N(\boldsymbol{\alpha}_1^*, \boldsymbol{\alpha}_0)\right). \tag{4.13}$$

This expression is the discretized version of the BPI. As usual when working with path integrals we switch to a synthetic but purely formal notation by introducing the continuous time $0 \le \tau \le \beta$ such that $\alpha_{i,\boldsymbol{k}\lambda} \equiv \alpha_{\boldsymbol{k}\lambda}(i\frac{\beta}{n})$ is transformed into a continuous and closed path $\alpha_{\boldsymbol{k}\lambda}(\tau)$ in the complex plane, $\alpha_{\boldsymbol{k}\lambda}(0) = \alpha_{\boldsymbol{k}\lambda}(\beta)$. In the limit $n \to \infty$ the product of infinitesimal evolutions in (4.13) becomes the imaginary-time propagator $U(\beta, 0) \equiv U(\beta)$ obeying the imaginary-time Schroedinger equation [6]

$$\hbar \frac{\partial}{\partial \tau} U(\tau) = -H_{L,R}^N(\boldsymbol{\alpha}(\tau))\, U(\tau). \tag{4.14}$$

The formal expression of $H_{L,R}^N(\boldsymbol{\alpha}(\tau))$ is equivalent to the one of $H_{L,R}^N$ except that we have to replace all the creation and annihilation operators $a_{\boldsymbol{k}\lambda}^\dagger$ and $a_{\boldsymbol{k}\lambda}$ by the (now time-dependent) c-numbers $\alpha_{\boldsymbol{k}\lambda}^*(\tau)$ and $\alpha_{\boldsymbol{k}\lambda}(\tau)$. Although the field operators are now simple c-numbers, $H_{L,R}^N(\boldsymbol{\alpha}(\tau))$ is still an operator due to its dependence on the position and momentum operators of the quantum particles. Consequently we have $[H_{L,R}^N(\boldsymbol{\alpha}(\tau_1)), H_{L,R}^N(\boldsymbol{\alpha}(\tau_2))] \neq 0$ and this implies that the propagator is given by the time-ordered exponential

$$U(\beta) = T\left[e^{-\frac{1}{\hbar}\int_0^{\hbar\beta} d\tau H_{L,R}^N(\boldsymbol{\alpha}(\tau))}\right] \tag{4.15}$$

according to standard perturbation theory. Furthermore, we note that the BPI reformulation yields $[H_{\boldsymbol{A}}(\boldsymbol{\alpha}(\tau)), H_0^{rad}(\boldsymbol{\alpha}(\tau))] = 0$ and hence the modified propagator

$$\hat{U}(\beta) = e^{-\frac{1}{\hbar}\int_0^{\hbar\beta} d\tau H_0^{rad}(\boldsymbol{\alpha}(\tau))} T\left[e^{-\frac{1}{\hbar}\int_0^{\hbar\beta} d\tau H_{\boldsymbol{A}}(\boldsymbol{\alpha}(\tau))}\right] \tag{4.16}$$

can be shown to satisfy the Schroedinger equation (4.14), too. We may therefore replace $U(\beta)$ by $\hat{U}(\beta)$. Introducing now the notation

$$\mathcal{D}\left[\boldsymbol{\alpha}(\cdot)\right] = \lim_{n\to\infty}\prod_{i=1}^{n}\prod_{\boldsymbol{k}\lambda}\int\frac{d^2\alpha_{i,\boldsymbol{k}\lambda}}{\pi} = \lim_{n\to\infty}\prod_{i=1}^{n}\int\mathcal{D}(\boldsymbol{\alpha}_i) \tag{4.17}$$

we can eventually write $\rho_{L,R}$ in a compact but formal form in terms of $\hat{U}(\beta)$

$$\rho_{L,R} = \frac{e^{-\beta C_N}}{Z_0^{rad}}\int\mathcal{D}\left[\boldsymbol{\alpha}(\cdot)\right]\, e^{-\int_0^{\hbar\beta} d\tau\left(\alpha^*(\tau)\frac{\partial}{\partial\tau}\alpha(\tau)\right)} e^{-\frac{1}{\hbar}\int_0^{\hbar\beta} d\tau H_0^{rad}(\boldsymbol{\alpha}(\tau))} T\left[e^{-\frac{1}{\hbar}\int_0^{\hbar\beta} d\tau H_{\boldsymbol{A}}(\boldsymbol{\alpha}(\tau))}\right]. \tag{4.18}$$

Finally, by performing the appropriate change of variables in the exponents of (4.18) we obtain

$$\rho_{L,R} = \frac{e^{-\beta C_N}}{Z_0^{rad}}\int\mathcal{D}\left[\boldsymbol{\alpha}(\cdot)\right]\, e^{-\int_0^1 ds\left(\alpha^*(s)\frac{\partial}{\partial s}\alpha(s)+\beta H_0^{rad}(\boldsymbol{\alpha}(s))\right)} T\left[e^{-\beta\int_0^1 ds H_{\boldsymbol{A}}(\boldsymbol{\alpha}(s))}\right]. \tag{4.19}$$

In this expression, the field operators have completely disappeared, but as mentioned above, $H_{\boldsymbol{A}}$ is still an operator. We avoid this difficulty by application of the FKI formula which allows us to convert $Z(T, N)$ into a purely classical object. After this, we will be able to perform the partial trace over the field degrees of freedom, i.e. to calculate the reduced thermal weight in the loop representation explicitly. In terms of $\rho_{L,R}$ the total partition function reads

$$\begin{aligned} Z(T,N) &= Tr_{mat}\,\rho_{L,R} = \left(\prod_{i=1}^{N}\int d\boldsymbol{r}_i\right) < \{\boldsymbol{r}_i\}|\rho_{L,R}|\{\boldsymbol{r}_i\} > \\ &= \frac{e^{-\beta C_N}}{Z_0^{rad}}\int\mathcal{D}\left[\boldsymbol{\alpha}(\cdot)\right] e^{-\int_0^{\beta\hbar} d\tau\left(\alpha^*(\tau)\frac{\partial}{\partial\tau}\alpha(\tau)\right)} e^{-\frac{1}{\hbar}\int_0^{\hbar\beta} d\tau H_0^{rad}(\boldsymbol{\alpha}(\tau))}\times \\ &\times\left[\left(\prod_{i=1}^{N}\int d\boldsymbol{r}_i\right) < \{\boldsymbol{r}_i\}|T\left[e^{-\frac{1}{\hbar}\int_0^{\hbar\beta} d\tau H_{\boldsymbol{A}}(\boldsymbol{\alpha}(\tau))}\right]|\{\boldsymbol{r}_i\} > \right]. \end{aligned} \tag{4.20}$$

We note that the matrix element in the square brackets of (4.20) is just the imaginary-time propagator corresponding to the Hamiltonian $H_{\boldsymbol{A}}(\alpha(\tau))$ and since the FKI formula is also applicable to time dependent vector potentials [2] we can rewrite the square brackets as

$$\prod_{i=1}^{N}\left(\frac{1}{2\pi\lambda_{\alpha_i}^2}\right)^{\frac{3}{2}}\int\mathcal{D}(\boldsymbol{\xi}_i)\int d\boldsymbol{r}_i e^{-\beta U_M(\mathcal{F}_1,\ldots,\mathcal{F}_N)}\, e^{-\beta U_{\boldsymbol{A}(s)}(\mathcal{F}_1,\ldots,\mathcal{F}_N,\boldsymbol{A}(\tau))} \tag{4.21}$$

where $U_M(\mathcal{F}_1,\ldots,\mathcal{F}_N)$ is defined in (3.18) and $U_{\boldsymbol{A}(s)}(\mathcal{F}_1,\ldots,\mathcal{F}_N,\boldsymbol{A}(s))$ is a modified version of its classical counterpart $U_{\boldsymbol{A}}$

$$U_{\boldsymbol{A}(s)}(\mathcal{F}_1,\ldots,\mathcal{F}_N,\boldsymbol{A}(s)) = -i\sum_{i=1}^{N}\frac{e_{\alpha_i}}{c\sqrt{\beta m_{\alpha_i}}}\int_0^1 d\boldsymbol{\xi}_i(s)\cdot\boldsymbol{A}(\boldsymbol{\alpha}(s),\boldsymbol{r}_i+\lambda_{\alpha_i}\boldsymbol{\xi}_i(s)). \quad (4.22)$$

This expression is almost identical with the matrix element (3.17) found in the classical case except that the vector potential has now an explicit time-dependence requiring a modification of $U_{\boldsymbol{A}}$ to $U_{\boldsymbol{A}(s)}$. We emphasize however, that this is merely an imaginary time not to be confused with the physical time. We are now in the position to write $Z(T,N)$ as a purely classical integral

$$\begin{aligned} Z(T,N) &= \prod_{i=1}^{N}\left(\frac{1}{2\pi\lambda_{\alpha_i}^2}\right)^{\frac{3}{2}}\int\mathcal{D}(\boldsymbol{\xi}_i)\int d\boldsymbol{r}_i\, e^{-\beta U_M(\mathcal{F}_1,\ldots,\mathcal{F}_N)}\times \\ &\times\left[\frac{e^{-\beta C_N}}{Z_0^{rad}}\int\mathcal{D}\left[\boldsymbol{\alpha}(\cdot)\right]e^{-\int_0^1 d\tau\left(\alpha^*_{\boldsymbol{k}\lambda}(\tau)\frac{\partial}{\partial\tau}\alpha_{\boldsymbol{k}\lambda}+\beta\sum_{\boldsymbol{k}\lambda}\hbar\omega_{\boldsymbol{k}}\alpha^*_{\boldsymbol{k}\lambda}(s)\alpha_{\boldsymbol{k}\lambda}(s)\right)}e^{-\beta U_{\boldsymbol{A}(s)}(\mathcal{F}_1,\ldots,\mathcal{F}_N,\boldsymbol{A}(\tau))}\right] \\ &\equiv \prod_{i=1}^{N}\left(\frac{1}{2\pi\lambda_{\alpha_i}^2}\right)^{\frac{3}{2}}\int\mathcal{D}(\boldsymbol{\xi}_i)\int d\boldsymbol{r}_i\, e^{-\beta U_M(\mathcal{F}_1,\ldots,\mathcal{F}_N)}\,\tilde{\rho}_{L,R}, \end{aligned} \quad (4.23)$$

where we denote the field-dependent part of the reduced thermal weight (in the filament representation) by $\tilde{\rho}_{L,R}$. The explicit calculation of $\tilde{\rho}_{L,R}$ will allow us to deduce the effective interaction potential between the particles and will be very important in the succeeding calculations. Furthermore, we observe that the first exponent in the bracket contains a quadratic form in $\boldsymbol{\alpha}$ and that the second exponent contains the prefactor i according to (4.22). We anticipate therefore the Fourier transform of a functional Gaussian integral, a situation similar to the one encountered in the classical case except that the simple Gaussian integrals are now functional Gaussian integrals. The calculations are straight forward and we refer to [6] for details. The Gaussian measure of the functional integral in (4.23) is normalized by the partition function of the free radiation field Z_0^{rad}. The calculation of Z_0^{rad} (as well as the following calculations) is performed in three steps: first we have to discretize the formal expressions, then calculate Gaussian integrals and finally pass again to the continuum limit. One finds

$$\begin{aligned} Z_0^{rad} &= \int\mathcal{D}[\boldsymbol{\alpha}(\cdot)]e^{-\int_0^1 d\tau\sum_{\boldsymbol{k}\lambda}\left(\alpha^*_{\boldsymbol{k}\lambda}(\tau)\frac{\partial}{\partial\tau}\alpha_{\boldsymbol{k}\lambda}(\tau)+\beta\hbar\omega_{\boldsymbol{k}}\alpha^*_{\boldsymbol{k}\lambda}(s)\alpha_{\boldsymbol{k}\lambda}(s)\right)} \\ &= \lim_{n\to\infty}\prod_{i=1}^{n}\prod_{\boldsymbol{k}\lambda}\int\frac{d^2\alpha_{\boldsymbol{k}\lambda}}{\pi}e^{-\sum_{\boldsymbol{k}\lambda}\sum_{i=1}^{n}\left(\alpha^*_{i,\boldsymbol{k}\lambda}\alpha_{i,\boldsymbol{k}\lambda}-\alpha^*_{i,\boldsymbol{k}\lambda}\alpha_{i-1,\boldsymbol{k}\lambda}+\frac{1}{n}\beta\hbar\omega_{\boldsymbol{k}}\alpha^*_{i,\boldsymbol{k}\lambda}\alpha_{i,\boldsymbol{k}\lambda}\right)} \\ &= \prod_{\boldsymbol{k}\lambda}\frac{1}{1-e^{-\beta\hbar\omega_{\boldsymbol{k}}}}. \end{aligned} \quad (4.24)$$

Since a Gaussian measure is completely determined by its covariance, it is interesting to give its expression

$$<\alpha_{\boldsymbol{k}\lambda}(s)\alpha^*_{\boldsymbol{k}'\lambda'}(s')>=\delta^{\boldsymbol{k}'\lambda'}_{\boldsymbol{k}\lambda}S^{-1}_{\boldsymbol{k}}(s,s'), \tag{4.25}$$

where

$$S^{-1}_{\boldsymbol{k}}(s,s')=e^{-\beta\hbar\omega_{\boldsymbol{k}}(s-s')}\left(\theta(s-s'-\eta)(1+n_{\boldsymbol{k}})+\theta(s'-s+\eta)n_{\boldsymbol{k}}\right), \quad \eta>0. \tag{4.26}$$

In (4.26), the photon mode occupation probability $n_{\boldsymbol{k}}$ obeying the Bose-Einstein statistics is

$$n_{\boldsymbol{k}}=\frac{1}{e^{\beta\hbar\omega_k-1}} \tag{4.27}$$

and the parameter η has been introduced to reproduce the correct equal-time covariance. At the end of each calculation we have to perform the limit $\eta\to 0$. Furhtermore, it can easily be shown that that $S^{-1}_{\boldsymbol{k}}(s,s')$ is the formal Green's function of the operator $\partial_s+\beta\hbar\omega$. Note that modes of the free quantum field are decoupled as can be seen in (4.25). The final step leading to the effective interaction potential consists now in the calculation of the Fourier transform of the Gaussian in $\tilde{\rho}_{L,R}$. In this aim we introduce foremost the quantity

$$J_{\boldsymbol{k}\lambda}(s)=\sum_{i=1}^{N}\sqrt{\frac{2\pi\hbar\beta e^2_{\alpha_i}}{m_{\alpha_i}\omega_{\boldsymbol{k}}R^3}}\frac{d\boldsymbol{\xi}^\mu_i(s)}{ds}\cdot\boldsymbol{e}^\mu_{\boldsymbol{k}\lambda}e^{-i\boldsymbol{k}\cdot(\boldsymbol{r}_i+\lambda_{\alpha_i}\boldsymbol{\xi}_i(s))}g(\boldsymbol{k}) \tag{4.28}$$

which gives a deeper insight into the mathematical structure of the exponent

$$\tilde{\rho}_{L,R}=\frac{e^{-\beta C_N}}{Z^{rad}_0}\int\mathcal{D}\left[\boldsymbol{\alpha}(\cdot)\right]e^{-\int_0^1 ds\sum_{\boldsymbol{k}\lambda}\left(\alpha^*_{\boldsymbol{k}\lambda}(s)\frac{\partial}{\partial s}\alpha_{\boldsymbol{k}\lambda}(s)+\beta\hbar\omega_{\boldsymbol{k}}\alpha^*_{\boldsymbol{k}\lambda}(s)\alpha_{\boldsymbol{k}\lambda}(s)-i\alpha_{\boldsymbol{k}\lambda}(s)J^*_{\boldsymbol{k}\lambda}(s)-i\alpha^*_{\boldsymbol{k}\lambda}(s)J_{\boldsymbol{k}\lambda}(s)\right)} \tag{4.29}$$

and performing the Gaussian integrals yields finally

$$\tilde{\rho}_{L,R}=e^{-\beta C_N}e^{-\sum_{\boldsymbol{k}\lambda}\int_0^1 ds\int_0^1 ds' J^*_{\boldsymbol{k}\lambda}(s)S^{-1}_{\boldsymbol{k}}(s,s')J_{\boldsymbol{k}\lambda}(s')}. \tag{4.30}$$

Rewriting this equation as

$$\tilde{\rho}_{L,R}=e^{-\beta C_N}e^{-\frac{\beta}{2}\sum_{i,j}^{N}e_{\alpha_i}e_{\alpha_j}W_m(i,j)}, \tag{4.31}$$

we are now able to deduce the interaction potential $W_m(i,j)$ between the particles. This interaction energy is mediated by the radiation field and and contains a self-energy contribution

$$W_m(i,j)=\frac{1}{\beta c^2\sqrt{m_{\alpha_i}m_{\alpha_j}}}\int\frac{d\boldsymbol{k}}{(2\pi)^3}e^{i\boldsymbol{k}(\boldsymbol{r}_i-\boldsymbol{r}_j)}\int_0^1 d\xi^\mu_i(s)e^{i\lambda_{\alpha_i}\boldsymbol{k}\cdot\boldsymbol{\xi}_i(s)}\int_0^1 d\xi^\nu_j(s')e^{-i\lambda_{\alpha_j}\boldsymbol{k}\cdot\boldsymbol{\xi}_j(s')}\mathcal{G}^{\mu\nu}(\boldsymbol{k},s,s'). \tag{4.32}$$

In (4.32) we have tacitely performed the thermodynamic limit $R \to \infty$ and the tensor $\mathcal{G}^{\mu\nu}(\boldsymbol{k}, s, s')$ is the quantum version of the time-independent $G^{\mu\nu}(\boldsymbol{k})$

$$\mathcal{G}^{\mu\nu}(\boldsymbol{k}, s, s') = 4\pi\beta\hbar c \frac{S_{\boldsymbol{k}}^{-1}(s, s')}{k} \delta_{tr}^{\mu\nu}(\boldsymbol{k}) g^2(\boldsymbol{k}). \tag{4.33}$$

Plugging (4.31) into (4.23) yields finally

$$\begin{aligned} Z(T, N) &= \prod_{i=1}^{N} \left(\frac{1}{2\pi\lambda_{\alpha_i}^2} \right)^{\frac{3}{2}} \int \mathcal{D}(\boldsymbol{\xi}_i) \int d\boldsymbol{r}_i \, e^{-\beta U_M(\mathcal{F}_1,\ldots,\mathcal{F}_N)} \, \tilde{\rho}_{L,R} \\ &= e^{-\beta C_N} \prod_{i=1}^{N} \left(\frac{1}{2\pi\lambda_{\alpha_i}^2} \right)^{\frac{3}{2}} \int \mathcal{D}(\boldsymbol{\xi}_i) \int d\boldsymbol{r}_i \, e^{-\beta U_M(\mathcal{F}_1,\ldots,\mathcal{F}_N)} e^{-\frac{\beta}{2} \sum_{i,j}^{N} e_{\alpha_i} e_{\alpha_j} W_m(i,j)}. \end{aligned} \tag{4.34}$$

4.2 The Second Order Field Moments

These preliminary considerations in mind we attack now the calculation of the four creator and annihilator correlationslation of the four creator and annihilator correlations

$$< a_{\boldsymbol{k}_1\lambda_1} a_{\boldsymbol{k}_2\lambda_2} >, \quad < a^*_{\boldsymbol{k}_1\lambda_1} a^*_{\boldsymbol{k}_2\lambda_2} >, \quad < a^*_{\boldsymbol{k}_1\lambda_1} a_{\boldsymbol{k}_2\lambda_2} > \quad \text{and} \quad < a_{\boldsymbol{k}_1\lambda_1} a^*_{\boldsymbol{k}_2\lambda_2} > . \tag{4.35}$$

The key point in this calculation is the field depending part of the reduced thermal weight $\tilde{\rho}_{L,R}$ and the integration leading from (4.29) to (4.30). Proceeding as in the classical case, we will use $\tilde{\rho}_{L,R}$ as the generating function for the field moments (4.35). Let's start off with the first average $< a_{\boldsymbol{k}_1\lambda_1} a_{\boldsymbol{k}_2\lambda_2} >$ which is given by

$$\begin{aligned} < a_{\boldsymbol{k}_1\lambda_1} a_{\boldsymbol{k}_2\lambda_2} > &= \frac{1}{Z_{L,R}^N} Tr \left(e^{-\beta H_{L,R}^N} a_{\boldsymbol{k}_1\lambda_1} a_{\boldsymbol{k}_2\lambda_2} \right) \\ &= \frac{1}{Z(T,N)} Tr_{mat} \left[\frac{e^{-\beta C_N}}{Z_0^{rad}} Tr_{rad} \left(e^{-\beta :H_{L,R}^N:} a_{\boldsymbol{k}_1\lambda_1} a_{\boldsymbol{k}_2\lambda_2} \right) \right] \\ &= \frac{1}{Z(T,N)} Tr_{mat} \left[\frac{e^{-\beta C_N}}{Z_0^{rad}} \int D(\boldsymbol{\alpha}) e^{-\boldsymbol{\alpha}^* \boldsymbol{\alpha}} < \boldsymbol{\alpha} | e^{-\beta :H_{L,R}^N:} a_{\boldsymbol{k}_1\lambda_1} a_{\boldsymbol{k}_2\lambda_2} | \boldsymbol{\alpha} > \right] \end{aligned} \tag{4.36}$$

We see that the term in brackets is nothing but a modified version of the reduced thermal weight $\rho_{L,R}$ and hence we can apply a BPI procedure similar to the one used in section 4.1 to calculate $Z(T, N)$. In particular, we use Trotter's formula (4.10) and take advantage of the normal order of the Hamiltonian to find

$$
\begin{aligned}
< a_{\boldsymbol{k}_1\lambda_1} a_{\boldsymbol{k}_2\lambda_2} > &= \frac{1}{Z(T,N)} Tr_{mat} \left[\frac{e^{-\beta C_N}}{Z_0^{rad}} \lim_{n\to\infty} \prod_{i=1}^{n} \int \mathcal{D}(\boldsymbol{\alpha}_i)\, e^{-\boldsymbol{\alpha}_i^* \boldsymbol{\alpha}_i} \times \right. \\
&\quad \left. \times < \boldsymbol{\alpha}_n| \left(1 - \frac{\beta}{n} : H_{L,R}^N :\right) |\boldsymbol{\alpha}_{n-1} > \ldots < \boldsymbol{\alpha}_1| \left(1 - \frac{\beta}{n} : H_{L,R}^N :\right) a_{\boldsymbol{k}_1\lambda_1} a_{\boldsymbol{k}_2\lambda_2} |\boldsymbol{\alpha}_0 > \right] \\
&= \frac{1}{Z(T,N)} Tr_{mat} \left[\frac{e^{-\beta C_N}}{Z_0^{rad}} \lim_{n\to\infty} \prod_{i=1}^{n} \int \mathcal{D}(\boldsymbol{\alpha}_i)\, e^{-\frac{\beta}{n} \boldsymbol{\alpha}_i^* \frac{(\boldsymbol{\alpha}_i - \boldsymbol{\alpha}_{i-1})}{\frac{\beta}{n}}} \times \right. \qquad (4.37) \\
&\quad \left. \times \alpha_{0,\boldsymbol{k}_1\lambda_1} \alpha_{0,\boldsymbol{k}_2\lambda_2} \left(1 - \frac{\beta}{n} H_{L,R}^N(\boldsymbol{\alpha}_n^*, \boldsymbol{\alpha}_{n-1})\right) \ldots \left(1 - \frac{\beta}{n} H_{L,R}^N(\boldsymbol{\alpha}_1^*, \boldsymbol{\alpha}_0)\right) \right].
\end{aligned}
$$

Again we identify $n \equiv 0$ and we emphasize that both $\alpha_{\boldsymbol{k}_1\lambda_1}$ and $\alpha_{\boldsymbol{k}_2\lambda_2}$ carry the index 0 since they have been obtained by the action of $a_{\boldsymbol{k}_1\lambda_1}$ and $a_{\boldsymbol{k}_2\lambda_2}$ on the ket $|\boldsymbol{\alpha}_0 >$. Taking the matter trace, introducing the FKI formula and using the definition (4.28) yields then

$$
\begin{aligned}
< a_{\boldsymbol{k}_1\lambda_1} a_{\boldsymbol{k}_2\lambda_2} > &= \frac{1}{Z(T,N)} \prod_{i=1}^{N} \left(\frac{1}{2\pi\lambda_{\alpha_i}^2}\right)^{\frac{3}{2}} \int \mathcal{D}(\boldsymbol{\xi}_i) \int d\boldsymbol{r}_i\; e^{-\beta U_M(\mathcal{F}_1,\ldots,\mathcal{F}_N)} \times \qquad (4.38) \\
&\quad \times \left[\frac{e^{-\beta C_N}}{Z_0^{rad}} \int \mathcal{D}\left[\boldsymbol{\alpha}(\cdot)\right] \alpha_{\boldsymbol{k}_1\lambda_1}(0) \alpha_{\boldsymbol{k}_2\lambda_2}(0) \times \right. \\
&\quad \left. \times\, e^{-\int_0^1 ds \sum_{\boldsymbol{k}\lambda} \left(\alpha_{\boldsymbol{k}\lambda}^*(s) \frac{\partial}{\partial s} \alpha_{\boldsymbol{k}\lambda}(s) + \beta\hbar\omega_{\boldsymbol{k}} \alpha_{\boldsymbol{k}\lambda}^*(s) \alpha_{\boldsymbol{k}\lambda}(s) - i\alpha_{\boldsymbol{k}\lambda}(s) J_{\boldsymbol{k}\lambda}^*(s) - i\alpha_{\boldsymbol{k}\lambda}^*(s) J_{\boldsymbol{k}\lambda}(s) \right)} \right].
\end{aligned}
$$

In this continuous-time notation we have to replace $\alpha_{n,\boldsymbol{k}\lambda}$ by $\alpha_{\boldsymbol{k}\lambda}(0)$. The BPI in this formula can be calculated by means of (4.30) and (4.29) and $\tilde{\rho}_{L,R}$ will play the role of a generating function. We introduce a family of smooth and regular functions $\{\mu_{\boldsymbol{k}\lambda}(s)\}_{\boldsymbol{k}\lambda}$ defined on the compact domain $[0,1]$. The square brackets in (4.38) can now be rewritten in terms of a double functional derivative

$$
\begin{aligned}
&- \left. \frac{\delta^2}{\delta\mu_{\boldsymbol{k}_1\lambda_1}^*(0) \delta\mu_{\boldsymbol{k}_2\lambda_2}^*(0)} \right|_{\mu_{\boldsymbol{k}\lambda}(s)\equiv 0} \frac{e^{-\beta C_N}}{Z_0^{rad}} \int \mathcal{D}\left[\boldsymbol{\alpha}(\cdot)\right] \times \qquad (4.39) \\
&\quad \times e^{-\int_0^1 ds \sum_{\boldsymbol{k}\lambda} \left(\alpha_{\boldsymbol{k}\lambda}^*(s) \frac{\partial}{\partial s} \alpha_{\boldsymbol{k}\lambda}(s) + \beta\hbar\omega_{\boldsymbol{k}} \alpha_{\boldsymbol{k}\lambda}^*(s) \alpha_{\boldsymbol{k}\lambda}(s) - i\alpha_{\boldsymbol{k}\lambda}(s)(J_{\boldsymbol{k}\lambda}^*(s) + \mu_{\boldsymbol{k}\lambda}^*(s)) - i\alpha_{\boldsymbol{k}\lambda}^*(s)(J_{\boldsymbol{k}\lambda}(s) + \mu_{\boldsymbol{k}\lambda}(s)) \right)} \\
&= - \left. \frac{\delta^2}{\delta\mu_{\boldsymbol{k}_1\lambda_1}^*(0) \delta\mu_{\boldsymbol{k}_2\lambda_2}^*(0)} \right|_{\mu_{\boldsymbol{k}\lambda}(s)\equiv 0} \frac{e^{-\beta C_N}}{Z_0^{rad}} \times \\
&\quad \times e^{-\sum_{\boldsymbol{k}\lambda} \int_0^1 ds \int_0^1 ds' S_{\boldsymbol{k}}^{-1}(s,s') \left(J_{\boldsymbol{k}\lambda}^*(s) J_{\boldsymbol{k}\lambda}(s') + J_{\boldsymbol{k}\lambda}^*(s) \mu_{\boldsymbol{k}\lambda}(s') + \mu_{\boldsymbol{k}\lambda}^*(s) J_{\boldsymbol{k}\lambda}(s') + \mu_{\boldsymbol{k}\lambda}^*(s) \mu_{\boldsymbol{k}\lambda}(s') \right)} \\
&= e^{-\beta C_N} e^{-\sum_{\boldsymbol{k}\lambda} \int_0^1 ds \int_0^1 ds' J_{\boldsymbol{k}\lambda}^*(s) S_{\boldsymbol{k}}^{-1}(s,s') J_{\boldsymbol{k}\lambda}(s')} \int_0^1 ds \int_0^1 ds' S_{\boldsymbol{k}_1}^{-1}(0,s') S_{\boldsymbol{k}_2}^{-1}(0,s) J_{\boldsymbol{k}_1\lambda_1}(s) J_{\boldsymbol{k}_2\lambda_2}(s') \\
&= \tilde{\rho}_{L,R} \int_0^1 ds \int_0^1 ds' S_{\boldsymbol{k}_1}^{-1}(0,s') S_{\boldsymbol{k}_2}^{-1}(0,s) J_{\boldsymbol{k}_1\lambda_1}(s) J_{\boldsymbol{k}_2\lambda_2}(s').
\end{aligned}
$$

We use (4.31) and plug this expression back into (4.38)

$$< a_{\boldsymbol{k}_1\lambda_1} a_{\boldsymbol{k}_2\lambda_2} > = -\frac{e^{-\beta C_N}}{Z(T,N)} \prod_{i=1}^{N} \left(\frac{1}{2\pi\lambda_{\alpha_i}^2}\right)^{\frac{3}{2}} \int \mathcal{D}(\boldsymbol{\xi}_i) \int d\boldsymbol{r}_i \, e^{-\beta U_M(\mathcal{F}_1,\ldots,\mathcal{F}_N)} e^{-\frac{\beta}{2}\sum_{i,j}^{N} e_{\alpha_i} e_{\alpha_j} W_m(i,j)} \times$$
$$\times \int_0^1 ds \int_0^1 ds' S_{\boldsymbol{k}_1}^{-1}(0,s) S_{\boldsymbol{k}_2}^{-1}(0,s') J_{\boldsymbol{k}_1\lambda_1}(s) J_{\boldsymbol{k}_2\lambda_2}(s'). \tag{4.40}$$

The remaining correlations in (4.35) can be calculated in a similar manner except that we have to be careful with the manipulations corresponding to those of (4.36). The second correlation reads

$$\begin{aligned}
< a_{\boldsymbol{k}_1\lambda_1}^{\dagger} a_{\boldsymbol{k}_2\lambda_2}^{\dagger} > &= \frac{1}{Z_{L,R}^N} Tr \left(e^{-\beta H_{L,R}^N} a_{\boldsymbol{k}_1\lambda_1}^{\dagger} a_{\boldsymbol{k}_2\lambda_2}^{\dagger} \right) \\
&= \frac{1}{Z(T,N)} Tr_{mat} \left[\frac{e^{-\beta C_N}}{Z_0^{rad}} Tr_{rad} \left(a_{\boldsymbol{k}_1\lambda_1}^{\dagger} a_{\boldsymbol{k}_2\lambda_2}^{\dagger} e^{-\beta : H_{L,R}^N :} \right) \right] \\
&= \frac{1}{Z(T,N)} Tr_{mat} \left[\frac{e^{-\beta C_N}}{Z_0^{rad}} \int D(\boldsymbol{\alpha}) e^{-\boldsymbol{\alpha}^*\boldsymbol{\alpha}} < \boldsymbol{\alpha} | a_{\boldsymbol{k}_1\lambda_1}^{\dagger} a_{\boldsymbol{k}_2\lambda_2}^{\dagger} e^{-\beta : H_{L,R}^N :} | \boldsymbol{\alpha} > \right] \\
&= \frac{1}{Z(T,N)} Tr_{mat} \left[\frac{e^{-\beta C_N}}{Z_0^{rad}} \lim_{n\to\infty} \prod_{i=1}^{n} \int \mathcal{D}(\boldsymbol{\alpha}_i) \, e^{-\boldsymbol{\alpha}_i^*\boldsymbol{\alpha}_i} \times \right. \\
&\quad \left. \times < \boldsymbol{\alpha}_n | a_{\boldsymbol{k}_1\lambda_1}^{\dagger} a_{\boldsymbol{k}_2\lambda_2}^{\dagger} \left(1 - \frac{\beta}{n} : H_{L,R}^N : \right) | \boldsymbol{\alpha}_{n-1} > \ldots < \boldsymbol{\alpha}_1 | \left(1 - \frac{\beta}{n} : H_{L,R}^N : \right) | \boldsymbol{\alpha}_0 > \right] \\
&= \frac{1}{Z(T,N)} Tr_{mat} \left[\frac{e^{-\beta C_N}}{Z_0^{rad}} \lim_{n\to\infty} \prod_{i=1}^{n} \int \mathcal{D}(\boldsymbol{\alpha}_i) \, e^{-\frac{\beta}{n}\boldsymbol{\alpha}_i^* \frac{(\boldsymbol{\alpha}_i - \boldsymbol{\alpha}_{i-1})}{\frac{\beta}{n}}} \times \right. \\
&\quad \left. \times \alpha_{n,\boldsymbol{k}_1\lambda_1}^* \alpha_{n,\boldsymbol{k}_2\lambda_2}^* \left(1 - \frac{\beta}{n} H_{L,R}^N(\boldsymbol{\alpha}_n^*, \boldsymbol{\alpha}_{n-1})\right) \ldots \left(1 - \frac{\beta}{n} H_{L,R}^N(\boldsymbol{\alpha}_1^*, \boldsymbol{\alpha}_0)\right) \right].
\end{aligned} \tag{4.41}$$

where we have used the cyclic property of the trace in the second equation. The remaining steps are completely analogous to those of $< a_{\boldsymbol{k}_1\lambda_1} a_{\boldsymbol{k}_2\lambda_2} >$ and we find

$$< a_{\boldsymbol{k}_1\lambda_1}^{\dagger} a_{\boldsymbol{k}_2\lambda_2}^{\dagger} > = -\frac{e^{-\beta C_N}}{Z(T,N)} \prod_{i=1}^{N} \left(\frac{1}{2\pi\lambda_{\alpha_i}^2}\right)^{\frac{3}{2}} \int \mathcal{D}(\boldsymbol{\xi}_i) \int d\boldsymbol{r}_i \, e^{-\beta U_M(\mathcal{F}_1,\ldots,\mathcal{F}_N)} e^{-\frac{\beta}{2}\sum_{i,j}^{N} e_{\alpha_i} e_{\alpha_j} W_m(i,j)} \times$$
$$\times \int_0^1 ds \int_0^1 ds' S_{\boldsymbol{k}_1}^{-1}(s,1) S_{\boldsymbol{k}_2}^{-1}(s',1) J_{\boldsymbol{k}_1\lambda_1}^*(s) J_{\boldsymbol{k}_2\lambda_2}^*(s'). \tag{4.42}$$

Then using once more the cyclic property of the trace we find

$$\begin{aligned}
< a_{\boldsymbol{k}_1\lambda_1} a^\dagger_{\boldsymbol{k}_2\lambda_2} > &= \frac{1}{Z^N_{L,R}} Tr\left(e^{-\beta H^N_{L,R}} a_{\boldsymbol{k}_1\lambda_1} a^\dagger_{\boldsymbol{k}_2\lambda_2}\right) \\
&= \frac{1}{Z(T,N)} Tr_{mat}\left[\frac{e^{-\beta C_N}}{Z^{rad}_0} Tr_{rad}\left(a^\dagger_{\boldsymbol{k}_2\lambda_2} e^{-\beta : H^N_{L,R}:} a_{\boldsymbol{k}_1\lambda_1}\right)\right] \qquad (4.43)\\
&= \frac{1}{Z(T,N)} \frac{e^{-\beta C_N}}{Z^{rad}_0} Tr_{mat}\Bigg[\lim_{n\to\infty} \prod_{i=1}^{n} \int \mathcal{D}(\boldsymbol{\alpha}_i)\, e^{-\frac{\beta}{n}\boldsymbol{\alpha}^*_i \frac{(\boldsymbol{\alpha}_i-\boldsymbol{\alpha}_{i-1})}{\frac{\beta}{n}}} \times \\
&\times \alpha_{0,\boldsymbol{k}_1\lambda_1} \alpha^*_{n,\boldsymbol{k}_2\lambda_2} \left(1 - \frac{\beta}{n} H^N_{L,R}(\boldsymbol{\alpha}^*_n, \boldsymbol{\alpha}_{n-1})\right) \dots \left(1 - \frac{\beta}{n} H^N_{L,R}(\boldsymbol{\alpha}^*_1, \boldsymbol{\alpha}_0)\right)\Bigg].
\end{aligned}$$

leading to the final result

$$\begin{aligned}
< a_{\boldsymbol{k}_1\lambda_1} a^\dagger_{\boldsymbol{k}_2\lambda_2} > &= \frac{e^{-\beta C_N}}{Z(T,N)} \prod_{i=1}^{N} \left(\frac{1}{2\pi\lambda^2_{\alpha_i}}\right)^{\frac{3}{2}} \int \mathcal{D}(\boldsymbol{\xi}_i) \int d\boldsymbol{r}_i\, e^{-\beta U_M(\mathcal{F}_1,\dots,\mathcal{F}_N)} e^{-\frac{\beta}{2}\sum^N_{i,j} e_{\alpha_i} e_{\alpha_j} W_m(i,j)} \times \\
&\times \left[\delta^{\boldsymbol{k}_2\lambda_2}_{\boldsymbol{k}_1\lambda_1} S^{-1}_{\boldsymbol{k}_1}(0,1) - \int_0^1 ds \int_0^1 ds' S^{-1}_{\boldsymbol{k}_1}(0,s) S^{-1}_{\boldsymbol{k}_2}(s',1) J_{\boldsymbol{k}_1\lambda_1}(s) J^*_{\boldsymbol{k}_2\lambda_2}(s')\right] \qquad (4.44)
\end{aligned}$$

The additional term proportional to $\delta^{\boldsymbol{k}_2\lambda_2}_{\boldsymbol{k}_1\lambda_1}$ is independent of $J_{\boldsymbol{k}\lambda}(s)$ and hence independent of the matter. According to (4.26) we have $S^{-1}_{\boldsymbol{k}}(0,1) = n_{\boldsymbol{k}} + 1$ and this leads together with the definition (4.34) to

$$\begin{aligned}
< a_{\boldsymbol{k}_1\lambda_1} a^\dagger_{\boldsymbol{k}_2\lambda_2} > &= \delta^{\boldsymbol{k}_2\lambda_2}_{\boldsymbol{k}_1\lambda_1}(n_{\boldsymbol{k}_1\lambda_1} + 1) - \frac{e^{-\beta C_N}}{Z(T,N)} \prod_{i=1}^{N} \left(\frac{1}{2\pi\lambda^2_{\alpha_i}}\right)^{\frac{3}{2}} \int \mathcal{D}(\boldsymbol{\xi}_i) \int d\boldsymbol{r}_i\, e^{-\beta U_M(\mathcal{F}_1,\dots,\mathcal{F}_N)} \times \\
&\times e^{-\frac{\beta}{2}\sum^N_{i,j} e_{\alpha_i} e_{\alpha_j} W_m(i,j)} \int_0^1 ds \int_0^1 ds' S^{-1}_{\boldsymbol{k}_1}(0,s) S^{-1}_{\boldsymbol{k}_2}(s',1) J_{\boldsymbol{k}_1\lambda_1}(s) J^*_{\boldsymbol{k}_2\lambda_2}(s').
\end{aligned} \qquad (4.45)$$

We check immediately that in the case $J_{\boldsymbol{k}\lambda}(s) \equiv 0$, i.e. in absence of the plasma, (4.45) yields the correct free field second order moment. The last correlation is then obtained using the bosonic commutation relation

$$\begin{aligned}
< a^\dagger_{\boldsymbol{k}_1\lambda_1} a_{\boldsymbol{k}_2\lambda_2} > &= < a_{\boldsymbol{k}_2\lambda_2} a^\dagger_{\boldsymbol{k}_1\lambda_1} > - \delta^{\boldsymbol{k}_2\lambda_2}_{\boldsymbol{k}_1\lambda_1} \\
&= \delta^{\boldsymbol{k}_2\lambda_2}_{\boldsymbol{k}_1\lambda_1} n_{\boldsymbol{k}_1\lambda_1} - \frac{e^{-\beta C_N}}{Z(T,N)} \prod_{i=1}^{N} \left(\frac{1}{2\pi\lambda^2_{\alpha_i}}\right)^{\frac{3}{2}} \int \mathcal{D}(\boldsymbol{\xi}_i) \int d\boldsymbol{r}_i\, e^{-\beta U_M(\mathcal{F}_1,\dots,\mathcal{F}_N)} \times \\
&\times e^{-\frac{\beta}{2}\sum^N_{i,j} e_{\alpha_i} e_{\alpha_j} W_m(i,j)} \int_0^1 ds \int_0^1 ds' S^{-1}_{\boldsymbol{k}_1}(s,1) S^{-1}_{\boldsymbol{k}_2}(0,s') J^*_{\boldsymbol{k}_1\lambda_1}(s) J_{\boldsymbol{k}_2\lambda_2}(s').
\end{aligned} \qquad (4.46)$$

4.3 The Field Correlations

With the results of the preceeding section we have now all the ingredients to calculate the field correlations $< A^\mu(\boldsymbol{x})A^\nu(\boldsymbol{y}) >_T$, $< E^\mu(\boldsymbol{x})E^\nu(\boldsymbol{y}) >_T$ and $< B^\mu(\boldsymbol{x})B^\nu(\boldsymbol{y}) >_T$ and we can proceed as in section 3.4. In the case of the vector potential we can write

$$\begin{aligned}
&< A^\mu(\boldsymbol{x})A^\nu(\boldsymbol{y}) >_T \\
&= \left(\frac{4\pi\hbar c^2}{R^3}\right)\sum_{\boldsymbol{k}\lambda}\sum_{\boldsymbol{k}'\lambda'}\frac{g(\boldsymbol{k})g(\boldsymbol{k}')}{2\sqrt{\omega_{\boldsymbol{k}}\omega_{\boldsymbol{k}'}}}\boldsymbol{e}^\mu_{\boldsymbol{k}\lambda}\boldsymbol{e}^\nu_{\boldsymbol{k}'\lambda'}\Big[e^{-i\boldsymbol{k}\boldsymbol{x}}e^{-i\boldsymbol{k}'\boldsymbol{y}} < a^\dagger_{\boldsymbol{k}\lambda}a^\dagger_{\boldsymbol{k}'\lambda'} > + \qquad (4.47)\\
&\quad + e^{-i\boldsymbol{k}\boldsymbol{x}}e^{+i\boldsymbol{k}'\boldsymbol{y}} < a^\dagger_{\boldsymbol{k}\lambda}a_{\boldsymbol{k}'\lambda'} > + e^{+i\boldsymbol{k}\boldsymbol{x}}e^{-i\boldsymbol{k}'\boldsymbol{y}} < a_{\boldsymbol{k}\lambda}a^\dagger_{\boldsymbol{k}'\lambda'} > + e^{+i\boldsymbol{k}\boldsymbol{x}}e^{+i\boldsymbol{k}'\boldsymbol{y}} < a_{\boldsymbol{k}\lambda}a_{\boldsymbol{k}'\lambda'} > \Big].
\end{aligned}$$

As we have seen in 4.2, there are again two kind of contributions: there are terms that sum up to a matter-dependent correlation $< A^\mu(\boldsymbol{x})A^\nu(\boldsymbol{y}) >^{mat}_T$ and there are contributions from the diagonal terms $< a^\dagger_{\boldsymbol{k}\lambda}a_{\boldsymbol{k}\lambda} >$ and $< a_{\boldsymbol{k}\lambda}a^\dagger_{\boldsymbol{k}\lambda} >$ that lead to the free field correlation $< A^\mu(\boldsymbol{x})A^\nu(\boldsymbol{y}) >^0_T$. The latter quantity is obtained by collecting the contributions of (4.45) and (4.46)

$$\begin{aligned}
< A^\mu(\boldsymbol{x})A^\nu(\boldsymbol{y}) >^0_T &= \left(\frac{4\pi\hbar c^2}{R^3}\right)\sum_{\boldsymbol{k}\lambda}\frac{g^2(\boldsymbol{k})}{2\omega_{\boldsymbol{k}}}\boldsymbol{e}^\mu_{\boldsymbol{k}\lambda}\boldsymbol{e}^\nu_{\boldsymbol{k}\lambda}\left((n_{\boldsymbol{k}}+1)e^{i\boldsymbol{k}(\boldsymbol{x}-\boldsymbol{y})} + n_{\boldsymbol{k}}e^{-i\boldsymbol{k}(\boldsymbol{x}-\boldsymbol{y})}\right)\\
&\overset{R\to\infty}{\to} \frac{\hbar c}{2}\int\frac{d\boldsymbol{k}}{(2\pi)^3}G^{\mu\nu}(\boldsymbol{k})k\left((n_{\boldsymbol{k}}+1)e^{i\boldsymbol{k}(\boldsymbol{x}-\boldsymbol{y})} + n_{\boldsymbol{k}}e^{-i\boldsymbol{k}(\boldsymbol{x}-\boldsymbol{y})}\right)\\
&= \frac{\hbar c}{2}\int\frac{d\boldsymbol{k}}{(2\pi)^3}G^{\mu\nu}(\boldsymbol{k})e^{i\boldsymbol{k}(\boldsymbol{x}-\boldsymbol{y})}k\,(2n_{\boldsymbol{k}}+1)\,. \qquad (4.48)
\end{aligned}$$

We are left with the matter-depending contributions. Unfortunately and in contrast to the classical case there are four different terms. The calculation is straight forward and we use (4.40), (4.42), (4.43) and (4.46) as well as the definitions (4.33) and (4.28) to obtain

$$\begin{aligned}
&< A^\mu(\boldsymbol{x})A^\nu(\boldsymbol{y}) >^{mat}_T \\
&= -\frac{1}{4}\frac{e^{-\beta C_N}}{Z(T,N)}\prod_{i=1}^N\left(\frac{1}{2\pi\lambda^2_{\alpha_i}}\right)^{\frac{3}{2}}\int\mathcal{D}(\boldsymbol{\xi}_i)\int d\boldsymbol{r}_i\, e^{-\beta U_M(\mathcal{F}_1,\ldots,\mathcal{F}_N)}e^{-\frac{\beta}{2}\sum_{i,j}^N e_{\alpha_i}e_{\alpha_j}W_m(i,j)}\times\\
&\quad\times\int\frac{d\boldsymbol{k}}{(2\pi)^3}e^{i\boldsymbol{k}\cdot\boldsymbol{x}}\int\frac{d\boldsymbol{k}'}{(2\pi)^3}e^{i\boldsymbol{k}'\cdot\boldsymbol{y}}\times\\
&\quad\times\sum_i\frac{e_{\alpha_i}}{c\sqrt{\beta m_{\alpha_i}}}\int_0^1 d\xi^\rho_i(s)e^{-i\boldsymbol{k}(\boldsymbol{r}_i+\lambda_{\alpha_i}\boldsymbol{\xi}_i(s))}\sum_j\frac{e_{\alpha_i}}{c\sqrt{\beta m_{\alpha_j}}}\int_0^1 d\xi^\sigma_j(s')e^{-i\boldsymbol{k}'(\boldsymbol{r}_j+\lambda_{\alpha_j}\boldsymbol{\xi}_j(s'))}\times\\
&\quad\times\Big[\mathcal{G}^{\mu\rho}(\boldsymbol{k},s,1)\,\mathcal{G}^{\nu\sigma}(\boldsymbol{k}',s',1) + \mathcal{G}^{\mu\rho}(\boldsymbol{k},s,1)\,\mathcal{G}^{\nu\sigma}(\boldsymbol{k}',0,s') +\\
&\qquad + \mathcal{G}^{\mu\rho}(\boldsymbol{k},0,s)\,\mathcal{G}^{\nu\sigma}(\boldsymbol{k}',s',1) + \mathcal{G}^{\mu\rho}(\boldsymbol{k},0,s)\,\mathcal{G}^{\nu\sigma}(\boldsymbol{k}',0,s')\Big]. \qquad (4.49)
\end{aligned}$$

Although $\mathcal{G}^{\mu\rho}(\boldsymbol{k},1,s) = \mathcal{G}^{\mu\rho}(\boldsymbol{k},0,s)$ and $\mathcal{G}^{\mu\rho}(\boldsymbol{k},s,1) = \mathcal{G}^{\mu\rho}(\boldsymbol{k},s,0)$ we have that $\mathcal{G}^{\mu\rho}(\boldsymbol{k},1,s) \neq \mathcal{G}^{\mu\rho}(\boldsymbol{k},s,1)$ and hence the four terms are not equal as it has been the case in the classical expression. Finally, we can rewrite (4.49) in a more compact form by means of the relation (2.25):

$$
\begin{aligned}
&< A^\mu(\boldsymbol{x}) A^\nu(\boldsymbol{y}) >_T^{mat} \\
&= -\frac{1}{4}\int \frac{d\boldsymbol{k}}{(2\pi)^3} e^{i\boldsymbol{k}\cdot\boldsymbol{x}} \int \frac{d\boldsymbol{k}'}{(2\pi)^3} e^{i\boldsymbol{k}'\cdot\boldsymbol{y}} \\
&\quad \times \left\langle \sum_i \frac{e_{\alpha_i}}{c\sqrt{\beta m_{\alpha_i}}} \int_0^1 d\xi_i^\rho(s) e^{-i\boldsymbol{k}(\boldsymbol{r}_i+\lambda_{\alpha_i}\boldsymbol{\xi}_i(s))} \sum_j \frac{e_{\alpha_j}}{c\sqrt{\beta m_{\alpha_j}}} \int_0^1 d\xi_j^\sigma(s') e^{-i\boldsymbol{k}'(\boldsymbol{r}_j+\lambda_{\alpha_j}\boldsymbol{\xi}_j(s'))} \right\rangle \times \\
&\quad \times \Big[\mathcal{G}^{\mu\rho}(\boldsymbol{k},s,1)\,\mathcal{G}^{\nu\sigma}(\boldsymbol{k}',s',1) + \mathcal{G}^{\mu\rho}(\boldsymbol{k},s,1)\,\mathcal{G}^{\nu\sigma}(\boldsymbol{k}',0,s') + \\
&\qquad + \mathcal{G}^{\mu\rho}(\boldsymbol{k},0,s)\,\mathcal{G}^{\nu\sigma}(\boldsymbol{k}',s',1) + \mathcal{G}^{\mu\rho}(\boldsymbol{k},0,s)\,\mathcal{G}^{\nu\sigma}(\boldsymbol{k}',0,s') \Big]. \qquad (4.50)
\end{aligned}
$$

Let us now analyze the transverse part of the electric field which is given by (2.8)

$$
\begin{aligned}
&< E_t^\mu(\boldsymbol{x}) E_t^\nu(\boldsymbol{y}) >_T = \\
&\left(\frac{4\pi\hbar c^2}{R^3}\right) \sum_{\boldsymbol{k}\lambda} \sum_{\boldsymbol{k}'\lambda'} \frac{g(\boldsymbol{k}) g(\boldsymbol{k}')}{2\sqrt{\omega_{\boldsymbol{k}}\omega_{\boldsymbol{k}'}}} \boldsymbol{e}_{\boldsymbol{k}\lambda}^\mu \boldsymbol{e}_{\boldsymbol{k}'\lambda'}^\nu \Big[-\left(\frac{\omega_{\boldsymbol{k}}\omega_{\boldsymbol{k}'}}{c^2}\right) e^{-i\boldsymbol{k}\boldsymbol{x}} e^{-i\boldsymbol{k}'\boldsymbol{y}} < a_{\boldsymbol{k}\lambda}^\dagger a_{\boldsymbol{k}'\lambda'}^\dagger > + \\
&\quad + \left(\frac{\omega_{\boldsymbol{k}}\omega_{\boldsymbol{k}'}}{c^2}\right) e^{-i\boldsymbol{k}\boldsymbol{x}} e^{+i\boldsymbol{k}'\boldsymbol{y}} < a_{\boldsymbol{k}\lambda}^\dagger a_{\boldsymbol{k}'\lambda'} > + \left(\frac{\omega_{\boldsymbol{k}}\omega_{\boldsymbol{k}'}}{c^2}\right) e^{+i\boldsymbol{k}\boldsymbol{x}} e^{-i\boldsymbol{k}'\boldsymbol{y}} < a_{\boldsymbol{k}\lambda} a_{\boldsymbol{k}'\lambda'}^\dagger > - \\
&\quad - \left(\frac{\omega_{\boldsymbol{k}}\omega_{\boldsymbol{k}'}}{c^2}\right) e^{+i\boldsymbol{k}\boldsymbol{x}} e^{+i\boldsymbol{k}'\boldsymbol{y}} < a_{\boldsymbol{k}\lambda} a_{\boldsymbol{k}'\lambda'} > \Big]. \qquad (4.51)
\end{aligned}
$$

Using the definition $\omega_{\boldsymbol{k}} = c|\boldsymbol{k}|$ we find then for the matter-independent correlations (in the thermodynamic limit $R \to \infty$)

$$
< E_t^\mu(\boldsymbol{x}) E_t^\nu(\boldsymbol{y}) >_T^0 \;=\; \frac{\hbar c}{2} \int \frac{d\boldsymbol{k}}{(2\pi)^3} G^{\mu\nu}(\boldsymbol{k}) e^{i\boldsymbol{k}(\boldsymbol{x}-\boldsymbol{y})} |k|^3 \left(2n_{\boldsymbol{k}}+1\right) \qquad (4.52)
$$

and for the matter depending part

$$
\begin{aligned}
&< E_t^\mu(\boldsymbol{x}) E_t^\nu(\boldsymbol{y}) >_T^{mat} \\
&= \frac{1}{4}\int \frac{d\boldsymbol{k}}{(2\pi)^3} e^{i\boldsymbol{k}\cdot\boldsymbol{x}} \int \frac{d\boldsymbol{k}'}{(2\pi)^3} e^{i\boldsymbol{k}'\cdot\boldsymbol{y}} (k\,k') \times \\
&\quad \times \left\langle \sum_i \frac{e_{\alpha_i}}{c\sqrt{\beta m_{\alpha_i}}} \int_0^1 d\xi_i^\rho(s) e^{-i\boldsymbol{k}(\boldsymbol{r}_i+\lambda_{\alpha_i}\boldsymbol{\xi}_i(s))} \sum_j \frac{e_{\alpha_j}}{c\sqrt{\beta m_{\alpha_j}}} \int_0^1 d\xi_j^\sigma(s') e^{-i\boldsymbol{k}'(\boldsymbol{r}_j+\lambda_{\alpha_j}\boldsymbol{\xi}_j(s'))} \right\rangle \times \\
&\quad \times \Big[\mathcal{G}^{\mu\rho}(\boldsymbol{k},s,1)\,\mathcal{G}^{\nu\sigma}(\boldsymbol{k}',s',1) - \mathcal{G}^{\mu\rho}(\boldsymbol{k},s,1)\,\mathcal{G}^{\nu\sigma}(\boldsymbol{k}',0,s') - \\
&\qquad - \mathcal{G}^{\mu\rho}(\boldsymbol{k},0,s)\,\mathcal{G}^{\nu\sigma}(\boldsymbol{k}',s',1) + \mathcal{G}^{\mu\rho}(\boldsymbol{k},0,s)\,\mathcal{G}^{\nu\sigma}(\boldsymbol{k}',0,s') \Big]. \qquad (4.53)
\end{aligned}
$$

This is indeed a very interesting result: in contrary to the classical case (3.47), the field correlations are influenced by the presence of the quantum charges and there is no decoupling. Eventually, the correlations of the magnetic induction $< B^\mu(\boldsymbol{x}) B^\nu(\boldsymbol{y}) >_T$ are obtained by taking the double curl of (4.49) and (4.48).

4.4 The Semi-Classical Limit $\hbar^{rad} \to 0$

If the calculations of the previous section are correct, we should be able to recover the classical field correlations in the semi-classical limit $\hbar^{rad} \to 0$ ($\hbar^{mat}$ being absorbed in λ, we shall henceforth denote $\hbar^{rad}$ simply by $\hbar$). Of course, we could perform the above limit in the final results of the field correlations (4.48), (4.50) and (4.52), (4.53) and compare them with their classical counterparts in chapter 3. However, according to the general formulas (4.47) and (4.51), it suffices to investigate the limit of the four different second order field moments multiplied by $\hbar$

$$\lim_{\hbar \to 0} \hbar < a_{\boldsymbol{k}_1\lambda_1} a_{\boldsymbol{k}_2\lambda_2} > . \tag{4.54}$$

One checks from e.g. (3.39) that their classical counterparts $\hbar < \alpha_{\boldsymbol{k}_1\lambda_1} \alpha_{\boldsymbol{k}_2\lambda_2} >$ etc. are indeed independent of $\hbar$.

Let us for instance consider the correlation $\hbar < a_{\boldsymbol{k}_1\lambda_1} a_{\boldsymbol{k}_2\lambda_2} >$ found in (4.40). The conjecture of Appendix A allows us to simplify the θ-functions appearing in the Green's functions $S_{\boldsymbol{k}}^{-1}(0, s)$ and $S_{\boldsymbol{k}}^{-1}(s, 0)$ and we obtain foremost

$$\begin{aligned}
&\lim_{\hbar \to 0} \hbar < a_{\boldsymbol{k}\lambda} a_{\boldsymbol{k}'\lambda'} >= \\
&= -\lim_{\hbar \to 0} \frac{e^{-\beta C_N}}{Z(T,N)} \prod_{i=1}^{N} \left(\frac{1}{2\pi \lambda_{\alpha_i}^2} \right)^{\frac{3}{2}} \int \mathcal{D}(\boldsymbol{\xi}_i) \int d\boldsymbol{r}_i \, e^{-\beta U_M(\mathcal{F}_1,\ldots,\mathcal{F}_N)} e^{-\frac{\beta}{2} \sum_{i,j}^{N} e_{\alpha_i} e_{\alpha_j} W_m(i,j)} \times \\
&\quad \times \sum_{i,j} \frac{2\pi\beta\hbar^2 e_{\alpha_i} e_{\alpha_j}}{\sqrt{m_{\alpha_i} m_{\alpha_j} w_{\boldsymbol{k}} w_{\boldsymbol{k}'}} R^3} g(\boldsymbol{k}) g(\boldsymbol{k}') \boldsymbol{e}^\mu_{\boldsymbol{k}\lambda} \boldsymbol{e}^\nu_{\boldsymbol{k}'\lambda'} \int_0^1 d\xi_i^\mu(s) \int_0^1 d\xi_j^\nu(s') \times \\
&\qquad \times e^{-i\boldsymbol{k}(\boldsymbol{r}_i + \lambda_{\alpha_i} \boldsymbol{\xi}_i(s))} e^{-i\boldsymbol{k}'(\boldsymbol{r}_j + \lambda_{\alpha_j} \boldsymbol{\xi}_j(s'))} \left[n_{\boldsymbol{k}} n_{\boldsymbol{k}'} e^{\beta\hbar\omega_{\boldsymbol{k}} s} e^{\beta\hbar\omega_{\boldsymbol{k}'} s'} \right] .
\end{aligned} \tag{4.55}$$

According to section 6.1 both the effective magnetic potential W_m and the partition function $Z(T, N)$ are transformed into their classical counterparts in the limit $\hbar \to 0$. Hence we are left with an $\hbar$-expansion of the square brackets in (4.55). Using the expansion of the bosonic occupation distribution

$$\begin{aligned}
n_{\boldsymbol{k}} &= \frac{1}{e^{\beta\hbar\omega_{\boldsymbol{k}}} - 1} \\
&= \frac{1}{\beta\hbar\omega_{\boldsymbol{k}}(1 + \underbrace{\frac{\beta\hbar\omega_{\boldsymbol{k}}}{2} + \frac{\beta\hbar\omega_{\boldsymbol{k}}}{3!} + \ldots}_{\equiv x})} \\
&= \frac{1}{\beta\hbar\omega_{\boldsymbol{k}}}(1 - x + x^2 - \ldots) \\
&= \frac{1}{\beta\hbar\omega_{\boldsymbol{k}}} - \frac{1}{2} + \frac{\beta\hbar\omega_{\boldsymbol{k}}}{12} + \mathcal{O}(\hbar^2,
\end{aligned} \tag{4.56}$$

we obtain eventually the limit

$$\begin{aligned}
&\lim_{\hbar\to 0} \hbar < a_{\boldsymbol{k}\lambda} a_{\boldsymbol{k}'\lambda'} >= \\
&= -\lim_{\hbar\to 0} \frac{e^{-\beta C_N}}{Z(T,N)} \prod_{i=1}^{N} \left(\frac{1}{2\pi\lambda_{\alpha_i}^2}\right)^{\frac{3}{2}} \int \mathcal{D}(\boldsymbol{\xi}_i) \int d\boldsymbol{r}_i\, e^{-\beta U_M(\mathcal{F}_1,\ldots,\mathcal{F}_N)} e^{-\frac{\beta}{2}\sum_{i,j}^{N} e_{\alpha_i} e_{\alpha_j} W_m(i,j)} \times \\
&\quad \times \sum_{i,j} \frac{2\pi\beta\hbar^2 e_{\alpha_i} e_{\alpha_j}}{\sqrt{m_{\alpha_i} m_{\alpha_j} w_{\boldsymbol{k}} w_{\boldsymbol{k}'}} R^3} g(\boldsymbol{k}) g(\boldsymbol{k}') \boldsymbol{e}^{\mu}_{\boldsymbol{k}\lambda} \boldsymbol{e}^{\nu}_{\boldsymbol{k}'\lambda'} \int_0^1 d\xi_i^{\mu}(s) \int_0^1 d\xi_j^{\nu}(s') \times \\
&\qquad \times e^{-i\boldsymbol{k}\cdot(\boldsymbol{r}_i + \lambda_{\alpha_i}\boldsymbol{\xi}_i(s))} e^{-i\boldsymbol{k}'(\boldsymbol{r}_j + \lambda_{\alpha_j}\boldsymbol{\xi}_j(s'))} \left[\frac{1}{(\beta\hbar)^2 \omega_{\boldsymbol{k}}\omega_{\boldsymbol{k}'}} + \mathcal{O}(\hbar^{-1})\right] \\
&= -\frac{e^{-\beta C_N}}{Z(T,N)} \prod_{i=1}^{N} \left(\frac{1}{2\pi\lambda_{\alpha_i}^2}\right)^{\frac{3}{2}} \int \mathcal{D}(\boldsymbol{\xi}_i) \int d\boldsymbol{r}_i\, e^{-\beta U_M(\mathcal{F}_1,\ldots,\mathcal{F}_N)} e^{-\frac{\beta}{2}\sum_{i,j}^{N} e_{\alpha_i} e_{\alpha_j} W_m(i,j)} \times \\
&\quad \times \sum_i \frac{e_{\alpha_i}}{c\sqrt{\beta m_{\alpha_i}}} \int_0^1 d\xi_i^{\mu}(s) e^{-i\boldsymbol{k}(\boldsymbol{r}_i + \lambda_{\alpha_i}\boldsymbol{\xi}_i(s))} \sum_j \frac{e_{\alpha_j}}{c\sqrt{\beta m_{\alpha_i}}} \int_0^1 d\xi_j^{\nu}(s') e^{-i\boldsymbol{k}'(\boldsymbol{r}_j + \lambda_{\alpha_j}\boldsymbol{\xi}_j(s'))} \times \\
&\qquad \times \frac{2\pi c^2}{R^3} \frac{g(\boldsymbol{k}) g(\boldsymbol{k}')}{(w_{\boldsymbol{k}} w_{\boldsymbol{k}'})^{3/2}} \boldsymbol{e}^{\mu}_{\boldsymbol{k}\lambda} \boldsymbol{e}^{\nu}_{\boldsymbol{k}'\lambda'}.
\end{aligned} \tag{4.57}$$

Using the definition (3.23) we recover the classical expression $\hbar < \alpha_{\boldsymbol{k}\lambda} \alpha_{\boldsymbol{k}'\lambda'} >$ of (3.39). The same arguments allow us to verify that the remaining second order field moments and a fortiori the field correlations have the correct semi-classical limit. Let us notice that the expansion (4.56) is equivalently valid for the high-temperature limit $\beta \to 0$ and the small $\boldsymbol{k}$ limit, a property that shall be useful in chapter 6.

Chapter 5

Correlations in the Quantum Fields: Perturbative Calculations

The correlation calculations for the full quantum system in chapter 4 have been performed by means of exact path integral methods. However, the FKI path integral and the BPI are rather sophisticated mathematical tools and it seems therefore legitimate to seek a confirmation of the final results by means of a conceptually less involved method. Such a method that is straight-forward and free of hidden subtleties is the well-known time-dependent perturbation theory. Obviously, we will only be able to make a low order expansion since the calculations get very complicated already for the second order term. But this is the usual prize to pay if one exchanges a mathematically elegant solution with one that is straight-forward and simple. Taking into consideration the kinetic term H_{kin}^N of the total Hamiltonian found in (2.9) one is tempted to use the coupling factor e/c as perturbative parameter. However, the speed of light c appears not only in this parameter and it is not always easy to see, once the calculations are done, whether a c is coming from the coupling constant or not. The same problem also arises if we seek the corresponding expansions of the exact results found in chapter 4. It is therefore easier to keep track of the perturbation if we choose the charge e (or more precisely the charges $\{e_\alpha\}$) as the perturbative parameter. Let us emphasize a final subtlety of this choice: the charges appearing in the Coulomb interaction cannot be considered as representing the perturbation parameters. Hence we are urged to distinguish the Coulomb charges from the charges arising in the field-matter coupling, let us call them coupling charges. However, this distinction will not cause any difficulties throughout the succeeding calculations.

Since the field correlations are completely determined by the second order field moments (4.35), we shall focus our interest on a second order expansion of these quantities. Let us now briefly outline the key features of the procedure leading to their perturbative expansion. For the sake of notational simplicity we introduce foremost a general observable $\hat{O} = \hat{O}(a_{\boldsymbol{k}\lambda}, a^{\dagger}_{\boldsymbol{k}'\lambda'})$ that depends solely on the field's degrees of freedom. Later on, we can replace $\hat{O}$ by the corresponding second order moments and achieve the calculations. In order to find the thermal average of $\hat{O}$ we have to calculate

$$< \hat{O} >= Tr(\hat{O}\rho^{tot}_{L,R}) = Tr_{rad}(Tr_{mat}(\rho^{tot}_{L,R})\hat{O}), \tag{5.1}$$

where $\rho^{tot}_{L,R}$ is the thermal weight (2.23). Hence our task is reduced to the calculation of the second order expansion of $\rho^{tot}_{L,R}$. Once we have found the expansion of $e^{-\beta H^N_{L,R}}$ by means of standard perturbation theory (section 5.1) we will focus on the partial matter trace $Tr_{mat}e^{-\beta H^N_{L,R}}$ (section 5.2). Again, we will not explicitly calculate it, but the results of section (3) will enable us express the corresponding quantities in terms of the FKI path integral formalism. In particular, this change of representation will be essential for the final comparison with the exact results of section 4.2. In section 5.3 we will focus on the field dependence of the expansion of $Tr_{mat}e^{-\beta H^N_{L,R}}$ and investigate the field trace $Tr_{rad}(Tr_{mat}e^{-\beta H^N_{L,R}}\hat{O})$. At this point we will replace the observable $\hat{O}$ by the four different types of second order field moments and calculate the corresponding partial traces by means of Wick's theorem. Using the correct expansion of the partition function $Z^N_{L,R}$ we can then derive the second order expansion of the thermal averages $< a_{\boldsymbol{k}\lambda}a_{\boldsymbol{k}'\lambda'} >$ etc. and hence of the field correlations.

5.1 The Formalism

Let us consider the full quantum Hamiltonian (2.9) and foremost rewrite the kinetic term

$$H^N_{kin} = \sum_{i=1} \left[\frac{\boldsymbol{p}_i^2}{2m} - \frac{e_{\alpha_i}}{mc}\boldsymbol{p}_i \cdot \boldsymbol{A}(\boldsymbol{r}_i) + \frac{e^2_{\alpha_i}}{c^2}\frac{\boldsymbol{A}^2(\boldsymbol{r}_i)}{2m} \right], \tag{5.2}$$

where we take advantage of the commutation relation $[\boldsymbol{p}_i, \boldsymbol{A}(\boldsymbol{r}_i)] = 0$, a consequence of the Coulomb gauge. As usual, we split the total Hamiltonian into a free part H_0 and an interaction part H_I

$$H^N_{L,R} = H_0 + H_I. \tag{5.3}$$

According to the definitions of section (4), the free Hamiltonian H_0 is

$$H_0 = H^{mat}_0 + H^{rad}_0 \tag{5.4}$$

with H^{rad}_0 defined in (2.9) and

$$H^{mat}_0 = \sum_{i=1}^{N} \frac{\boldsymbol{p}_i^2}{2m} + H^N_C + H^N_{pot}. \tag{5.5}$$

The interaction Hamiltonian is given by

$$H_I = H_I^{(1)} + H_I^{(2)},$$

where

$$H_I^{(1)} = -\frac{1}{mc}\sum_{i=1}^{N} e_{\alpha_i}\boldsymbol{p}_i \cdot \boldsymbol{A}(\boldsymbol{r}_i) \quad (5.6)$$

$$H_I^{(2)} = \frac{1}{2mc}\sum_{i=1}^{N} e_{\alpha_i}^2 \boldsymbol{A}^2(\boldsymbol{r}_i). \quad (5.7)$$

The imaginary time propagator $U(\beta, 0)$ corresponding to $H_{L,R}^N$ can now be expanded in a standard perturbation series as follows

$$\begin{aligned} U(\beta,0) = e^{-\beta H_{L,R}^N} &= e^{-\beta H_0}\left(\mathbf{1} - \int_0^\beta d\tau H_I(\tau) + \int_0^\beta d\tau_2 \int_0^{\tau_2} d\tau_1 H_I(\tau_2)H_I(\tau_1) + \right) + \mathcal{O}(e^3) \\ &= e^{-\beta H_0}\left(\mathbf{1} - \int_0^\beta d\tau (H_I^{(1)}(\tau) + H_I^{(2)}(\tau)) + \int_0^\beta d\tau_2 \int_0^{\tau_2} d\tau_1 H_I^{(1)}(\tau_2)H_I^{(1)}(\tau_1)\right) + \mathcal{O}(e^3). \end{aligned} \quad (5.8)$$

The second equality is due to the fact that $H_I^{(1)}$ is linear in the charges $\{e_\alpha\}$ whereas $H_I^{(2)}$ is quadratic. In (5.8), the imaginary time dependence of the operators is given by the usual mapping

$$H \mapsto H(\tau) = e^{\tau H_0} H e^{-\tau H_0}. \quad (5.9)$$

At this point we would like to anticipate the following simplification: according to the definition (5.6), $H_I^{(1)}$ is actually linear in $a_{\boldsymbol{k}\lambda}$ and $a_{\boldsymbol{k}\lambda}^\dagger$ and produces therefore in the calculation of the expanded partition function

$$Z_{L,R}^N = Tr_{rad} Tr_{mat} U(\beta, 0) \quad (5.10)$$

the generic terms $Tr_{rad}(e^{-\beta H_0^{rad}} a_{\boldsymbol{k}\lambda})$ and $Tr_{rad}(e^{-\beta H_0^{rad}} a_{\boldsymbol{k}\lambda}^\dagger)$. According to the symmetry arguments of appendix (C) these terms are both zero. Furthermore, when replacing $\hat{O}$ by the second order field moments, the partial trace

$$Tr_{rad}(U(\beta, 0)\hat{O}) \quad (5.11)$$

will contain cubic terms in $a_{\boldsymbol{k}\lambda}$ and $a_{\boldsymbol{k}\lambda}$ that vanish by similar symmetry arguments. Hence, in the expansion (5.8), the term

$$-e^{-\beta H_0}\int_0^\beta d\tau H_I^{(1)} \quad (5.12)$$

will not contribute to our calculations. On the other hand, the remaining terms will actually give non-zero contributions and before we proceed to their calculation, we rewrite them explicitly as bilinear expressions in the creation and annihilation operators. The integral over $H_I^{(2)}(\tau)$ can be reformulated as

$$\int_0^\beta d\tau H_I^{(2)}(\tau) = \sum_{i,j}^N e_{\alpha_i} e_{\alpha_j} \sum_{\boldsymbol{k}\lambda} \sum_{\boldsymbol{k}'\lambda'} \delta_{ij} \Big[a_{\boldsymbol{k}\lambda}^\dagger a_{\boldsymbol{k}'\lambda'}^\dagger S^{1ij}_{\boldsymbol{k}\lambda\boldsymbol{k}'\lambda'} + a_{\boldsymbol{k}\lambda}^\dagger a_{\boldsymbol{k}'\lambda'} S^{2ij}_{\boldsymbol{k}\lambda\boldsymbol{k}'\lambda'} + \\ + a_{\boldsymbol{k}\lambda} a_{\boldsymbol{k}'\lambda'}^\dagger S^{3ij}_{\boldsymbol{k}\lambda\boldsymbol{k}'\lambda'} + a_{\boldsymbol{k}\lambda} a_{\boldsymbol{k}'\lambda'} S^{4ij}_{\boldsymbol{k}\lambda\boldsymbol{k}'\lambda'} \Big], \qquad (5.13)$$

where we have introduced

$$\begin{aligned} S^{1ij}_{\boldsymbol{k}\lambda\boldsymbol{k}'\lambda'} &= \frac{\pi\hbar}{\sqrt{m_{\alpha_i} m_{\alpha_j}} R^3} \int_0^\beta d\tau e^{\hbar\tau(\omega_{\boldsymbol{k}}+\omega_{\boldsymbol{k}'})} \frac{g(\boldsymbol{k})g(\boldsymbol{k}')}{\omega_{\boldsymbol{k}}\omega_{\boldsymbol{k}'}} \boldsymbol{e}_{\boldsymbol{k}\lambda} \cdot \boldsymbol{e}_{\boldsymbol{k}'\lambda'} e^{\tau H_0^{mat}} e^{-i\boldsymbol{r}_i\cdot\boldsymbol{k}} e^{-i\boldsymbol{r}_j\cdot\boldsymbol{k}'} e^{-\tau H_0^{mat}} \\ S^{2ij}_{\boldsymbol{k}\lambda\boldsymbol{k}'\lambda'} &= \frac{\pi\hbar}{\sqrt{m_{\alpha_i} m_{\alpha_j}} R^3} \int_0^\beta d\tau e^{\hbar\tau(\omega_{\boldsymbol{k}}-\omega_{\boldsymbol{k}'})} \frac{g(\boldsymbol{k})g(\boldsymbol{k}')}{\omega_{\boldsymbol{k}}\omega_{\boldsymbol{k}'}} \boldsymbol{e}_{\boldsymbol{k}\lambda} \cdot \boldsymbol{e}_{\boldsymbol{k}'\lambda'} e^{\tau H_0^{mat}} e^{-i\boldsymbol{r}_i\cdot\boldsymbol{k}} e^{i\boldsymbol{r}_j\cdot\boldsymbol{k}'} e^{-\tau H_0^{mat}} \\ S^{3ij}_{\boldsymbol{k}\lambda\boldsymbol{k}'\lambda'} &= \frac{\pi\hbar}{\sqrt{m_{\alpha_i} m_{\alpha_j}} R^3} \int_0^\beta d\tau e^{-\hbar\tau(\omega_{\boldsymbol{k}}-\omega_{\boldsymbol{k}'})} \frac{g(\boldsymbol{k})g(\boldsymbol{k}')}{\omega_{\boldsymbol{k}}\omega_{\boldsymbol{k}'}} \boldsymbol{e}_{\boldsymbol{k}\lambda} \cdot \boldsymbol{e}_{\boldsymbol{k}'\lambda'} e^{\tau H_0^{mat}} e^{i\boldsymbol{r}_i\cdot\boldsymbol{k}} e^{-i\boldsymbol{r}_j\cdot\boldsymbol{k}'} e^{-\tau H_0^{mat}} \\ S^{4ij}_{\boldsymbol{k}\lambda\boldsymbol{k}'\lambda'} &= \frac{\pi\hbar}{\sqrt{m_{\alpha_i} m_{\alpha_j}} R^3} \int_0^\beta d\tau e^{-\hbar\tau(\omega_{\boldsymbol{k}}+\omega_{\boldsymbol{k}'})} \frac{g(\boldsymbol{k})g(\boldsymbol{k}')}{\omega_{\boldsymbol{k}}\omega_{\boldsymbol{k}'}} \boldsymbol{e}_{\boldsymbol{k}\lambda} \cdot \boldsymbol{e}_{\boldsymbol{k}'\lambda'} e^{\tau H_0^{mat}} e^{i\boldsymbol{r}_i\cdot\boldsymbol{k}} e^{i\boldsymbol{r}_j\cdot\boldsymbol{k}'} e^{-\tau H_0^{mat}}. \end{aligned} \qquad (5.14)$$

The exponentials of the type $e^{\hbar\tau(\omega_{\boldsymbol{k}}+\omega_{\boldsymbol{k}'})}$ represent the imaginary time evolution of the creation and annihilation operators according to

$$\begin{aligned} e^{\tau H_0^{rad}} a_{\boldsymbol{k}\lambda} e^{-\tau H_0^{rad}} &= a_{\boldsymbol{k}\lambda}(\tau) = a_{\boldsymbol{k}\lambda} e^{-\tau\hbar\omega_{\boldsymbol{k}}} \\ e^{\tau H_0^{rad}} a_{\boldsymbol{k}\lambda}^\dagger e^{-\tau H_0^{rad}} &= a_{\boldsymbol{k}\lambda}^\dagger(\tau) = a_{\boldsymbol{k}\lambda}^\dagger e^{\tau\hbar\omega_{\boldsymbol{k}}}. \end{aligned} \qquad (5.15)$$

Finally, the double integral in (5.8) can be restated by means of a θ-function and we obtain

$$\int_0^\beta d\tau_1 \int_0^\beta d\tau_2\, \theta(\tau_1 - \tau_2)\, H_I^{(1)}(\tau_1) H_I^{(1)}(\tau_2) = \\ = \sum_{i,j}^N e_{\alpha_i} e_{\alpha_j} \sum_{\boldsymbol{k}\lambda} \sum_{\boldsymbol{k}'\lambda'} \Big[a_{\boldsymbol{k}\lambda}^\dagger a_{\boldsymbol{k}'\lambda'}^\dagger T^{1ij}_{\boldsymbol{k}\lambda\boldsymbol{k}'\lambda'} + a_{\boldsymbol{k}\lambda}^\dagger a_{\boldsymbol{k}'\lambda'} T^{2ij}_{\boldsymbol{k}\lambda\boldsymbol{k}'\lambda'} + \\ + a_{\boldsymbol{k}\lambda} a_{\boldsymbol{k}'\lambda'}^\dagger T^{3ij}_{\boldsymbol{k}\lambda\boldsymbol{k}'\lambda'} + a_{\boldsymbol{k}\lambda} a_{\boldsymbol{k}'\lambda'} T^{4ij}_{\boldsymbol{k}\lambda\boldsymbol{k}'\lambda'} \Big], \qquad (5.16)$$

where

$$T^{1ij}_{\boldsymbol{k}\lambda\boldsymbol{k}'\lambda'} = \frac{2\pi\hbar}{m_{\alpha_i}m_{\alpha_j}R^3}\int_0^\beta d\tau_1\int_0^\beta d\tau_2\,\theta(\tau_1-\tau_2)e^{\hbar(\tau_1\omega_{\boldsymbol{k}}+\tau_2\omega_{\boldsymbol{k}'})}\frac{g(\boldsymbol{k})g(\boldsymbol{k}')}{\omega_{\boldsymbol{k}}\omega_{\boldsymbol{k}'}}\times$$
$$\times e^{\tau_1 H_0^{mat}}\left(\boldsymbol{e}_{\boldsymbol{k}\lambda}\cdot\boldsymbol{p}_i e^{-i\boldsymbol{k}\cdot\boldsymbol{r}_i}\right)e^{-(\tau_1-\tau_2)H_0^{mat}}\left(\boldsymbol{e}_{\boldsymbol{k}'\lambda'}\cdot\boldsymbol{p}_j e^{-i\boldsymbol{k}'\cdot\boldsymbol{r}_j}\right)e^{-\tau_2 H_0^{mat}}$$
$$T^{2ij}_{\boldsymbol{k}\lambda\boldsymbol{k}'\lambda'} = \frac{2\pi\hbar}{m_{\alpha_i}m_{\alpha_j}R^3}\int_0^\beta d\tau_1\int_0^\beta d\tau_2\,\theta(\tau_1-\tau_2)e^{\hbar(\tau_1\omega_{\boldsymbol{k}}-\tau_2\omega_{\boldsymbol{k}'})}\frac{g(\boldsymbol{k})g(\boldsymbol{k}')}{\omega_{\boldsymbol{k}}\omega_{\boldsymbol{k}'}}\times$$
$$\times e^{\tau_1 H_0^{mat}}\left(\boldsymbol{e}_{\boldsymbol{k}\lambda}\cdot\boldsymbol{p}_i e^{-i\boldsymbol{k}\cdot\boldsymbol{r}_i}\right)e^{-(\tau_1-\tau_2)H_0^{mat}}\left(\boldsymbol{e}_{\boldsymbol{k}'\lambda'}\cdot\boldsymbol{p}_j e^{i\boldsymbol{k}'\cdot\boldsymbol{r}_j}\right)e^{-\tau_2 H_0^{mat}}$$
$$T^{3ij}_{\boldsymbol{k}\lambda\boldsymbol{k}'\lambda'} = \frac{2\pi\hbar}{m_{\alpha_i}m_{\alpha_j}R^3}\int_0^\beta d\tau_1\int_0^\beta d\tau_2\,\theta(\tau_1-\tau_2)e^{-\hbar(\tau_1\omega_{\boldsymbol{k}}-\tau_2\omega_{\boldsymbol{k}'})}\frac{g(\boldsymbol{k})g(\boldsymbol{k}')}{\omega_{\boldsymbol{k}}\omega_{\boldsymbol{k}'}}\times$$
$$\times e^{\tau_1 H_0^{mat}}\left(\boldsymbol{e}_{\boldsymbol{k}\lambda}\cdot\boldsymbol{p}_i e^{i\boldsymbol{k}\cdot\boldsymbol{r}_i}\right)e^{-(\tau_1-\tau_2)H_0^{mat}}\left(\boldsymbol{e}_{\boldsymbol{k}'\lambda'}\cdot\boldsymbol{p}_j e^{-i\boldsymbol{k}'\cdot\boldsymbol{r}_j}\right)e^{-\tau_2 H_0^{mat}}$$
$$T^{4ij}_{\boldsymbol{k}\lambda\boldsymbol{k}'\lambda'} = \frac{2\pi\hbar}{m_{\alpha_i}m_{\alpha_j}R^3}\int_0^\beta d\tau_1\int_0^\beta d\tau_2\,\theta(\tau_1-\tau_2)e^{-\hbar(\tau_1\omega_{\boldsymbol{k}}+\tau_2\omega_{\boldsymbol{k}'})}\frac{g(\boldsymbol{k})g(\boldsymbol{k}')}{\omega_{\boldsymbol{k}}\omega_{\boldsymbol{k}'}}\times$$
$$\times e^{\tau_1 H_0^{mat}}\left(\boldsymbol{e}_{\boldsymbol{k}\lambda}\cdot\boldsymbol{p}_i e^{i\boldsymbol{k}\cdot\boldsymbol{r}_i}\right)e^{-(\tau_1-\tau_2)H_0^{mat}}\left(\boldsymbol{e}_{\boldsymbol{k}'\lambda'}\cdot\boldsymbol{p}_j e^{i\boldsymbol{k}'\cdot\boldsymbol{r}_j}\right)e^{-\tau_2 H_0^{mat}} \tag{5.17}$$

In equations (5.13) and (5.16) the matter dependence is now entirely contained in the coefficients $S^{lij}_{\boldsymbol{k}\lambda\boldsymbol{k}'\lambda'}$ and $T^{lij}_{\boldsymbol{k}\lambda\boldsymbol{k}'\lambda'}$ and hence we can write the partial matter trace of $U(\beta,0)$ as

$$\rho_m \equiv Tr_{mat}U(\beta,0) = Tr_{mat}e^{-\beta H^N_{L,R}} = \rho_m^{(0)} + \rho_m^{(1)} + \rho_m^{(2)} + \mathcal{O}(e^3), \tag{5.18}$$

where ρ_m is now a field-depending quantity. Note that $\rho_m^{(1)}$ stands for the terms linear in $a_{\boldsymbol{k}\lambda}$, $a^\dagger_{\boldsymbol{k}\lambda}$ (that will not contribute to the final result according to the above comment). Finally,

$$\rho_m^{(0)} = e^{-\beta H_0^{rad}}Tr_{mat}e^{-\beta H_0^{mat}}, \tag{5.19}$$

and

$$\rho_m^{(2)} = e^{-\beta H_0^{rad}}\sum_{i,j}^N\sum_{\boldsymbol{k}\lambda}\sum_{\boldsymbol{k}'\lambda'}\Big[a^\dagger_{\boldsymbol{k}\lambda}a^\dagger_{\boldsymbol{k}'\lambda'}\mathcal{M}^{1ij}_{\boldsymbol{k}\lambda\boldsymbol{k}'\lambda'} + a^\dagger_{\boldsymbol{k}\lambda}a_{\boldsymbol{k}'\lambda'}\mathcal{M}^{2ij}_{\boldsymbol{k}\lambda\boldsymbol{k}'\lambda'} +$$
$$+a_{\boldsymbol{k}\lambda}a^\dagger_{\boldsymbol{k}'\lambda'}\mathcal{M}^{3ij}_{\boldsymbol{k}\lambda\boldsymbol{k}'\lambda'} + a_{\boldsymbol{k}\lambda}a_{\boldsymbol{k}'\lambda'}\mathcal{M}^{4ij}_{\boldsymbol{k}\lambda\boldsymbol{k}'\lambda'}\Big] \tag{5.20}$$

with

$$\mathcal{M}^{lij}_{\boldsymbol{k}\lambda\boldsymbol{k}'\lambda'} = Tr_{mat}\left\{e^{-\beta H_0^{mat}}\left(T^{lij}_{\boldsymbol{k}\lambda\boldsymbol{k}'\lambda'} + \delta_{ij}S^{lij}_{\boldsymbol{k}\lambda\boldsymbol{k}'\lambda'}\right)\right\} \qquad l = 1,2,3,4. \tag{5.21}$$

The advantage of the representation (5.18) is that we have formally separated the field and the matter dependence. We focus now on the quantities $\mathcal{M}^{lij}_{\boldsymbol{k}\lambda\boldsymbol{k}'\lambda'}$ and seek formulation in terms of the loop formalism.

5.2 The Matter Trace

Actually, the goal of this section is not the explicit calculation of the matter traces in (5.18) because this is, as mentioned beforehand, a fairly impossible mission. However, if we want to achieve the aim of this chapter, i.e. the perturbative inspection of the exact results, we have to ensure that the two methods lead to commensurable mathematical objects. Since chapter 4 has been based on the FKI path integral formulation of the matter's degrees of freedom, we would like to translate the present objects into this very formalism. Having a look at e.g. (5.17) we see that the usual application of closure relations and Trotter's formula will not lead to a straightforward transformation to the path integral formalism. We are therefore urged to adopt a more sophisticated approach by taking advantage of the semi-classical results of chapter 3. Thereby the basic idea is to make an expansion of $Tr_{mat}(e^{-\beta H_{L,R}^N})$ by means of the thermal weight in in the filament representation (3.17) and to extract the coefficients (5.21) by means of appropriate comparisons. Since ρ_m (5.18) is an object obtained in the full quantum formalism, we have to perform at first the semi-classical limit $\hbar^{rad} \to 0$ in order to render such a comparison possible. For the sake of clarity, we define

$$\rho_m^{cl} \equiv \lim_{\hbar^{rad}\to 0} \rho_m = \rho_m^{cl(0)} + \rho_m^{cl(1)} + \rho_m^{cl(2)} + \mathcal{O}(e^3), \tag{5.22}$$

which we can reformulate by means of the results of section 3.2. Since the potential U_A defined in (3.19) is linear in the coupling charges $\{e_\alpha\}$ we obtain the following expansion

$$\begin{aligned} \rho_m^{cl} &= e^{-\beta H_0^{rad}} Tr_{mat} e^{-\beta U_M} e^{-\beta U_A} \\ &= e^{-\beta H_0^{rad}} Tr_{mat} e^{-\beta U_M} \left(\mathbf{1} - \beta U_A + \beta^2 U_A^2) + \mathcal{O}(e^3\right) \end{aligned} \tag{5.23}$$

According to (5.22) we have thus

$$\rho_m^{cl(0)} = e^{-\beta H_0^{rad}} Tr_{mat} e^{-\beta U_M} \tag{5.24}$$

and

$$\rho_m^{cl(2)} = \rho_m^{cl(21)} + \rho_m^{cl(22)} + \rho_m^{cl(23)} + \rho_m^{cl(24)}, \tag{5.25}$$

where the four quadratic terms in (5.25) are given by

$$\begin{aligned} \rho_m^{cl(21)} = -\sum_{i,j}^N e_{\alpha_i} e_{\alpha_j} \sum_{\boldsymbol{k}\lambda} \sum_{\boldsymbol{k}'\lambda'} e^{-\beta H_0^{rad}} (\alpha_{\boldsymbol{k}\lambda}^* \alpha_{\boldsymbol{k}'\lambda'}^*) \int_0^1 ds \int_0^1 ds' \prod_{i=1}^N \left(\frac{1}{2\pi\lambda_{\alpha_i}}\right)^{\frac{3}{2}} \int \mathcal{D}(\boldsymbol{\xi}_i) \int d\boldsymbol{r}_i \times \\ \times \left\{ e^{-\beta U_M} \left(\frac{d\boldsymbol{\xi}_i(s_1)}{ds_1} \cdot \boldsymbol{e}_{\boldsymbol{k}\lambda}\right) \left(\frac{d\boldsymbol{\xi}_j(s_2)}{ds_2} \cdot \boldsymbol{e}_{\boldsymbol{k}'\lambda'}\right) e^{-i\boldsymbol{k}(\boldsymbol{r}_i + \lambda_{\alpha_i}\boldsymbol{\xi}_i(s))} e^{-i\boldsymbol{k}'(\boldsymbol{r}_j + \lambda_{\alpha_j}\boldsymbol{\xi}_j(s'))} C(k,k') \right\}, \end{aligned} \tag{5.26}$$

$$\rho_m^{cl(22)} = -\sum_{i,j}^{N} e_{\alpha_i} e_{\alpha_j} \sum_{\boldsymbol{k}\lambda} \sum_{\boldsymbol{k}'\lambda'} e^{-\beta H_0^{rad}} (\alpha^*_{\boldsymbol{k}\lambda} \alpha_{\boldsymbol{k}'\lambda'}) \int_0^1 ds \int_0^1 ds' \prod_{i=1}^{N} \left(\frac{1}{2\pi\lambda_{\alpha_i}}\right)^{\frac{3}{2}} \int \mathcal{D}(\boldsymbol{\xi}_i) \int d\boldsymbol{r}_i \times$$
$$\times \left\{ e^{-\beta U_M} \left(\frac{d\boldsymbol{\xi}_i(s_1)}{ds_1} \cdot \boldsymbol{e}_{\boldsymbol{k}\lambda}\right) \left(\frac{d\boldsymbol{\xi}_j(s_2)}{ds_2} \cdot \boldsymbol{e}_{\boldsymbol{k}'\lambda'}\right) e^{-i\boldsymbol{k}(\boldsymbol{r}_i + \lambda_{\alpha_i}\boldsymbol{\xi}_i(s))} e^{i\boldsymbol{k}'(\boldsymbol{r}_j + \lambda_{\alpha_j}\boldsymbol{\xi}_j(s'))} C(k,k') \right\}, \tag{5.27}$$

$$\rho_m^{cl(23)} \quad -\sum_{i,j}^{N} e_{\alpha_i} e_{\alpha_j} \sum_{\boldsymbol{k}\lambda} \sum_{\boldsymbol{k}'\lambda'} e^{-\beta H_0^{rad}} (\alpha_{\boldsymbol{k}\lambda} \alpha^*_{\boldsymbol{k}'\lambda'}) \int_0^1 ds \int_0^1 ds' \prod_{i=1}^{N} \left(\frac{1}{2\pi\lambda_{\alpha_i}}\right)^{\frac{3}{2}} \int \mathcal{D}(\boldsymbol{\xi}_i) \int d\boldsymbol{r}_i \times$$
$$\times \left\{ e^{-\beta U_M} \left(\frac{d\boldsymbol{\xi}_i(s_1)}{ds_1} \cdot \boldsymbol{e}_{\boldsymbol{k}\lambda}\right) \left(\frac{d\boldsymbol{\xi}_j(s_2)}{ds_2} \cdot \boldsymbol{e}_{\boldsymbol{k}'\lambda'}\right) e^{i\boldsymbol{k}(\boldsymbol{r}_i + \lambda_{\alpha_i}\boldsymbol{\xi}_i(s))} e^{-i\boldsymbol{k}'(\boldsymbol{r}_j + \lambda_{\alpha_j}\boldsymbol{\xi}_j(s'))} C(k,k') \right\}, \tag{5.28}$$

$$\rho_m^{cl(24)} = -\sum_{i,j}^{N} e_{\alpha_i} e_{\alpha_j} \sum_{\boldsymbol{k}\lambda} \sum_{\boldsymbol{k}'\lambda'} e^{-\beta H_0^{rad}} (\alpha_{\boldsymbol{k}\lambda} \alpha_{\boldsymbol{k}'\lambda'}) \int_0^1 ds \int_0^1 ds' \prod_{i=1}^{N} \left(\frac{1}{2\pi\lambda_{\alpha_i}}\right)^{\frac{3}{2}} \int \mathcal{D}(\boldsymbol{\xi}_i) \int d\boldsymbol{r}_i \times$$
$$\times \left\{ e^{-\beta U_M} \left(\frac{d\boldsymbol{\xi}_i(s_1)}{ds_1} \cdot \boldsymbol{e}_{\boldsymbol{k}\lambda}\right) \left(\frac{d\boldsymbol{\xi}_j(s_2)}{ds_2} \cdot \boldsymbol{e}_{\boldsymbol{k}'\lambda'}\right) e^{i\boldsymbol{k}(\boldsymbol{r}_i + \lambda_{\alpha_i}\boldsymbol{\xi}_i(s))} e^{i\boldsymbol{k}'(\boldsymbol{r}_j + \lambda_{\alpha_j}\boldsymbol{\xi}_j(s'))} C(k,k') \right\}. \tag{5.29}$$

The constant $C(k,k')$ appearing in $\rho_m^{cl(21)}$, $\rho_m^{cl(22)}$, $\rho_m^{cl(23)}$ and $\rho_m^{cl(24)}$ is defined by

$$C(k,k') = \frac{\pi\hbar\beta}{\sqrt{m_{\alpha_1} m_{\alpha_2}} R^3} \frac{g(\boldsymbol{k})g(\boldsymbol{k}')}{\sqrt{\omega_{\boldsymbol{k}}\omega_{\boldsymbol{k}'}}}. \tag{5.30}$$

Note that we are not interested in $\rho_m^{cl(1)}$ since its quantum counterpart $\rho_m^{(1)}$ does not give any contribution to the final averages (cf comment above). Before we can use the expansion (5.22) we have to take the semi-classical limit $\hbar^{rad} \to 0$ of (5.18) and replace the creation and annihilation operators $a_{\boldsymbol{k}\lambda}$, $a^\dagger_{\boldsymbol{k}\lambda}$ by the corresponding c-numbers $\alpha_{\boldsymbol{k}\lambda}$, $\alpha^*_{\boldsymbol{k}\lambda}$. Care needs to be taken with the $\hbar$ appearing in the prefactors of (5.14) and (5.17). It originates from the prefactor of the vector potential (2.6) and appears therefore in both the quantum and the classical expression as can be seen in (5.30). Hence the semi-classical limit is simply reduced to the cancellation of the $\hbar$ appearing in the imaginary time evolution exponentials $e^{-\tau\hbar\omega_{\boldsymbol{k}}}$ in (5.14) and (5.17). This being said, we can now start to perform a direct comparison between (5.23) and the semi-classical limit of (5.18). Considering the zero order terms (5.19) and (5.24) we deduce immediately that

$$Z_0^{mat} \equiv Tr_{mat} e^{-\beta H_0^{mat}} = Tr_{mat} e^{-\beta U_M} = \prod_{i=1}^{N} \left(\frac{1}{2\pi\lambda_{\alpha_i}}\right)^{\frac{3}{2}} \int \mathcal{D}(\boldsymbol{\xi}_i) \int d\boldsymbol{r}_i e^{-\beta U_M}. \tag{5.31}$$

The comparison of the second order terms (5.20) and (5.25) is less obvious. Comparing separately the terms in $\alpha^*_{\boldsymbol{k}\lambda}\alpha^*_{\boldsymbol{k}'\lambda'}$ etc and switching back to the quantum formalism yields eventually the following expressions for the matter traces (5.21)

$$\mathcal{M}^{1ij}_{\boldsymbol{k}\lambda\boldsymbol{k}'\lambda'} = \tag{5.32}$$
$$-\prod_{i=1}^{N}\left(\frac{1}{2\pi\lambda_{\alpha_i}}\right)^{\frac{3}{2}}\int\mathcal{D}(\boldsymbol{\xi}_i)\int d\boldsymbol{r}_i e^{-\beta U_M}C(k,k')\times$$
$$\left(\int_0^1 d\boldsymbol{\xi}_i(s)\cdot\boldsymbol{e}_{\boldsymbol{k}\lambda}e^{-i\boldsymbol{k}(\boldsymbol{r}_i+\lambda_{\alpha_i}\boldsymbol{\xi}_i(s))}e^{\beta\hbar\tau\omega_{\boldsymbol{k}}s}\right)\left(\int_0^1 d\boldsymbol{\xi}_j(s')\cdot\boldsymbol{e}_{\boldsymbol{k}'\lambda'}e^{-i\boldsymbol{k}'(\boldsymbol{r}_j+\lambda_{\alpha_j}\boldsymbol{\xi}_j(s'))}e^{\beta\hbar\tau\omega_{\boldsymbol{k}'}s'}\right)$$

$$\mathcal{M}^{2ij}_{\boldsymbol{k}\lambda\boldsymbol{k}'\lambda'} =$$
$$-\prod_{i=1}^{N}\left(\frac{1}{2\pi\lambda_{\alpha_i}}\right)^{\frac{3}{2}}\int\mathcal{D}(\boldsymbol{\xi}_i)\int d\boldsymbol{r}_i e^{-\beta U_M}C(k,k')\times \tag{5.33}$$
$$\left(\int_0^1 d\boldsymbol{\xi}_i(s)\cdot\boldsymbol{e}_{\boldsymbol{k}\lambda}e^{-i\boldsymbol{k}(\boldsymbol{r}_i+\lambda_{\alpha_i}\boldsymbol{\xi}_i(s))}e^{\beta\hbar\tau\omega_{\boldsymbol{k}}s}\right)\left(\int_0^1 d\boldsymbol{\xi}_j(s')\cdot\boldsymbol{e}_{\boldsymbol{k}'\lambda'}e^{i\boldsymbol{k}'(\boldsymbol{r}_j+\lambda_{\alpha_j}\boldsymbol{\xi}_j(s'))}e^{-\beta\hbar\tau\omega_{\boldsymbol{k}'}s'}\right)$$

$$\mathcal{M}^{3ij}_{\boldsymbol{k}\lambda\boldsymbol{k}'\lambda'} = \tag{5.34}$$
$$-\prod_{i=1}^{N}\left(\frac{1}{2\pi\lambda_{\alpha_i}}\right)^{\frac{3}{2}}\int\mathcal{D}(\boldsymbol{\xi}_i)\int d\boldsymbol{r}_i e^{-\beta U_M}C(k,k')\times$$
$$\left(\int_0^1 d\boldsymbol{\xi}_i(s)\cdot\boldsymbol{e}_{\boldsymbol{k}\lambda}e^{i\boldsymbol{k}(\boldsymbol{r}_i+\lambda_{\alpha_i}\boldsymbol{\xi}_i(s))}e^{-\beta\hbar\tau\omega_{\boldsymbol{k}}s}\right)\left(\int_0^1 d\boldsymbol{\xi}_j(s')\cdot\boldsymbol{e}_{\boldsymbol{k}'\lambda'}e^{-i\boldsymbol{k}'(\boldsymbol{r}_j+\lambda_{\alpha_j}\boldsymbol{\xi}_j(s'))}e^{\beta\hbar\tau\omega_{\boldsymbol{k}'}s'}\right)$$

$$\mathcal{M}^{4ij}_{\boldsymbol{k}\lambda\boldsymbol{k}'\lambda'} = \tag{5.35}$$
$$-\prod_{i=1}^{N}\left(\frac{1}{2\pi\lambda_{\alpha_i}}\right)^{\frac{3}{2}}\int\mathcal{D}(\boldsymbol{\xi}_i)\int d\boldsymbol{r}_i e^{-\beta U_M}C(k,k')\times$$
$$\left(\int_0^1 d\boldsymbol{\xi}_i(s)\cdot\boldsymbol{e}_{\boldsymbol{k}\lambda}e^{i\boldsymbol{k}(\boldsymbol{r}_i+\lambda_{\alpha_i}\boldsymbol{\xi}_i(s))}e^{-\beta\hbar\tau\omega_{\boldsymbol{k}}s}\right)\left(\int_0^1 d\boldsymbol{\xi}_j(s')\cdot\boldsymbol{e}_{\boldsymbol{k}'\lambda'}e^{i\boldsymbol{k}'(\boldsymbol{r}_j+\lambda_{\alpha_j}\boldsymbol{\xi}_j(s'))}e^{-\beta\hbar\tau\omega_{\boldsymbol{k}'}s'}\right).$$

Having now translated the matter-depending part of (5.18) into the path integral formalism we can focus our attention on the radiation part.

5.3 The Field Trace

The next step consists in taking the radiation trace of the expansion (5.18) multiplied by the observable, $Tr_{rad}(\rho_m\hat{O})$, up to the second order. According to (5.18) and the comment made in section 5.1 this involves the following types of traces

$$Tr_{rad}e^{-\beta H_0^{rad}}\hat{O}$$
$$Tr_{rad}e^{-\beta H_0^{rad}}b_1 b_2\hat{O}, \tag{5.36}$$

where b_1 and b_2 are elements of $\{a_{\boldsymbol{k}\lambda}\}$ and $\{a^\dagger_{\boldsymbol{k}\lambda}\}$. Let us now start with the investigation of the first one of the second order field moments, i.e. let $\hat{O} \equiv a_{\boldsymbol{k}_1\lambda_1}a_{\boldsymbol{k}_2\lambda_2}$. Using once again the results of appendix C we can immediately see that

$$Tr_{rad}e^{-\beta H_0^{rad}}a_{\boldsymbol{k}_1\lambda_1}a_{\boldsymbol{k}_2\lambda_2} = 0. \tag{5.37}$$

In the case of the quartic terms

$$Tr_{rad}e^{-\beta H_0^{rad}}b_1 b_2 a_{\boldsymbol{k}_1\lambda_1}a_{\boldsymbol{k}_2\lambda_2} \tag{5.38}$$

the number of creators has to be equal to the number of annihilators for a non-zero contribution and hence the only term that survives is

$$\begin{aligned} &Tr_{rad}e^{-\beta H_0^{rad}}a^\dagger_{\boldsymbol{k}\lambda}a^\dagger_{\boldsymbol{k}'\lambda'}a_{\boldsymbol{k}_1\lambda_1}a_{\boldsymbol{k}_2\lambda_2} = \\ &= \sum_{\{n_{\boldsymbol{k}\lambda}\}} e^{-\beta\hbar\sum_{\boldsymbol{k}\lambda}\omega_{\boldsymbol{k}}n_{\boldsymbol{k}\lambda}} < \{n_{\boldsymbol{k}\lambda}\}|a^\dagger_{\boldsymbol{k}\lambda}a^\dagger_{\boldsymbol{k}'\lambda'}a_{\boldsymbol{k}_1\lambda_1}a_{\boldsymbol{k}_2\lambda_2}|\{n_{\boldsymbol{k}\lambda}\} > \\ &= \sum_{\{n_{\boldsymbol{k}\lambda}\}} e^{-\beta\hbar\sum_{\boldsymbol{k}\lambda}\omega_{\boldsymbol{k}}n_{\boldsymbol{k}\lambda}} \left[\left(1-\delta^{\boldsymbol{k}_2\lambda_2}_{\boldsymbol{k}_1\lambda_1}\right)\left(\delta^{\boldsymbol{k}_1\lambda_1}_{\boldsymbol{k}\lambda}\delta^{\boldsymbol{k}_2\lambda_2}_{\boldsymbol{k}'\lambda'} + \delta^{\boldsymbol{k}_2\lambda_2}_{\boldsymbol{k}\lambda}\delta^{\boldsymbol{k}_1\lambda_1}_{\boldsymbol{k}'\lambda'}\right)n_{\boldsymbol{k}}n_{\boldsymbol{k}'} + \delta^{\boldsymbol{k}_2\lambda_2}_{\boldsymbol{k}_1\lambda_1}\left(\delta^{\boldsymbol{k}_1\lambda_1}_{\boldsymbol{k}\lambda}\delta^{\boldsymbol{k}_2\lambda_2}_{\boldsymbol{k}'\lambda'}n_{\boldsymbol{k}}(n_{\boldsymbol{k}}-1)\right)\right]. \end{aligned} \tag{5.39}$$

Before we can proceed to the expansion of the thermal average $< a_{\boldsymbol{k}\lambda}a_{\boldsymbol{k}'\lambda'} >$, we have to expand the normalization $1/Z^N_{L,R}$ appearing in the thermal weight $\rho^{tot}_{L,R}$ up to second order in $\{e_\alpha\}$. Using the sum rule for the geometric series, we find

$$\begin{aligned} \frac{1}{Z^N_{L,R}} &= \frac{1}{Z_0}\Big(1 + \underbrace{\frac{Z_1}{Z_0} + \frac{Z_2}{Z_0} + \mathcal{O}(e^3)}_{:=x}\Big) \\ &= \frac{1}{Z_0}(1 - x + x^2 - \ldots) \\ &= \frac{1}{Z_0}\left(1 - \frac{Z_2}{Z_0}\right) + \mathcal{O}(e^3), \end{aligned} \tag{5.40}$$

where we have used the fact that $Z_1 = 0$. $Z_0 = Z_0^{rad}Z_0^{mat}$ is the free partition function according to the definitions (5.31) and (4.24). The second order term is given by

$$\begin{aligned} Z_2 &= \sum_{i,j}^N e_{\alpha_i}e_{\alpha_j}\sum_{\boldsymbol{k}\lambda}\sum_{\boldsymbol{k}'\lambda'}\left[Tr_{rad}(e^{-\beta H_0^{rad}}a^\dagger_{\boldsymbol{k}\lambda}a_{\boldsymbol{k}'\lambda'})\mathcal{M}^{2ij}_{\boldsymbol{k}\lambda\boldsymbol{k}'\lambda'} + Tr_{rad}(e^{-\beta H_0^{rad}}a_{\boldsymbol{k}\lambda}a^\dagger_{\boldsymbol{k}'\lambda'})\mathcal{M}^{3ij}_{\boldsymbol{k}\lambda\boldsymbol{k}'\lambda'}\right] \\ &= Z_0^{rad}\sum_{i,j}^N e_{\alpha_i}e_{\alpha_j}\sum_{\boldsymbol{k}\lambda}\left[n_{\boldsymbol{k}\lambda}\mathcal{M}^{2ij}_{\boldsymbol{k}\lambda\boldsymbol{k}\lambda} + (1+n_{\boldsymbol{k}\lambda})\mathcal{M}^{3ij}_{\boldsymbol{k}\lambda\boldsymbol{k}\lambda}\right]. \end{aligned} \tag{5.41}$$

The second equality is obtained by means of the following relation

$$\sum_{\{n_{\boldsymbol{k}\lambda}\}} e^{-\beta\hbar\sum_{\boldsymbol{k}\lambda} n_{\boldsymbol{k}\lambda}\omega_{\boldsymbol{k}}} n_{\boldsymbol{k}_1\lambda_1} = Z_0^{rad} n_{\boldsymbol{k}_1\lambda_1}. \tag{5.42}$$

At this very point we are able to state the final result. Using (5.18) together with (5.37), (5.39) and (5.42) yields the second order expansion

$$\begin{aligned} < a_{\boldsymbol{k}_1\lambda_1} a_{\boldsymbol{k}_2\lambda_2} > &= \frac{1}{Z_0}\left(1 - \frac{Z_2}{Z_0}\right) Tr_{rad}(\rho_m) \\ &= \frac{1}{Z_0}\sum_{i,j}^{N} e_{\alpha_i} e_{\alpha_j} \sum_{\boldsymbol{k}\lambda}\sum_{\boldsymbol{k}'\lambda'}\sum_{\{n_{\boldsymbol{k}\lambda}\}} e^{-\beta\hbar\sum_{\boldsymbol{k}\lambda} n_{\boldsymbol{k}\lambda}\omega_{\boldsymbol{k}}} \mathcal{M}^{1ij}_{\boldsymbol{k}\lambda\boldsymbol{k}'\lambda'} \times \\ &\quad \times \left[\left(1 - \delta^{\boldsymbol{k}_2\lambda_2}_{\boldsymbol{k}_1\lambda_1}\right)\left(\delta^{\boldsymbol{k}_1\lambda_1}_{\boldsymbol{k}\lambda}\delta^{\boldsymbol{k}_2\lambda_2}_{\boldsymbol{k}'\lambda'} + \delta^{\boldsymbol{k}_2\lambda_2}_{\boldsymbol{k}\lambda}\delta^{\boldsymbol{k}_1\lambda_1}_{\boldsymbol{k}'\lambda'}\right) n_{\boldsymbol{k}} n_{\boldsymbol{k}'} + \delta^{\boldsymbol{k}_2\lambda_2}_{\boldsymbol{k}_1\lambda_1}\left(\delta^{\boldsymbol{k}_1\lambda_1}_{\boldsymbol{k}\lambda}\delta^{\boldsymbol{k}_2\lambda_2}_{\boldsymbol{k}'\lambda'} n_{\boldsymbol{k}}(n_{\boldsymbol{k}} - 1)\right)\right] \\ &\quad\ + \mathcal{O}(e^3) \end{aligned} \tag{5.43}$$

Let us investigate separately the two terms in the square brackets: the first takes care of the off-diagonal contributions $\boldsymbol{k}_1\lambda_1 \neq \boldsymbol{k}_2\lambda_2$ and we find by means of (5.42) and the symmetry relation $\mathcal{M}^{1ij}_{\boldsymbol{k}\lambda\boldsymbol{k}'\lambda'} = \mathcal{M}^{1ji}_{\boldsymbol{k}'\lambda'\boldsymbol{k}\lambda}$

$$< a_{\boldsymbol{k}_1\lambda_1} a_{\boldsymbol{k}_2\lambda_2} >= \frac{2}{Z_0^{mat}} \sum_{i,j}^{N} e_{\alpha_i} e_{\alpha_j} n_{\boldsymbol{k}_1\lambda_1} n_{\boldsymbol{k}_2\lambda_2} \mathcal{M}^{1ij}_{\boldsymbol{k}_1\lambda_1\boldsymbol{k}_2\lambda_2}, \qquad \boldsymbol{k}_1\lambda_1 \neq \boldsymbol{k}_2\lambda_2. \tag{5.44}$$

The second term in the square brackets of (5.43) takes care of the diagonal contribution $\boldsymbol{k}_1\lambda_1 = \boldsymbol{k}_2\lambda_2$; using the equality

$$\sum_{\{n_{\boldsymbol{k}\lambda}\}} n^2_{\boldsymbol{k}\lambda} e^{-\beta\hbar\sum_{\boldsymbol{k}\lambda} n_{\boldsymbol{k}\lambda}\omega_{\boldsymbol{k}}} = Z_0^{rad}(2n^2_{\boldsymbol{k}\lambda} + n_{\boldsymbol{k}\lambda}) \tag{5.45}$$

we find actually the same contribution as in (5.44) evaluated in $\boldsymbol{k}_1\lambda_1 = \boldsymbol{k}_2\lambda_2$. According to (5.32) we obtain the final expression for (5.43)

$$\begin{aligned} < a_{\boldsymbol{k}_1\lambda_1} a_{\boldsymbol{k}_2\lambda_2} > &= -\frac{1}{Z_0^{mat}} \sum_{i,j}^{N} \frac{2\pi\hbar\beta e_{\alpha_i} e_{\alpha_j}}{\sqrt{m_{\alpha_i} m_{\alpha_j}} R^3} \frac{g(\boldsymbol{k}_1) g(\boldsymbol{k}_2)}{\sqrt{\omega_{\boldsymbol{k}_1}\omega_{\boldsymbol{k}_2}}} n_{\boldsymbol{k}_1\lambda_1} n_{\boldsymbol{k}_2\lambda_2} \times \\ &\quad \times \prod_{i=1}^{N}\left(\frac{1}{2\pi\lambda_{\alpha_i}}\right)^{\frac{3}{2}} \int \mathcal{D}(\boldsymbol{\xi}_i) \int d\boldsymbol{r}_i e^{-\beta U_M} \left(\int_0^1 d\boldsymbol{\xi}_i(s) \cdot \boldsymbol{e}_{\boldsymbol{k}_1\lambda_1} e^{-i\boldsymbol{k}_1(\boldsymbol{r}_i + \lambda_{\alpha_i}\boldsymbol{\xi}_i(s))} e^{\beta\hbar\tau\omega_{\boldsymbol{k}_1} s}\right) \times \\ &\quad \times \left(\int_0^1 d\boldsymbol{\xi}_j(s') \cdot \boldsymbol{e}_{\boldsymbol{k}_2\lambda_2} e^{-i\boldsymbol{k}_2(\boldsymbol{r}_j + \lambda_{\alpha_j}\boldsymbol{\xi}_j(s'))} e^{\beta\hbar\tau\omega_{\boldsymbol{k}_2} s'}\right). \end{aligned} \tag{5.46}$$

Let us compare this result to the second order expansion of the exact expression (4.40) found by means of the BPI technique. In (4.40), $J_{\boldsymbol{k}\lambda}(s)$ and $J_{\boldsymbol{k}'\lambda'}(s')$ are both linear in the charge we retain merely the zero-order contributions from the exponentials $e^{-\beta W_m}$ and e^{-C_N} as well as from the inverse partition function $1/Z(T,N)$. The latter can be verified to have the following expansion

$$\frac{1}{Z(T,N)} = \frac{1}{Z_0^{mat}} + \mathcal{O}(e^2) \tag{5.47}$$

and hence it is easy to verify that we recover the exact same expression as in (5.46).

The remaining three moments are obtained by means of similar calculations. However, the field trace of the quartic terms

$$Tr_{rad}\left(e^{-\beta H_0^{rad}} b_1 b_2 b_3 b_4\right), \qquad b_i \in \left\{\{a_{\boldsymbol{k}\lambda}\}, \{a^\dagger_{\boldsymbol{k}\lambda}\}\right\} \tag{5.48}$$

are substantially more complicated if the operators don't appear in the normal order as in the case of (5.39). In point of fact, one has to apply Wick's theorem in order to reduce the expression (5.48) to a sum of normal ordered quantities. These expansions are rather tedious and we don't present the calculations here. However, we have checked that the final second order expansions coincided with the corresponding expansions of the exact results (4.42), (4.45) and (4.46). Consequently, we can confirm that the perturbative expansions of the field correlations (up to second order in the coupling charges) agree with the results of chapter 4.

Chapter 6

Correlations in the Quantum Fields: Long Distance Asymptotics

The results of the preceding sections are given in form of exact Fourier transforms, e.g. the vector potential correlations (4.50) and (4.48). So if we would like to analyze the result in real space we have to perform these Fourier transforms. Due to the complicated integrands involving in particular the cut-off function $g(\boldsymbol{k})$, this seems to be a fairly impossible task in the general case. However, there is a special case where we can considerably simplify our calculations: if we consider the large distance asymptotics $|\boldsymbol{r}_i - \boldsymbol{r}_j| \to \infty$, we need to retain solely the leading terms of a small $\boldsymbol{k}$ expansion of the integrands (for a mathematical justification of this statement, please refer to [11]). From a physical point of view, such an analysis tells us much about the nature of the correlations between points that are separated by distances much larger than the microscopic scale. We can find out whether there are long range correlations or whether there is some sort of screening mechanism that implies an exponential decay. Before we investigate the field correlations, we focus on the magnetic interaction potential W_m given in (4.32).

6.1 The Magnetic Interaction Potential W_m

In the semi-classical case, the effective magnetic potential W_m is known to reveal a dipolar decay in the long distance asymptotics [5]. It is the goal of this section to find out whether the quantum nature of the field causes a modification of this inverse power law decay or not. Before we can analyze the $\boldsymbol{k} \to 0$ behavior of (4.32), we have to perform an appropriate expansion of the function

$$\mathcal{G}^{\mu\nu}(\boldsymbol{k}, s, s') = 4\pi\beta\hbar c \frac{S_{\boldsymbol{k}}^{-1}(s,s')}{k} \delta_{tr}^{\mu\nu}(\boldsymbol{k}) g^2(\boldsymbol{k}). \tag{6.1}$$

and hence of the Green's function $S_{\boldsymbol{k}}^{-1}(s,s')$. In this aim we introduce the time difference $\tau \equiv s - s'$ and rewrite

$$S_{\boldsymbol{k}}^{-1}(s,s') \equiv S_{\boldsymbol{k}}^{-1}(\tau) = e^{-\beta\hbar\omega_{\boldsymbol{k}}\tau}\left(\theta(\tau-\eta)(1+n_{\boldsymbol{k}}) + \theta(\eta-\tau)n_{\boldsymbol{k}}\right) = e^{-\beta\hbar\omega_{\boldsymbol{k}}\tau}(\theta(\tau-\eta)+n_{\boldsymbol{k}}) \tag{6.2}$$

Noting that the $\hbar$ expansion of the occupation number $n_{\boldsymbol{k}}$ in (4.56) is equivalent to a small $\boldsymbol{k}$ expansion we find the following expansion of the Green's function

$$\begin{aligned} S_{\boldsymbol{k}}^{-1}(\tau) = & \frac{1}{\beta\hbar\omega_{\boldsymbol{k}}} + \left[\theta(\tau-\eta) - \frac{1}{2} - \tau\right] + \\ & + \beta\hbar\omega_{\boldsymbol{k}}\left[\frac{1}{12} - \tau\theta(\tau-\eta) + \frac{\tau}{2} + \frac{1}{2}\tau^2\right] + \mathcal{O}(k^2). \end{aligned} \tag{6.3}$$

Plugging this expansion into (6.1) and using the small $\boldsymbol{k}$ behavior of the cut-off function, $g(\boldsymbol{k}) \sim 1$, we can calculate the asymptotic behavior of $W_m(i,j)$. However, decomposing the Green's function into its even and odd parts under time reversal $(\tau \mapsto -\tau)$

$$S_{\boldsymbol{k}}^{-1}(\tau) = S_{\boldsymbol{k}}^{-1\,(even)}(\tau) + S_{\boldsymbol{k}}^{-1\,(odd)}(\tau), \tag{6.4}$$

one checks from (4.32) that the contribution $W_m^{odd}(i,j)$ arising from $S_{\boldsymbol{k}}^{-1\,(odd)}$ is anti-symmetric under the exchange of i and j, i.e. $W_m^{odd}(i,j) = -W_m^{odd}(j,i)$. Obviously, only the symmetric part of $W_m(i,j)$ does contribute to the sum of the pairwise interactions and hence it is sufficient to keep only the even part $W_m^{even}(i,j)$ coming from

$$\begin{aligned} S_{\boldsymbol{k}}^{-1\,(even)}(\tau) = & \frac{1}{\beta\hbar\omega_{\boldsymbol{k}}} + \frac{1}{2}\left[\theta(\tau-\eta) + \theta(-\tau-\eta) - 1\right] + \\ & + \frac{\beta\hbar\omega_{\boldsymbol{k}}}{2}\left[\frac{1}{6} - \tau\theta(\tau-\eta) + \tau\theta(-\tau-\eta) + \tau^2\right] + \mathcal{O}(k^2) \\ = & \frac{1}{\beta\hbar\omega_{\boldsymbol{k}}} + \frac{\beta\hbar\omega_{\boldsymbol{k}}}{2}\left[\frac{1}{6} - |\tau| + \tau^2\right] + \mathcal{O}(k^2) \end{aligned} \tag{6.5}$$

The last equality is not strictly rigorous, anticipating the $\eta \to 0$ limit and ignoring the fact that for $\tau = 0$ there is a constant term $-1/2$ which survives. The right prescription in this matter is to keep the θ-functions as well as the parameter η until the very end and to perform eventually the limit $\eta \to 0$. However, using the conjecture of appendix A we anticipate the validity of (6.5) and we analyze its contribution to the interaction potential $W_m^{even}(i,j)$. Plugging the first term into (6.1) yields

$$\mathcal{G}^{\mu\nu,even}(\boldsymbol{k},\tau) = G^{\mu\nu}(\boldsymbol{k}) + \mathcal{O}(k^0) \tag{6.6}$$

and thus the first approximation to $W_m^{even}(i,j)$, let us call it $W_{m,1}^{even}(i,j)$, is the classical interaction potential (3.31). Hence $W_m^{even}(i,j)$ has a leading order term in r^{-3}. The second term of the expansion, $W_{m,2}^{even}(i,j)$ (i.e the first approximation to the classical potential) is slightly more complicated and demands a thorough analysis. Using (4.32) we find

$$\begin{aligned} W_{m,2}^{even}(i,j) &= \frac{1}{\beta\sqrt{m_{\alpha_i}m_{\alpha_j}}c^2}\int\frac{d\boldsymbol{k}}{(2\pi)^3}e^{i\boldsymbol{k}(\boldsymbol{r}_i-\boldsymbol{r}_j)}\int_0^1 d\xi_i^\mu(s)e^{i\lambda_{\alpha_i}\boldsymbol{k}\cdot\boldsymbol{\xi}_i(s)}\int_0^1 d\xi_j^\nu(s')e^{-i\lambda_{\alpha_j}\boldsymbol{k}\cdot\boldsymbol{\xi}_j(s')}\times \\ &\quad\times 2\pi(\beta\hbar c)^2 g^2(\boldsymbol{k})\delta_{tr}^{\mu\nu}(\boldsymbol{k})\left[\frac{1}{6}-|s-s'|+(s-s')^2\right] \\ &\sim 2\pi\lambda_{\alpha_i}\lambda_{\alpha_j}\int\frac{d\boldsymbol{k}}{(2\pi)^3}e^{i\boldsymbol{k}(\boldsymbol{r}_i-\boldsymbol{r}_j)}\int_0^1 d\xi_i^\mu(s)\int_0^1 d\xi_j^\nu(s')\times \\ &\quad\times\left\{\left(\delta^{\mu\nu}-\frac{k^\mu k^\nu}{k^2}\right)+i\lambda_{\alpha_i}\boldsymbol{k}\cdot\boldsymbol{\xi}_i(s)\left(\delta^{\mu\nu}-\frac{k^\mu k^\nu}{k^2}\right)-i\lambda_{\alpha_j}\boldsymbol{k}\cdot\boldsymbol{\xi}_j(s')\left(\delta^{\mu\nu}-\frac{k^\mu k^\nu}{k^2}\right)\right\}\times \\ &\quad\times\left[\frac{1}{6}-|s-s'|+(s-s')^2\right]. \end{aligned} \tag{6.7}$$

The approximation is again based on the expansion of the exponentials and on $g(\boldsymbol{k})\sim 1$. Note that this term is explicitly depending on $\hbar$ and hence in the classical field limit it vanishes (and so do all higher order terms). Furthermore, we would like to emphasize that we cannot use Itô's lemma due to the time-dependence of the square brackets. The Fourier transform of the transverse Dirac-Delta $\delta_{tr}^{\mu\nu}(\boldsymbol{r})$ appearing in the first term of (6.7) is

$$\int\frac{d\boldsymbol{k}}{(2\pi)^3}e^{i\boldsymbol{k}(\boldsymbol{r}_i-\boldsymbol{r}_j)}\left(\delta^{\mu\nu}-\frac{k^\mu k^\nu}{k^2}\right)=\delta^{\mu\nu}\delta(\boldsymbol{r}_i-\boldsymbol{r}_j)+\frac{1}{4\pi}\partial_\mu\partial_\nu\frac{1}{r}. \tag{6.8}$$

Since $r\to\infty$, the first term disappears and we are left with an algebraic decay in $\sim r^{-3}$ which is due to the transversality of $\delta_{tr}^{\mu\nu}(\boldsymbol{r})$. The remaining two terms of (6.7) yield according to

$$\int\frac{d\boldsymbol{k}}{(2\pi)^3}e^{i\boldsymbol{k}(\boldsymbol{r}_i-\boldsymbol{r}_j)}(i\boldsymbol{\xi}\cdot\boldsymbol{k})\left(\delta^{\mu\nu}-\frac{k^\mu k^\nu}{k^2}\right)=-(\boldsymbol{\xi}_i(s)\cdot\nabla_{\boldsymbol{r}})\,\partial_\mu\partial_\nu\left(\frac{1}{4\pi r}\right) \tag{6.9}$$

a contribution that decays as $\sim r^{-4}$ for fixed loop shapes $\boldsymbol{\xi}_i$ and $\boldsymbol{\xi}_j$. Hence the leading order term of $W^{even}_{m,2}(i,j)$ is

$$\begin{aligned}
W^{even}_{m,2}(i,j) &= \frac{1}{2}\lambda_{\alpha_i}\lambda_{\alpha_j}\int_0^1 d\xi_i^\mu(s)\int_0^1 d\xi_j^\nu(s')\left[\frac{1}{6}-|s-s'|+(s-s')^2\right]\partial_\mu\partial_\nu\frac{1}{r}+\mathcal{O}(r^{-4})\\
&= \frac{1}{2}\lambda_{\alpha_i}\lambda_{\alpha_j}\int_0^1 ds\int_0^1 ds'\xi_i^\mu(s)\xi_j^\nu(s')\frac{d^2}{dsds'}\left[\frac{1}{6}-|s-s'|+(s-s')^2\right]\partial_\mu\partial_\nu\frac{1}{r}+\mathcal{O}(r^{-4})\\
&= \frac{1}{2}\int_0^1 ds\int_0^1 ds'\left(\lambda_{\alpha_i}\boldsymbol{\xi}_i(s)\cdot\nabla\right)\left(\lambda_{\alpha_j}\boldsymbol{\xi}_j(s')\cdot\nabla\right)\left[2\delta(s-s')-2\right]\frac{1}{r}+\mathcal{O}(r^{-4})\\
&= \int_0^1 ds\int_0^1 ds'\left(\delta(s-s')-1\right)\left(\lambda_{\alpha_i}\boldsymbol{\xi}_i(s)\cdot\nabla\right)\left(\lambda_{\alpha_j}\boldsymbol{\xi}_j(s')\cdot\nabla\right)\frac{1}{r}+\mathcal{O}(r^{-4}).
\end{aligned} \tag{6.10}$$

Bearing in mind that the total interaction between two loops is the sum of the Coulomb potential V_C and the magnetic potential W_m, this is actually a very interesting result. To see why, we have to split the equal-time Coulomb potential V_C defined in (3.20) into two parts

$$V_C(i,j) = V_{cl}(i,j) + W_C(i,j), \tag{6.11}$$

where V_{cl} is a genuine classical electrostatic Coulomb potential

$$V_{cl}(i,j) = \int_0^1 ds\int_0^1 ds'\frac{1}{\left|\boldsymbol{r}_j+\lambda_{\alpha_j}\boldsymbol{\xi}_j(s')-\boldsymbol{r}_i-\lambda_{\alpha_i}\boldsymbol{\xi}_i(s)\right|} \tag{6.12}$$

and $W_C(i,j) = V_C(i,j) - V_{cl}(i,j)$ is the purely quantum-mechanical contribution. The asymptotic behaviour of $W_C(i,j)$ is according to [5] opposite to the one of $W^{even}_{m,2}$ and we can conclude that the quantum contribution to the Coulomb potential W_C is cancelled by the magnetic contribution $W^{even}_{m,2}(i,j)$ in the large distance limit $r\to\infty$. Hence the quantum nature of radiation field destroys the dipolar nature of the quantum Coulomb interaction.

Let us finally collect the different contributions to the asymptotics of the magnetic potential $W^{even}_m(i,j)$. The decay is dipolar as in the case of the classical radiation field and to the leading order we find (cf [5] for the asymptotics of the classical term $W^{even}_{m,1}$)

$$\begin{aligned}
W^{even}_m(i,j) &= W^{even}_{m,1}(i,j)+W^{even}_{m,2}(i,j)+\mathcal{O}(\frac{1}{r^4})\\
&\overset{r\to\infty}{\sim} \int_0^1 ds\int_0^1 ds'\left(\delta(s-s')-1\right)\left(\lambda_{\alpha_i}\boldsymbol{\xi}_i(s)\cdot\nabla\right)\left(\lambda_{\alpha_j}\boldsymbol{\xi}_j(s')\cdot\nabla\right)\frac{1}{r}-\\
&\quad -\frac{1}{2\beta\sqrt{m_{\alpha_i}m_{\alpha_j}}c^2}\int_0^1 d\xi_i^\mu(s)\int_0^1 d\xi_j^\mu(s')\left(\lambda_{\alpha_i}\boldsymbol{\xi}_i(s)\cdot\nabla\right)\left(\lambda_{\alpha_j}\boldsymbol{\xi}_j(s')\cdot\nabla\right)\frac{1}{r}\left(\delta^{\mu\nu}+\frac{\tilde{r}^\mu\tilde{r}^\nu}{r}\right),
\end{aligned} \tag{6.13}$$

6.2 The Electric Field Correlations

Now we investigate the large distance correlations of the transverse electric field $< E_t^\mu(\boldsymbol{x}) E_t^\nu(\boldsymbol{y}) >_T$. We consider foremost the matter-depending part (4.53). Using the conjecture of appendix A), we replace the Green's functions $S_{\boldsymbol{k}}^{-1}$ appearing in the stochastic line integrals by

$$\begin{aligned} S_{\boldsymbol{k}}^{-1}(1,s) = S_{\boldsymbol{k}}^{-1}(0,s) &\to e^{\beta\hbar\omega_{\boldsymbol{k}} s} n_{\boldsymbol{k}} \\ S_{\boldsymbol{k}}^{-1}(s,1) = S_{\boldsymbol{k}}^{-1}(s,0) &\to e^{-\beta\hbar\omega_{\boldsymbol{k}} s}(n_{\boldsymbol{k}}+1) \end{aligned} \tag{6.14}$$

and we obtain

$$\begin{aligned} &< E_t^\mu(\boldsymbol{x}) E_t^\nu(\boldsymbol{y}) >_T^{mat} \\ &= \frac{1}{4}\int \frac{d\boldsymbol{k}}{(2\pi)^3} e^{i\boldsymbol{k}\cdot\boldsymbol{x}} \int \frac{d\boldsymbol{k}'}{(2\pi)^3} e^{i\boldsymbol{k}'\cdot\boldsymbol{y}} \\ &\quad\times \left\langle \sum_i \frac{e_{\alpha_i}}{c\sqrt{\beta m_{\alpha_i}}} \int_0^1 d\xi_i^\rho(s) e^{-i\boldsymbol{k}(\boldsymbol{r}_i+\lambda_{\alpha_i}\boldsymbol{\xi}_i(s))} \sum_j \frac{e_{\alpha_j}}{c\sqrt{\beta m_{\alpha_j}}} \int_0^1 d\xi_j^\sigma(s') e^{-i\boldsymbol{k}'(\boldsymbol{r}_j+\lambda_{\alpha_j}\boldsymbol{\xi}_j(s'))} \right\rangle \times \\ &\quad\times (4\pi\hbar c\beta)^2 g^2(\boldsymbol{k}) g^2(\boldsymbol{k}') \delta_{tr}^{\mu\rho}(\boldsymbol{k}) \delta_{tr}^{\nu\sigma}(\boldsymbol{k}') \Big[(n_{\boldsymbol{k}}+1)(n_{\boldsymbol{k}'}+1) e^{-\beta\hbar(\omega_{\boldsymbol{k}} s+\omega_{\boldsymbol{k}'} s')} - \\ &\quad - n_{\boldsymbol{k}}(n_{\boldsymbol{k}'}+1) e^{-\beta\hbar(\omega_{\boldsymbol{k}'} s'-\omega_{\boldsymbol{k}} s)} - (n_{\boldsymbol{k}}+1) n_{\boldsymbol{k}'} e^{-\beta\hbar(\omega_{\boldsymbol{k}} s-\omega_{\boldsymbol{k}'} s')} + n_{\boldsymbol{k}} n_{\boldsymbol{k}'} e^{\beta\hbar(\omega_{\boldsymbol{k}} s+\omega_{\boldsymbol{k}'} s')} \Big]. \end{aligned} \tag{6.15}$$

We consider now the first of the four terms of (6.15) which we label $< E^\mu(\boldsymbol{x}) E^\nu(\boldsymbol{y}) >_T^{mat(1)}$ (the remaining three terms can be dealt with in a similar manner)

$$\begin{aligned} &< E^\mu(\boldsymbol{x}) E^\nu(\boldsymbol{y}) >_T^{mat(1)} \\ &= \frac{1}{4}\int \frac{d\boldsymbol{k}}{(2\pi)^3} e^{i\boldsymbol{k}\cdot\boldsymbol{x}} \int \frac{d\boldsymbol{k}'}{(2\pi)^3} e^{i\boldsymbol{k}'\cdot\boldsymbol{y}} \left[4\pi\hbar\beta c g^2(\boldsymbol{k}) \delta_{tr}^{\mu\rho}(\boldsymbol{k})(n_{\boldsymbol{k}}+1)\right] \left[4\pi\hbar\beta c g^2(\boldsymbol{k}') \delta_{tr}^{\mu\sigma}(\boldsymbol{k}')(n_{\boldsymbol{k}'}+1)\right] \times \\ &\quad\times \Bigg\langle \sum_i \frac{e_{\alpha_i}}{c\sqrt{\beta m_{\alpha_i}}} \int_0^1 d\xi_i^\rho(s) e^{-i\boldsymbol{k}(\boldsymbol{r}_i+\lambda_{\alpha_i}\boldsymbol{\xi}_i(s))} e^{-\beta\hbar\omega_{\boldsymbol{k}} s} \times \\ &\qquad\qquad \times \sum_j \frac{e_{\alpha_j}}{c\sqrt{\beta m_{\alpha_j}}} \int_0^1 d\xi_j^\sigma(s') e^{-i\boldsymbol{k}'(\boldsymbol{r}_j+\lambda_{\alpha_j}\boldsymbol{\xi}_j(s'))} e^{-\beta\hbar\omega_{\boldsymbol{k}'} s'} \Bigg\rangle. \end{aligned} \tag{6.16}$$

For the sake of compactness we introduce now the quantities:

$$\begin{aligned} \widehat{G}^{\mu\nu}(\boldsymbol{k}) &= 4\pi\hbar\beta c g^2(\boldsymbol{k}) \delta_{tr}^{\mu\nu}(\boldsymbol{k})(n_{\boldsymbol{k}}+1), \\ \widehat{j}^\mu(\mathcal{F}_i,\boldsymbol{k}) &= \frac{e_{\alpha_i}}{c\sqrt{\beta m_{\alpha_i}}} \int_0^1 d\xi_i^\mu(s) e^{-i\boldsymbol{k}(\boldsymbol{r}_i+\lambda_{\alpha_i}\boldsymbol{\xi}_i(s))} e^{-\beta\hbar\omega_{\boldsymbol{k}} s}, \\ \widehat{\mathcal{J}}^\mu(\boldsymbol{k}) &= \sum_{i=1}^N \widehat{j}^\mu(\mathcal{F}_i,\boldsymbol{k}). \end{aligned} \tag{6.17}$$

which allow us to rewrite (6.16) in a concise way:

$$< E^\mu(\boldsymbol{x})E^\nu(\boldsymbol{y}) >_T^{mat(1)} = \frac{1}{4}\int \frac{d\boldsymbol{k}}{(2\pi)^3} e^{i\boldsymbol{k}\cdot\boldsymbol{x}} \int \frac{d\boldsymbol{k}'}{(2\pi)^3} e^{i\boldsymbol{k}'\cdot\boldsymbol{y}} \widehat{G}^{\mu\rho}(\boldsymbol{k})\widehat{G}^{\nu\sigma}(\boldsymbol{k}') \left\langle \widehat{\mathcal{J}}^\rho(\boldsymbol{k})\widehat{\mathcal{J}}^\sigma(\boldsymbol{k}')\right\rangle . \tag{6.18}$$

Introducing the loop-density

$$\hat{\rho}(\mathcal{F}) = \sum_{i=1}^{N} \delta(\mathcal{F}, \mathcal{F}_i) \tag{6.19}$$

we can rewrite $\widehat{\mathcal{J}}^\rho(\boldsymbol{k}) = \int d\mathcal{F}\widehat{j}^\rho(\mathcal{F}, \boldsymbol{k})\hat{\rho}(\mathcal{F})$, where $\int d\mathcal{F}$ is the integral over the entire phase space of filaments. Hence we can express the thermal average in (6.18) as

$$\begin{aligned} < \widehat{\mathcal{J}}^\rho(\boldsymbol{k})\widehat{\mathcal{J}}^\sigma(\boldsymbol{k}') > &= \int d\mathcal{F}_1 \int d\mathcal{F}_2 \widehat{j}^\rho(\mathcal{F}_1, \boldsymbol{k})\widehat{j}^\sigma(\mathcal{F}_2, \boldsymbol{k}') < \hat{\rho}(\mathcal{F}_1)\hat{\rho}(\mathcal{F}_2) > \qquad (6.20)\\ &= (2\pi)^3\delta(\boldsymbol{k}+\boldsymbol{k}') \sum_{\alpha_1,\alpha_2} \int \mathcal{D}(\boldsymbol{\xi}_1) \int \mathcal{D}(\boldsymbol{\xi}_2) \times \\ &\quad \times \widehat{\mathcal{T}}^\rho(\alpha_1, \boldsymbol{\xi}_1, \boldsymbol{k}) \left(\widehat{\mathcal{T}}^\sigma(\alpha_2, \boldsymbol{\xi}_2, \boldsymbol{k})\right)^* n(\alpha_1, \boldsymbol{\xi}_1, \alpha_2, \boldsymbol{\xi}_2, \boldsymbol{k}), \end{aligned}$$

where we have introduced the loop-loop correlation function

$$n(\alpha_1, \boldsymbol{r}_1, \boldsymbol{\xi}_1, \alpha_2, \boldsymbol{r}_2, \boldsymbol{\xi}_2) = n(\mathcal{F}_1, \mathcal{F}_2) =< \hat{\rho}(\mathcal{F}_1)\hat{\rho}(\mathcal{F}_2) > \tag{6.21}$$

and we have set

$$\widehat{\mathcal{T}}^\rho(\alpha_i, \boldsymbol{\xi}_i, \boldsymbol{k}) = \int_0^1 d\xi_i^\rho(s) e^{-i\boldsymbol{k}(\lambda_i\boldsymbol{\xi}_i(s) - i\beta c s\hbar\hat{\boldsymbol{k}})}. \tag{6.22}$$

Note that the $\delta(\boldsymbol{k}+\boldsymbol{k}')$ appearing in (6.20) is a consequence of the translational invariance of the system. We may already take advantage of this Dirac-Delta and simplify the correlation (6.18)

$$< E^\mu(\boldsymbol{x})E^\nu(\boldsymbol{y}) >_T^{mat(1)} = \frac{1}{4}\int \frac{d\boldsymbol{k}}{(2\pi)^3} e^{i\boldsymbol{k}\cdot(\boldsymbol{x}-\boldsymbol{y})} \widehat{G}^{\mu\rho}(\boldsymbol{k})\widehat{G}^{\nu\sigma}(\boldsymbol{k})\mathcal{Q}^{\rho\sigma}(\boldsymbol{k}), \tag{6.23}$$

where we have introduced the tensor

$$\mathcal{Q}^{\rho\sigma}(\boldsymbol{k}) = \sum_{\alpha_1,\alpha_2} \int \mathcal{D}(\boldsymbol{\xi}_1) \int \mathcal{D}(\boldsymbol{\xi}_2) \widehat{\mathcal{T}}^\rho(\alpha_1, \boldsymbol{\xi}_1, \boldsymbol{k}) \left(\widehat{\mathcal{T}}^\sigma(\alpha_2, \boldsymbol{\xi}_2, \boldsymbol{k})\right)^* n(\alpha_1, \boldsymbol{\xi}_1, \alpha_2, \boldsymbol{\xi}_2, \boldsymbol{k}). \tag{6.24}$$

Since we are interested in the large distance asymptotics, we have to analyze this tensor for $\boldsymbol{k} \sim 0$. Because of isotropy, $\mathcal{Q}^{\rho\sigma}(\boldsymbol{k})$ transforms covariantly under rotations and using some further arguments found in [5], we can approximate it as

$$\mathcal{Q}^{\rho\sigma}(\boldsymbol{k}) = a^{(1)}k^2[1 + \mathcal{O}(k)]\delta^{\rho\sigma} \overset{k\to 0}{\sim} a^{(1)}k^2. \tag{6.25}$$

Before we can calculate the final Fourier transform in (6.23) we have to complete the small k expansion of the integrand, i.e. we have to incorporate the expansion

$$(n_{\boldsymbol{k}}+1)^2 = \left(\frac{1}{\beta\hbar\omega_{\boldsymbol{k}}}\right)^2 + \mathcal{O}(k^{-1}). \tag{6.26}$$

Eventually, equations (6.25) and (6.26) allow us to extract the most singular term in the integrand of (6.18)

$$\begin{aligned} < E^\mu(\boldsymbol{x})E^\nu(\boldsymbol{y}) >_T^{mat(1)} &= \frac{1}{4}\int \frac{d\boldsymbol{k}}{(2\pi)^3} e^{i\boldsymbol{k}\cdot(\boldsymbol{x}-\boldsymbol{y})}\delta_{tr}^{\mu\nu}(\boldsymbol{k})g^2(\boldsymbol{k})\left[(4\pi)^2 a^{(1)} + \mathcal{O}(k)\right] \\ &\overset{k\to 0}{\sim} 4\pi^2 a^{(1)} \int \frac{d\boldsymbol{k}}{(2\pi)^3} e^{i\boldsymbol{k}\cdot(\boldsymbol{x}-\boldsymbol{y})}\delta_{tr}^{\mu\nu}(\boldsymbol{k})g^2(\boldsymbol{k}) \\ &\sim \pi a^{(1)}\partial_\mu\partial_\nu\frac{1}{r}, \qquad r\to\infty \end{aligned} \tag{6.27}$$

In order to find the r-independent quantity $a^{(1)} = a^{(1)}(\hbar,\beta,\rho)$ we proceed as follows: first we take the integrand of (6.23) and use for $\mathcal{Q}^{\rho\sigma}(\boldsymbol{k})$ two equivalent expressions: once the leading order for the small k expansion of (6.24) and once the leading order term of approximation (6.25). The two resulting expressions are by construction equal and hence we obtain the following equation (using $\delta_{tr}^{\mu\sigma}(\boldsymbol{k})\delta_{tr}^{\nu\sigma}(\boldsymbol{k}) = \delta_{tr}^{\mu\nu}(\boldsymbol{k})$)

$$\begin{aligned} &(4\pi\beta\hbar c)^2 k^2 (n_{\boldsymbol{k}}+1)^2 a^{(1)}\delta_{tr}^{\mu\nu}(\boldsymbol{k}) \\ &= (4\pi\beta\hbar c)^2 k^2 (n_{\boldsymbol{k}}+1)^2 \sum_{\alpha_1,\alpha_2}\int \mathcal{D}(\boldsymbol{\xi}_1)\int \mathcal{D}(\boldsymbol{\xi}_2)\int_0^1 d\xi_1^\rho(s_1)\int_0^1 d\xi_2^\rho(s_2)\left[\hat{\boldsymbol{k}}\cdot(\lambda_1\boldsymbol{\xi}_1 - i\beta\hbar c s_1\hat{\boldsymbol{k}})\right]\times \\ &\quad\times\left[\hat{\boldsymbol{k}}\cdot(\lambda_2\boldsymbol{\xi}_2 + i\beta\hbar c s_2\hat{\boldsymbol{k}})\right] n(\alpha_1,\boldsymbol{\xi}_1,\alpha_2,\boldsymbol{\xi}_2,\boldsymbol{k}=0)\delta_{tr}^{\mu\rho}(\boldsymbol{k})\delta_{tr}^{\nu\sigma}(\boldsymbol{k}). \end{aligned} \tag{6.28}$$

Taking then the trace (multiplication by $\delta_{tr}^{\mu\nu}(\boldsymbol{k})$) on both sides of (6.28) we obtain finally

$$\begin{aligned} a^{(1)}(\rho,\beta,\hbar) = &\frac{1}{2}\sum_{\alpha,\alpha'}\frac{e_\alpha e_{\alpha'}\lambda_\alpha\lambda_{\alpha'}}{\beta\sqrt{m_\alpha m_{\alpha'}}c^2}\int D(\xi)\int D(\xi')\times \\ &\times\int_0^1 d\xi^\rho(s)\int_0^1 d\xi'^\sigma(s')\,\delta_{tr}^{\rho\sigma}(\hat{\boldsymbol{k}})\,n(\alpha,\boldsymbol{\xi},\alpha',\boldsymbol{\xi}',\boldsymbol{k}=0)\times \\ &\times\left(\hat{\boldsymbol{k}}\cdot\boldsymbol{\xi}(s) - \frac{i}{\lambda_\alpha}\beta\hbar c s\right)\left(\hat{\boldsymbol{k}}\cdot\boldsymbol{\xi}'(s') + \frac{i}{\lambda_{\alpha'}}\beta\hbar c s'\right), \end{aligned} \tag{6.29}$$

The remaining three terms in (6.15) are calculated in a similar manner. Before we can give the asymptotics of the total correlation function, we need to investigate the free-field contribution whose exact expression if found in (4.52). Taking the limit $\boldsymbol{k}\to 0$ we can expand the integrand and obtain the leading order correlation for

$r \to \infty$

$$\begin{aligned} < E_t^\mu(\boldsymbol{x}) E_t^\nu(\boldsymbol{y}) >_T^0 &= \frac{4\pi}{\beta} \int \frac{d\boldsymbol{k}}{(2\pi)^3} e^{i\boldsymbol{k}(\boldsymbol{x}-\boldsymbol{y})} \delta_{tr}^{\mu\nu}(\boldsymbol{k}) g(\boldsymbol{k})(1 + \mathcal{O}(k)) \\ &\overset{k\to 0}{\sim} \frac{4\pi}{\beta} \int \frac{d\boldsymbol{k}}{(2\pi)^3} e^{i\boldsymbol{k}(\boldsymbol{x}-\boldsymbol{y})} \delta_{tr}^{\mu\nu}(\boldsymbol{k}) \\ &\sim \frac{1}{\beta} \partial_\mu \partial_\nu \frac{1}{r}, \qquad r \to \infty \end{aligned} \tag{6.30}$$

Summing up the different contributions yields finally

$$\begin{aligned} < E_t^\mu(\boldsymbol{x}) E_t^\nu(\boldsymbol{y}) >_T &= \sum_{i=1}^{4} < E_t^\mu(\boldsymbol{x}) E_t^\nu(\boldsymbol{y}) >_T^{mat(i)} + < E_t^\mu(\boldsymbol{x}) E_t^\nu(\boldsymbol{y}) >_T^0 \\ &= \left[\Omega + \frac{1}{\beta}\right] \partial_\mu \partial_\nu \frac{1}{r} + \mathcal{O}(r^{-4}), \qquad r \to \infty \end{aligned} \tag{6.31}$$

where

$$\Omega \equiv \Omega(\rho_\alpha, \beta, \hbar) = \pi(a^{(1)} - a^{(2)} - a^{(3)} + a^{(4)}) \tag{6.32}$$

and

$$\begin{aligned} a^{(i)}(\rho_\alpha, \beta, \hbar) = \frac{1}{2} \sum_{\alpha,\alpha'} & \frac{e_\alpha e_{\alpha'} \lambda_\alpha \lambda_{\alpha'}}{\beta \sqrt{m_\alpha m_{\alpha'}} c^2} \int D(\xi) \int D(\xi') \times \\ & \times \int_0^1 d\xi^\mu(s) \int_0^1 d\xi'^\nu(s') \delta_{tr}^{\mu\nu}(\hat{\boldsymbol{k}})\, n(\alpha, \boldsymbol{\xi}, \alpha', \boldsymbol{\xi}', \boldsymbol{k} = 0) \times \\ & \times \left(\hat{\boldsymbol{k}} \cdot \boldsymbol{\xi}(s) - \sigma^{(i)} \frac{i}{\lambda_\alpha} \beta \hbar c s\right) \left(\hat{\boldsymbol{k}} \cdot \boldsymbol{\xi}'(s') + \eta^{(i)} \frac{i}{\lambda_{\alpha'}} \beta \hbar c s'\right), \end{aligned} \tag{6.33}$$

with $\sigma^{(1)} = \sigma^{(2)} = -\sigma^{(3)} = -\sigma^{(4)} = 1$ and $\eta^{(1)} = \eta^{(3)} = -\eta^{(2)} = -\eta^{(4)} = 1$. We see that summing up the four contributions (6.33) yields

$$\begin{aligned} \Omega = 2\pi \sum_{\alpha,\alpha'} & e_\alpha e_{\alpha'} \lambda_\alpha \lambda_{\alpha'} \int D(\xi) \int D(\xi') \times \\ & \times \int_0^1 d\xi^\mu(s) \int_0^1 d\xi'^\nu(s')\, (ss')\, \delta_{tr}^{\mu\nu}(\hat{\boldsymbol{k}})\, n(\alpha, \boldsymbol{\xi}, \alpha', \boldsymbol{\xi}', \boldsymbol{k} = 0) \\ = 2\pi \sum_{\alpha,\alpha'} & e_\alpha e_{\alpha'} \int D(\xi) \int D(\xi') \times \\ & \times \int_0^1 ds \int_0^1 ds'\, \lambda_\alpha \xi^\mu(s) \lambda_{\alpha'} \xi'^\nu(s')\, \delta_{tr}^{\mu\nu}(\hat{\boldsymbol{k}})\, n(\alpha, \boldsymbol{\xi}, \alpha', \boldsymbol{\xi}', \boldsymbol{k} = 0), \end{aligned} \tag{6.34}$$

where the second equality is obtained by an integration by parts and the de Broglie wavelengths λ_α and $\lambda_{\alpha'}$ are the result of a recombination of $\hbar^{rad}$ with the particles' masses. Hence in the limit $\hbar^{rad} \to 0$ these quantities vanish and we recover the semi-classical result of section (3)

$$< E_t^\mu(\boldsymbol{x}) E_t^\nu(\boldsymbol{y}) >_T = \frac{1}{\beta}\,\partial_\mu \partial_\nu \frac{1}{r} + \mathcal{O}(r^{-4}). \tag{6.35}$$

The matter-dependence of the transverse field correlations (6.31) is entirely contained in the coefficient $\Omega(\rho_\alpha, \hbar, \beta)$. According to (6.34) , Ω involves the loop density correlation function $n(\alpha, \boldsymbol{\xi}, \alpha', \boldsymbol{\xi}', \boldsymbol{k} = 0)$ and hence we can find a density expansion by means of Mayer diagrammatics (cf appendix D.3). To first order in the particle densities, the expansion reads (cf appendix D.2)

$$\Omega \sim \frac{\pi}{9} \sum_\alpha e_\alpha^2 \lambda_\alpha^2 \rho_\alpha. \tag{6.36}$$

Note that the asymptotics for the longitudinal electric field correlations are given by (3.54), where we have to use the quantum-mechanical charge-charge correlation function $S(\boldsymbol{r})$.

Chapter 7

Macroscopic Theory versus Microscopic Theory

7.1 The Macroscopic Approach by Landau and Lifshitz

In their famous Course of Theoretical Phyiscs [14], Landau and Lifshitz (LL) have developed a macroscopic theory that enables the calculation of the electromagnetic field correlations in presence of matter. Their approach is based on the macroscopic description of the medium by means of a complex dielectric function $\epsilon(\boldsymbol{x}, t)$ which determines the electric field response

$$\boldsymbol{D}(\boldsymbol{x}, t) = \int\int d\boldsymbol{x}' dt' \epsilon(\boldsymbol{x} - \boldsymbol{x}', t - t') \boldsymbol{E}(\boldsymbol{x}', t'). \tag{7.1}$$

LL choose to work in the framework of a local theory, i.e. they use a spatial $\delta(\boldsymbol{x} - \boldsymbol{x}')$ dependence of ϵ and consequently the double Fourier transform $\epsilon(\omega)$ is independent of $\boldsymbol{k}$. Another feature is the negligence of the magnetic response of the medium, i.e. they set the magnetic permeability μ equal to its vacuum value (in Gaussian units, this corresponds to $\mu \equiv 1$). This means that their theory incorporates merely the Coulomb interactions within the bulk whereas the magnetic interactions are completely disregarded. Using linear response theory and in particular the fluctuation-dissipation theorem, they determine the equal-time correlations of the electromagnetic field $< E^\mu(\boldsymbol{x}), E^\nu(\boldsymbol{y}) >$ and $< B^\mu(\boldsymbol{x}), B^\nu(\boldsymbol{y}) >$. We shall content ourselves with the announcement of the final results and refer to [14, 15] for details of the derivation. In the case of the electric field, LL obtain the following correlation

$$< E^{\mu}(\boldsymbol{x})E^{\nu}(\boldsymbol{y}) >= \int \frac{d\omega}{2\pi} \int \frac{d\boldsymbol{k}}{(2\pi)^3} e^{i\boldsymbol{k}\cdot\boldsymbol{r}} \mathcal{E}^{\mu\nu}(\boldsymbol{k},\omega), \tag{7.2}$$

where the double Fourier transform (calculated at $t = 0$ due to ubiquitousness) of the electric field correlation function $\mathcal{E}^{\mu\nu}(\boldsymbol{k},\omega)$ can be decomposed into its longitudinal and transverse parts [13]

$$\mathcal{E}^{\mu\nu}(\boldsymbol{k},\omega) = \mathcal{E}_l^{\mu\nu}(\boldsymbol{k},\omega) + \mathcal{E}_t^{\mu\nu}(\boldsymbol{k},\omega), \tag{7.3}$$

where the longitudinal part is

$$\mathcal{E}_l^{\mu\nu}(\boldsymbol{k},\omega) = -4\pi\hbar \coth\frac{\hbar\omega}{2T} \mathrm{Im}\frac{1}{\epsilon(\omega)}\frac{k^{\mu}k^{\nu}}{k^2}, \tag{7.4}$$

and the transverse part is

$$\mathcal{E}_t^{\mu\nu}(\boldsymbol{k},\omega) = -4\pi\hbar \coth\frac{\hbar\omega}{2T} \mathrm{Im}\frac{\omega^2/c^2}{(\omega^2/c^2)\epsilon(\omega) - k^2}\left(\delta^{\mu\nu} - \frac{k^{\mu}k^{\nu}}{k^2}\right). \tag{7.5}$$

Since we are in particular interested in the long distance asymptotics of the field correlations (7.2), we have to consider the singular terms of a small $\boldsymbol{k}$-expansion in (7.4) and (7.5). The Fourier transform of (7.4) yields immediately

$$< E_l^{\mu}(\boldsymbol{x}), E_l^{\nu}(\boldsymbol{y}) >\sim \left(\int \frac{d\omega}{2\pi}\hbar \coth\frac{\hbar\omega}{2T}\mathrm{Im}\frac{1}{\epsilon(\omega)}\right)\partial_{\mu}\partial_{\nu}\frac{1}{r}, \qquad r \to \infty. \tag{7.6}$$

In (7.5), we use the small $\boldsymbol{k}$-expansion

$$\frac{1}{b-k^2} = \frac{1}{b} + \mathcal{O}(k^2), \qquad b \neq 0, \tag{7.7}$$

to extract (for $\omega \neq 0$) the most singular term which is verified to be the opposite of (7.6):

$$< E_t^{\mu}(\boldsymbol{x}), E_t^{\nu}(\boldsymbol{y}) >\sim -\left(\int \frac{d\omega}{2\pi}\hbar \coth\frac{\hbar\omega}{2T}\mathrm{Im}\frac{1}{\epsilon(\omega)}\right)\partial_{\mu}\partial_{\nu}\frac{1}{r}, \qquad r \to \infty. \tag{7.8}$$

We note furthermore that the rest $\mathcal{O}(k^2)$ in (7.7) is analytic in $\boldsymbol{k}$ (due to the parity of the function) and hence the remaining terms of the expansion yield after Fourier transformation real space contributions that decay faster than any inverse power of r. However, since the expansion (7.7) is not applicable in $\omega = 0$, the nature of the field correlations depends on the behavior of the dielectric function for $\omega \to 0$. Although the frequency-dependence of $\epsilon(\omega)$ is only known for some specific models, we conjecture that the limit

$$\lim_{\omega\to 0}\frac{\epsilon(\omega)}{\omega^2} \tag{7.9}$$

is finite and that the previous arguments can be extended to the whole frequency spectrum. Since (7.6) and (7.8) cancel each other, the total correlation $< E^{\mu}(\boldsymbol{x})E^{\nu}(\boldsymbol{y}) >$ has no algebraic tail in the long distance asymptotics. If the matter could be described in terms of classical mechanics, this result would actually look reasonable in virtue of the Debye-Hückel theory which predicts screening in classical plasmas. However, taking into consideration the quantum nature of the matter as well as the fact that quantum fluctuations destroy Debye screening [17], we do have some doubts about the accuracy of such an exponential decay. All the more, it is a very interesting task to investigate the results of the LL theory from a microscopic point of view. Considering the results of chapters 3 and 4 we should in of fact be able to perform a direct comparison.

7.2 Comparability of the two Theories

Before we can perform a conclusive comparison, we have to analyze carefully the comparability of the macroscopic and the microscopic approaches. A first concern is the description by the medium in terms of the dielectric function $\epsilon(\omega)$. Indeed, we know that in a plasma of free charges we have $\epsilon(\omega \to 0) = \infty$, i.e. the static response of a conductor is diverging. This divergence renders the direct comparison subtle as can be seen in e.g. (7.6): the integral over ω can be divergent if ϵ is not sufficiently regular in the vicinity of 0. A second delicate point is apparently the $\boldsymbol{k}$-independence of the dielectric function $\epsilon(\omega)$, a physically not justified working hypothesis of LL. Following the arguments of [13] we can ensure that a $\boldsymbol{k}$-dependence of the dielectric function would not influence the leading order terms of (7.4) and (7.5) such that the dipolar terms (7.6) and (7.5) would persist to cancel each other. On the other hand, the introduction of a $\boldsymbol{k}$-dependent $\epsilon(\boldsymbol{k}, \omega)$ introduces higher order algebraic terms as can be seen in the particular case of the jellium [5]. Therefore, if we extend the LL theory to a non-local theory, its essence is actually not the prediction of a complete screening but merely the cancellation of the dipolar terms in $\sim r^{-3}$. This realization is crucial for a correct comparison with the microscopic model: indeed, the latter theory is highly non-local and thus we can at most expect the disappearance of the dipolar terms. Another concern about the LL theory is the negligence of the magnetic contributions $\mu \equiv 1$. According to (2.4), the potential vector $\boldsymbol{A}$ and hence the transverse electric field are not decoupled from the Coulomb potential. Hence, we risk to loose certain Coulomb contributions to the final field correlations if we try to eliminate the magnetic contributions by setting ab initio $\boldsymbol{A} \equiv 0$. In point of fact, considering the complete Hamiltonian (2.9), we will find both, Coulombic and magnetic contributions to the leading order correlation. The correct translation of $\mu \equiv 1$ into the framework of the microscopic model as well as the accurate extraction of the pure Coulomb effects from the final correlations will be a crucial point of the succeeding discussion.

Let us foremost consider the situation encountered in chapter 3, i.e. a quantum plasma immersed into a classical radiation field. According to (3.50) and (3.54) we have the total correlation (the cross-terms vanish)

$$< E^{\mu}(\boldsymbol{x})E^{\nu}(\boldsymbol{y}) >\sim \partial_\mu\partial_\nu \frac{1}{r}\left(\frac{2\pi}{3}\int d\boldsymbol{r}|r|^2 S(\boldsymbol{r}) + \frac{1}{\beta}\right), \qquad r\to\infty. \tag{7.10}$$

This term vanishes in the classical limit in virtue of the Stillinger-Lovett sum rule [16] but persists in the presence of quantum charges as can be seen by considering the one component plasma (OCP) [5]. Therefore, in the semi-classical case, we cannot confirm the dipolar decay predicted by LL. As suggested by [13] the classical nature of the radiation field may not be justified and a conclusive statement can only be given after having analyzed the results in the full quantum system.

In the full quantum system the electric field correlations are according to (3.54) and (6.31)

$$< E^{\mu}(\boldsymbol{x})E^{\nu}(\boldsymbol{y}) >\sim \partial_\mu\partial_\nu \frac{1}{r}\left(\Omega + \frac{1}{\beta} + \frac{2\pi}{3}\int d\boldsymbol{r}|r|^2 S(\boldsymbol{r})\right), \qquad r\to\infty, \tag{7.11}$$

where Ω is defined in (6.34). As already emphasized, a direct comparison with the LL theory urges us to extract the Coulombic contribution of (7.11). In order to get rid of all the relativistic effects (i.e. magnetic effects) we propose therefore to perform the limit $c\to\infty$ in the brackets of (7.11)

$$\lim_{c\to\infty}\left(\Omega + \frac{1}{\beta} + \frac{2\pi}{3}\int d\boldsymbol{r}|r|^2 S(\boldsymbol{r})\right), \tag{7.12}$$

such that only the instantaneous Coulomb interaction survives. If this limit is indeed zero, then we can confirm the prediction of LL, if it is not, we have to discuss the consequences. However, this limit is rather subtle and we rewrite (7.12) foremost in terms of the loop formalism

$$\begin{aligned}\lim\nolimits_{c\to\infty}&\left[\frac{4\pi}{3}\int d\boldsymbol{r}\int D(\xi)\int D(\xi')\sum_{\alpha,\alpha'} e_\alpha e_{\alpha'} n(\alpha,\boldsymbol{\xi},\alpha',\boldsymbol{\xi}',\boldsymbol{r})\int_0^1 ds\int_0^1 ds'\,\lambda_\alpha\lambda_{\alpha'}\boldsymbol{\xi}(s)\cdot\boldsymbol{\xi}'(s') \;+\right.\\ &\left.+\frac{1}{\beta} + \frac{2\pi}{3}\int d\boldsymbol{r}|r|^2 S(\boldsymbol{r})\right].\end{aligned} \tag{7.13}$$

We note that the c-dependences of Ω and the second moment of the structure function $S(\boldsymbol{r})$ are comprised in the loop density correlation function n which has been defined in (4.37). More precisely, the c-dependence is in the Gibbs factor of the magnetic potential $e^{-\beta W_m(i,j)}$. In appendix B we prove the non-commutativity of the following limits

$$\begin{aligned}&\lim_{r\to\infty}\lim_{c\to\infty} W_m(r) = 0\\ &\lim_{c\to\infty}\lim_{r\to\infty} W_m(r) \neq 0.\end{aligned} \tag{7.14}$$

Consequently, it is not obvious that we can interchange the limit $c \to \infty$ with the integral $\int d\boldsymbol{r}$ in (7.13). Although a rigorous proof is missing, we make at this point the conjecture that the interchange is mathematically correct. Like this, the verification of LL's prediction is reduced to the examination of the following equality

$$-\frac{2\pi}{3}\int d\boldsymbol{r}|r|^2 S(\boldsymbol{r}) =$$
$$\frac{1}{\beta}+\frac{4\pi}{3}\int d\boldsymbol{r}\int D(\xi)\int D(\xi')\sum_{\alpha,\alpha'} e_\alpha e_{\alpha'} n(\alpha,\boldsymbol{\xi},\alpha',\boldsymbol{\xi}',\boldsymbol{r})\int_0^1 ds\int_0^1 ds'\,\lambda_\alpha\lambda_{\alpha'}\boldsymbol{\xi}(s)\cdot\boldsymbol{\xi}'(s'), \tag{7.15}$$

where $n(\alpha,\boldsymbol{\xi},\alpha',\boldsymbol{\xi}'\boldsymbol{k})$ is now, due to the limit $c \to \infty$, the purely Coulombic correlation function, i.e. the radiation field is neglected. Indeed, if this equation is satisfied, we can confirm that the dipolar contribution to the electric field correlations vanishes; otherwise the microscopic theory falsifies the macroscopic approach. First of all, we note that in the classical limit $\hbar \to 0$ (7.15) corresponds to the Stillinger-Lovett sum rule and hence we are very tempted to prove it in the quantum case by means of an appropriate sum rule, too. So far, we haven't yet been able to prove the conjecture in question, but in order to justify our anticipation of its validity, we we would like to check it up to $\mathcal{O}(\hbar^2)$ in the special cases of the OCP and the MCP (multi-component plasma).

7.3 Special Cases OCP and MCP

The second moment of the structure function is well-known in the case of the quantum OCP and reads [5]

$$-\frac{2\pi}{3}\int d\boldsymbol{r}|r|^2 S(\boldsymbol{r}) = \frac{\hbar\omega_p}{2}\coth\left(\frac{\hbar\omega_p\beta}{2}\right) = \frac{1}{\beta}+\frac{\beta}{3}\left(\frac{\hbar\omega_p}{2}\right)^2+\mathcal{O}(\hbar^4), \tag{7.16}$$

where ω_p is the plasmon frequency

$$\omega_p^2 = \frac{4\pi\rho e^2}{m}. \tag{7.17}$$

Since the second term on the rhs of (7.15) is proportional to λ^2 we need merely the zero-order term of the $\hbar$-expansion of $n(\alpha,\boldsymbol{\xi},\alpha',\boldsymbol{\xi}'\boldsymbol{r})$ which is known to be the classical particle density correlation function

$$n(\boldsymbol{\xi},\boldsymbol{\xi}',\boldsymbol{r}) = n_{cl}(\boldsymbol{r})+\mathcal{O}(\hbar^2) = \rho_{cl}^{(2)}(\boldsymbol{r})+\delta(\boldsymbol{\xi},\boldsymbol{\xi}')\delta(\boldsymbol{r})\rho+\mathcal{O}(\hbar^2). \tag{7.18}$$

The $\boldsymbol{\xi}$- and $\boldsymbol{\xi}'$-independent term $\rho_{cl}^{(2)}$ will not contribute to the rhs of (7.15) as can be seen by means of simple symmetry arguments: $\int \mathcal{D}(\boldsymbol{\xi})\boldsymbol{\xi}(s) = -\int \mathcal{D}(-\boldsymbol{\xi})\boldsymbol{\xi}(s) = -\int \mathcal{D}(\boldsymbol{\xi})\boldsymbol{\xi}(s) = 0$. Hence we are left with the coincident points contribution of (7.18) which yields the following $\hbar$-expansion of the rhs in (7.15)

$$\frac{1}{\beta} + \frac{4\pi}{3}\rho e^2\lambda^2 \int \mathcal{D}(\boldsymbol{\xi}) \int \mathcal{D}(\boldsymbol{\xi}') \int_0^1 ds \int_0^1 ds' \boldsymbol{\xi}(s)\cdot\boldsymbol{\xi}(s') + \mathcal{O}(\hbar^2) \tag{7.19}$$
$$= \frac{1}{\beta} + \frac{4\pi}{3}\rho e^2\lambda^2 \int_0^1 ds \int_0^1 ds' \left(min(s,s') - ss'\right) + \mathcal{O}(\hbar^2) = \frac{1}{\beta} + \frac{\beta}{3}\left(\frac{\pi\rho\hbar^2 e^2}{m}\right) + \mathcal{O}(\hbar^2).$$

Consequently, the equality (7.15) is verified for the quantum OCP up to $\hbar^2$. The simple expression for the second moment (7.16) is due to a collective oscillation of the particles with frequency ω_p. This so-called plasmon oscillations are due to the fact that all particles carry the same mass. However, as soon as the particles have different inertial properties the plasmon oscillations disappear and the relation (7.16) does not hold any more. An analysis of the MCP urges us therefore to seek an $\hbar$-expansion of the lhs in (7.15). In this aim, we introduce at first the Wigner-Kirkwood expansion in $\hbar$ for the two-particle density [17]

$$\rho^{(2)}(\alpha,\alpha',\boldsymbol{r}) = \rho_{cl}^{(2)}(\alpha,\alpha',\boldsymbol{r}) + \hbar^2\rho_2^{(2)}(\alpha,\alpha',\boldsymbol{r}) + \mathcal{O}(\hbar^4), \tag{7.20}$$

where the first correction is given by

$$\rho_2^{(2)}(\alpha,\alpha',\boldsymbol{r}) = \frac{\beta}{24}\left(\frac{1}{m_\alpha} + \frac{1}{m_{\alpha'}}\right)\nabla^2\rho_{cl}^{(2)}(\alpha,\alpha',\boldsymbol{r}). \tag{7.21}$$

Whenever a quantity is classical, we indicate it explicitely (e.g. $\rho_{cl}^{(2)}$), otherwise the object is quantum-mechanical. Using (7.21) we can now expand the lhs of (7.15) as follows

$$-\frac{2\pi}{3}\int d\boldsymbol{r}|r|^2 S(\boldsymbol{r}) =$$
$$= -\frac{2\pi}{3}\int d\boldsymbol{r}|r|^2 \sum_{\alpha\alpha'} e_\alpha e_{\alpha'} n_T(\alpha,\alpha',\boldsymbol{r})$$
$$= -\frac{2\pi}{3}\int d\boldsymbol{r}|r|^2 \sum_{\alpha\alpha'} e_\alpha e_{\alpha'} \left[\rho^{(2)}(\alpha,\alpha',\boldsymbol{r}) - \rho_\alpha\rho_{\alpha'} + \delta(\boldsymbol{r})\delta_{\alpha\alpha'}\rho_\alpha\right]$$
$$= -\frac{2\pi}{3}\int d\boldsymbol{r}|r|^2 \sum_{\alpha\alpha'} e_\alpha e_{\alpha'} \left[\rho_{cl}^{(2)}(\alpha,\alpha',\boldsymbol{r}) + \hbar^2\rho_2^{(2)}(\alpha,\alpha',\boldsymbol{r}) - \rho_\alpha\rho_{\alpha'} + \delta(\boldsymbol{r})\delta_{\alpha\alpha'}\rho_\alpha\right] + \mathcal{O}(\hbar^4)$$
$$= -\frac{2\pi}{3}\int d\boldsymbol{r}|r|^2 \sum_{\alpha\alpha'} e_\alpha e_{\alpha'}\, n_T^{cl}(\alpha,\alpha',\boldsymbol{r}) -$$
$$-\frac{2\pi}{3}\hbar^2\int d\boldsymbol{r}|r|^2 \sum_{\alpha\alpha'} e_\alpha e_{\alpha'} \left(\frac{\beta}{24}\left(\frac{1}{m_\alpha} + \frac{1}{m_{\alpha'}}\right)\nabla^2\rho_{cl}^{(2)}(\alpha,\alpha',\boldsymbol{r})\right) + \mathcal{O}(\hbar^4). \tag{7.22}$$

The first term in (7.22) is simply the second moment of the classical structure function and according to the Stillinger Lovett sum rule it equals $\frac{1}{\beta}$. We focus now on the second term that demands a more detailed analysis. Noticing that in a homogeneous plasma the particle density ρ_α is $\boldsymbol{r}$-independent, we can rewrite

$$-\frac{\pi\beta}{36}\hbar^2 \int d\boldsymbol{r}|r|^2 \sum_{\alpha\alpha'} e_\alpha e_{\alpha'} \left(\frac{1}{m_\alpha} + \frac{1}{m_{\alpha'}}\right) \left[\nabla^2 n_T^{cl}(\alpha, \alpha', \boldsymbol{r}) - \nabla^2 \delta(\boldsymbol{r})\delta_{\alpha\alpha'}\rho_\alpha\right]. \tag{7.23}$$

The first term in the previous expression involves the classical particle density correlation function which is known to decay exponentially. Hence, applying twice an integration by parts we find

$$-\frac{\pi\beta}{6}\hbar^2 \int d\boldsymbol{r} \sum_{\alpha\alpha'} e_\alpha e_{\alpha'} \left(\frac{1}{m_\alpha} + \frac{1}{m_{\alpha'}}\right) n_T^{cl}(\alpha, \alpha', \boldsymbol{r}) \tag{7.24}$$

which is zero according to [16]. After a double integration by parts of the $\delta(\boldsymbol{r})$-term in (7.23) we can summarize the expansion (7.22)

$$-\frac{2\pi}{3} \int d\boldsymbol{r}|r|^2 S(\boldsymbol{r}) = \frac{1}{\beta} + \frac{\pi\beta\hbar^2}{3} \sum_{\alpha} \frac{e_\alpha^2 \rho_\alpha}{m_\alpha} + \mathcal{O}(\hbar^4). \tag{7.25}$$

A calculation similar to the one performed in (7.19) yields finally the same expansion for the rhs of (7.15). This allows us to conclude that the MCP satisfies the LL conjecture (7.15) up to $\hbar^2$, too.

Currently, we cannot say more about (7.15) and it is an open question whether this assertion is true or false. Using linear response theory and screening arguments applied to a gas of Coulomb loops, we have managed to derive different but very similar relations. Encouraged by these observations, we continue for the time being our quest for a general proof of (7.15) instead of envisaging its falsification.

Chapter 8

Conclusion

Within the scope of this chapter, we would like to review the main objectives outlined in the introduction. In the first part, we resume the impact of the quantization of the radiation field. In doing so, we are mainly interested in the novelties implicated by the consideration of the full quantum formalism in comparison to the semi-classical framework. In a second part, we revisit the direct comparison between our exact microscopic developments and the macroscopic theory of Landau and Lifshitz.

8.1 Impact of the Field Quantization

The main goal of this thesis was to review and explore some exact results in the framework of thermal QED. Our microscopic model of a quantum plasma in thermal equilibrium with the photon gas is a rather novel approach to a better comprehension of the field-matter interactions. Since we abstain from a priori approximations, our approach is mathematically rather involved and it is legitimate to discuss the necessity of such a detailed description. Thereby, we have to distinguish two situations. In point of fact, if we are solely interested in the particle correlations, we may ask whether the consideration of the field is necessary at all. Indeed, within the semi-classical framework [5], it is found that in a non-relativistic quantum plasma at ambient temperature, the magnetic contributions to the particle correlations are largely negligible. This can be shown on the basis of simple physical arguments, cf [6]. However, the unimportance of the classical field is not necessarily true for the quantum field and as we will explain below, the magnetic contributions become indeed important. On the other hand, if we are interested in field-related quantities, e.g. the field fluctuations, the crucial question is whether the classical field is an appropriate approximation or not. In [5], it is argued that the use of a classical radiation field is acceptable if we are interested in long distance asymptotics. Below, we will judge this hypotheses in the light of our quantum results.

In point of fact, the total Hamiltonian given in (2.9) is supposed to provide an accurate microscopic description of non-relativistic matter in both the solid and fluid phase. However, the techniques and reasonings we use in our calculations implicate a certain restriction of our model's domain of validity. Let us briefly discuss them. A first restriction is due to the repeated exploitation of the system's translational invariance. In other words, we exclude the existence of discrete translational symmetries and hence the solid phase. Another subtlety is due to the choice of our mathematical tools, namely the joint path integral representation of the thermal Gibbs weight. Although these techniques enable us to perform exact calculations in an elegant manner, they implicate a major restriction, namely the demand for a low density regime. Let us see why. First, the bosonic path integral allows us to perform the partial trace over the field's degrees of freedom and to reduce the field-matter coupling to an effective magnetic particle interaction W_m. The subsequent application of the Feynman-Kac-Itô formula transforms the quantum particles into a collection of charged random loops. These loops are purely classical objects and we have therefore transformed the original quantum phase space into a classical phase space of loops. This very fact is a main advantage if we proceed to explicit calculations. We can treat the arising classical statistical integrals by the dint of Mayer diagrammatics. In order to ensure the integrability of each single diagram, it is necessary the the loop extensions λ are much smaller than the distance a between the particles: $\lambda << a$. We conclude that our approach is only useful in a low density regime.

Let us now commence the discussion of the effective magnetic interaction potential W_m. As can be seen in the explicit expression (4.32), this potential resembles structurally its semi-classical counterpart W_m^{cl} found in (3.31), the main difference being the imaginary-time Green's function $S_{\boldsymbol{k}}^{-1}$. Regarding the particle correlations, the presence of the magnetic potential does not alter the $\sim r^{-6}$ decay in the long distance asymptotics. This can be understood by the following argument: as shown in [17], the breakdown of the Debye screening in a quantum Coulomb gas in absence of the radiation field is caused be the asymptotic dipolar decay of the quantum contribution to the Coulomb interaction, W_C. Since W_m reveals an asymptotic dipolar decay, too, it will merely aid the Coulombic part W_C to destroy the exponential clustering, however without changing the inverse power law. Concerning this matter, we resigned to perform explicit calculations, but noticing that the semi-classical counterpart W_m^{cl} has also a dipolar decay, we can follow word by word the arguments of section 5 in [5] to show that the two-particle correlations decay indeed as $\sim r^{-6}$. On the other hand, a closer look at the leading order term of W_m in (6.13) reveals a real novelty in comparison to the classical radiation field. Whereas the first term $W_{m,1}^{even}$ is simply the effective potential encountered in the semi-classical framework, the second term $W_{m,2}^{even}$ has the exactly opposite asymptotic behavior of the quantum Coulomb contribution W_C. The resulting screening of the leading order term of W_C has indeed a drastic impact on the particle correlations: the leading order term in $\sim r^{-6}$ is now purely magnetic. Thus, in contrast to the semi-classical case, the magnetic contributions start to compete the Coulombic contributions and the quantized radiation field cannot be neglected any more.

Let us now focus on the electric field correlations which constitute to some extent the objects of main interest. A first conclusion concerns the correlations in the transverse electric field. In the semi-classical framework, we have obtained a rather surprising result (3.47): although the charged particles certainly interact with the electric field, the transverse correlations decouple from the matter and coincide with the free field correlations. Considering then the full quantum model, we have shown that the corresponding correlations do now comprise a non-vanishing matter-depending term. Hence we can conclude that the cited decoupling must be a mere curiosity of the field's classical nature. Furthermore, we have shown that the electric field correlations reveal a dipolar asymptotic decay. The explicit expression of this dipolar term plays a significant role in the comparison to the macroscopic theory (cf next section).

Finally, we would like to get back to the initial question about the necessity of a full quantum model. In the light of the above-mentioned novelties due to the field's quantum nature, we conclude that the full quantum description is indeed essential for a correct description of both the field and particle correlations. In addition, we hope that these results allow us to perform a conclusive assessment of the LL theory.

8.2 Open Questions regarding the LL Theory

As outlined in the introduction, it is a big challenge for theoretical physicists to either corroborate or falsify existing macroscopic theories and the inherent hypotheses by the dint of microscopic models. Thanks to the results obtained in chapter 4, we should technically be able to perform such an examination in the case of the macroscopic theory on field fluctuations developed by Landau and Lifshitz. Let us briefly review the key features of their approach. Characterizing the medium by means of a complex, frequency-depending dielectric function $\epsilon(\omega)$, they use linear response theory and the dissipation-fluctuation theorem to show that both the transverse and longitudinal electric field correlations have a $\sim r^{-3}$ decay in the long distance asymptotics. Surprisingly, these two terms cancel each other such that the total electric field correlations decay exponentially. We have consequently tried, on the basis of our microscopic calculations, to recover this annihilation of the dipolar terms, i.e. to prove the conjecture (7.15). Unfortunately, we are presently not able to definitely corroborate or falsify the LL theory. However, encouraged by $\mathcal{O}(\hbar^2)$-checks in conjunction with the astonishing similarity of (7.15) with usual sum rules in charged fluids, we believe that this conjecture may indeed be true.

In any case, we may make some comments regarding the validity of LL's working hypotheses outlined in section 7.2. The first major hypothesis by LL is the $\boldsymbol{k}$-independence of the dielectric function $\epsilon(\omega)$ which signifies that their theory is actually a local theory. Based on this assertion, LL predict an exponential decay of the total electric field correlations $< E^{\mu}(\boldsymbol{x})E^{\nu}(\boldsymbol{y}) >$. On the other hand, our microscopic investigation anticipates an asymptotic algebraic decay in $\sim r^{-3}$. Even if we discard the magnetic contributions by taking the limit $c \to \infty$ (rendering our theory consistent with LL, cf second hypothesis), we are left with an algebraic tail. From the point of view of our microscopic model, we must therefore conclude that LL's locality assumption (causing the exponential decay) is an unacceptable simplification of the underlying physical reality. However, this is not a generally conclusive assessment: since our plasma of freely moving charges is a macroscopic conductor, a description in terms of the response function ϵ is only meaningful for high frequencies and in the static limit $\omega \to 0$, ϵ diverges. This fact renders a conclusive comparison rather difficult. The second major hypothesis made by LL is the negligence of the magnetic contributions, an assertion achieved by setting $\mu \equiv 1$. From our model's point of view, it is intuitively clear that the freely moving Coulomb charges create currents and are therefore subjected to mutual magnetic interactions. Moreover, considering the results of chapter 4, we see that there are magnetic contributions to the electric field correlations even in the leading order term (decaying as $\sim r^{-3}$). Hence, it is impossible to justify this choice of LL by means of physical arguments. But again, this is merely the conductor's viewpoint and we can imagine the existence of insulators which satisfy $\mu \sim 1$. We have to accredit that the inclusion of the magnetic effects into the LL theory seems to be a truly non-trivial problem and hence we can comprehend their simplification in the first instance.

Finally, we would like to make some suggestions for future efforts. As mentioned above, the chosen mathematical tools restrict the validity of our calculations to the low density regime. It would certainly be interesting to develop techniques valid for higher densities, allowing therefore the description of possible phase transitions and the emergence of the solid phase. In order to complete the comparison with the macroscopic LL theory, it would furthermore be instructive to calculate the asymptotic correlations of the magnetic field, too. Regarding the conjecture (7.15), we keep on seeking a physical argument which would allow us to prove its validity.

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Appendix A

The Green's Function $S_{\boldsymbol{k}}^{-1}$

In sections (6.2) and (4.4) we have encountered two different types of stochastic line integrals containing the Green's functions $S_{\boldsymbol{k}}^{-1}(1,s)$ and $S_{\boldsymbol{k}}^{-1}(s,1)$, explicitly

$$\begin{aligned} &\int_0^1 ds S_{\boldsymbol{k}}^{-1}(1,s) J_{\boldsymbol{k}\lambda}(s) \\ &= \underbrace{\int_0^1 dse^{\beta\hbar\omega_{\boldsymbol{k}}(1-s)}\theta(1-s-\eta)(1+n_{\boldsymbol{k}})J_{\boldsymbol{k}\lambda}(s)}_{:=I_1(\eta)} + \underbrace{\int_0^1 dse^{-\beta\hbar\omega_{\boldsymbol{k}}(1-s)}\theta(s-1+\eta)n_{\boldsymbol{k}}J_{\boldsymbol{k}\lambda}(s)}_{:=I_2(\eta)} \end{aligned} \tag{A.1}$$

and

$$\begin{aligned} &\int_0^1 ds S_{\boldsymbol{k}}^{-1}(s,1) J_{\boldsymbol{k}\lambda}(s) \\ &= \underbrace{\int_0^1 dse^{-\beta\hbar\omega_{\boldsymbol{k}}(s-1)}\theta(s-1-\eta)(1+n_{\boldsymbol{k}})J_{\boldsymbol{k}\lambda}(s)}_{:=I_3(\eta)} + \underbrace{\int_0^1 dse^{-\beta\hbar\omega_{\boldsymbol{k}}(s-1)}\theta(1-s+\eta)n_{\boldsymbol{k}}J_{\boldsymbol{k}\lambda}(s)}_{:=I_4(\eta)}. \end{aligned} \tag{A.2}$$

Since $s \in [0,1]$ and $\eta > 0$ we can immediately see that

$$\begin{aligned} \lim_{\eta\to 0^+} I_3 &\equiv 0 \\ \lim_{\eta\to 0^+} I_4 &= \int_0^1 dse^{-\beta\hbar\omega_{\boldsymbol{k}}(s-1)}n_{\boldsymbol{k}}J_{\boldsymbol{k}\lambda}(s). \end{aligned} \tag{A.3}$$

However, the case of (A.2) is not that easy to handle. In point of fact, if $J_{\boldsymbol{k}\lambda}(s)$ were a regular function, we could easily prove that

$$\begin{aligned} \lim_{\eta\to 0^+} I_2 &\equiv 0 \\ \lim_{\eta\to 0^+} I_1 &= \int_0^1 dse^{-\beta\hbar\omega_{\boldsymbol{k}}(1-s)}(1+n_{\boldsymbol{k}})J_{\boldsymbol{k}\lambda}(s). \end{aligned} \tag{A.4}$$

But $J_{\boldsymbol{k}}(s)$ is far from being a regular function and the validity of (A.4) is foremost an open question. Therefore, for the sake of a rigorous calculation, we should actually keep the η-depending integrands of I_1 and I_2 as they are given in (A.1) and only take the limit $\eta \to 0$ at the very end, i.e. after having integrated over the Brownian bridges.

Appendix B

The Magnetic potential W_m in the $c \to \infty$ limit

In both the semi-classical and the quantum pictures the effective magnetic potentials W_m^{cl} and W_m^{qm} are relativistic objects due to their dependence upon the speed of light c. The semi-classical formula (3.31) has a simple $\sim 1/c^2$ proportionality such that

$$\lim_{c\to\infty} W_m^{cl}(\boldsymbol{r}) = 0. \tag{B.1}$$

On the other hand, its counterpart arising in the full quantum system has a less obvious c-dependence as can be seen in (4.32). Let us therefore examine more closely the limit $c \to \infty$ of $W_m^{qm}(\boldsymbol{r})$. In this aim, we introduce the scaling $\boldsymbol{q} \equiv c\boldsymbol{k}$ and using the definition (3.30) we can rewrite

$$\begin{aligned} W_m^{qm}(\boldsymbol{r}) = & \frac{1}{c^3}\frac{4\pi\hbar}{\sqrt{m_\alpha m_{\alpha'}}}\int\frac{d\boldsymbol{q}}{(2\pi)^3}e^{i\frac{\boldsymbol{q}}{c}\cdot\boldsymbol{r}}\int_0^1 d\xi^\mu(s)e^{i\lambda_\alpha\frac{\boldsymbol{q}}{c}\cdot\boldsymbol{\xi}(s)}\int_0^1 d\xi'^\nu e^{-i\lambda_{\alpha'}\frac{\boldsymbol{q}}{c}\cdot\boldsymbol{\xi}'(s')}\times \\ & \times\frac{S_{\boldsymbol{q}}^{-1}(s,s')}{q}\delta_{tr}^{\mu\nu}(\boldsymbol{q})g^2\left(\frac{q}{c}\right). \end{aligned} \tag{B.2}$$

Then, thanks to the inequality

$$|W_m^{qm}(\boldsymbol{r})| \le \frac{1}{c^3}\frac{4\pi\hbar}{\sqrt{m_\alpha m_{\alpha'}}}\int\frac{d\boldsymbol{q}}{(2\pi)^3}\int_0^1 d\xi^\mu(s)\int_0^1 d\xi'^\nu(s')\left|\frac{S_{\boldsymbol{q}}^{-1}(s,s')}{q}\delta_{tr}^{\mu\nu}(\boldsymbol{q})\right|g^2\left(\frac{q}{c}\right) \tag{B.3}$$

we can deduce (using $g(q/c) \le 1$) that

$$\lim_{c\to\infty} W_m^{qm}(\boldsymbol{r}) = 0. \tag{B.4}$$

However, we note that (B.4) is only verified for a fixed and finite $\boldsymbol{r}$. In point of fact, we derive from the long distance asymptotics (6.13) that

$$\lim_{c\to\infty}\lim_{r\to\infty} W_m^{qm}(\boldsymbol{r}) \neq 0 \tag{B.5}$$

and hence we have to be careful when taking the $c \to \infty$ limit of an integral over the effective potential $W_m^{qm}(\boldsymbol{r})$, e.g. (7.13).

Appendix C

Symmetry Considerations

C.1 Symmetry Transformations of the pure Quantum System

Many properties of a physical system can be deduced by means of symmetry considerations, avoiding thereby explicit calculations. In this section, we focus on three different systems: the free quantum radiation field, an N-particle plasma coupled to one field mode and finally the full system characterized by $H^N_{L,R}$ given in (2.9). In doing so, we look for appropriate symmetry transformations and investigate the consequences for the thermal average of observables, in particular the second order field moments (4.35).

Let us briefly review the concept of a gauge transformation of the first kind [12] in the case of the free radiation field. It is a global transformation defined as the multiplication of a one-particle state by a constant phase factor

$$|\phi> \mapsto e^{-i\alpha}|\phi>, \qquad \alpha \in \mathbb{R}. \tag{C.1}$$

The corresponding unitary operator on the Fock space $\mathcal{F}_{photon}$ is given by

$$\mathrm{U}_\alpha = e^{-i\alpha \mathrm{N}} \tag{C.2}$$

where $\mathrm{N} = \sum_{\boldsymbol{k}\lambda} a^\dagger_{\boldsymbol{k}\lambda} a_{\boldsymbol{k}\lambda}$ is the number operator. It is easy to show that

$$\begin{aligned} \mathrm{U}^\dagger_\alpha a_{\boldsymbol{k}\lambda} \mathrm{U}_\alpha &= e^{-i\alpha} a_{\boldsymbol{k}\lambda} \\ \mathrm{U}^\dagger_\alpha a^\dagger_{\boldsymbol{k}\lambda} \mathrm{U}_\alpha &= e^{i\alpha} a^\dagger_{\boldsymbol{k}\lambda} \end{aligned} \tag{C.3}$$

and consequently U_α is a symmetry transformation of the free radiation field, i.e.

$$\mathrm{U}^\dagger_\alpha H^{rad}_0 \mathrm{U}_\alpha = H^{rad}_0. \tag{C.4}$$

In addition, this relation is equivalent to $[H_0^{rad}, \mathrm{N}] = 0$ and hence the particle number in the free field is conserved. Let us now examine the consequence of this symmetry for the second order field moments. In point of fact, using the thermal Gibbs weight ρ we find by means of (C.3) and the unitary property $\mathrm{U}_\alpha^\dagger \mathrm{U}_\alpha = \mathbf{1}$

$$\begin{aligned} < a_{\boldsymbol{k}\lambda} a_{\boldsymbol{k}'\lambda'} > &= Tr(\rho a_{\boldsymbol{k}\lambda} a_{\boldsymbol{k}'\lambda'}) = Tr(\mathrm{U}_\alpha^\dagger \mathrm{U}_\alpha \rho a_{\boldsymbol{k}\lambda} a_{\boldsymbol{k}'\lambda'}) \\ &= Tr(\mathrm{U}_\alpha^\dagger \rho \mathrm{U}_\alpha \mathrm{U}_\alpha^\dagger a_{\boldsymbol{k}\lambda} a_{\boldsymbol{k}'\lambda'} \mathrm{U}_\alpha) = e^{-2i\alpha} < a_{\boldsymbol{k}\lambda} a_{\boldsymbol{k}'\lambda'} > . \end{aligned} \tag{C.5}$$

But α is arbitrary and hence we conclude that

$$< a_{\boldsymbol{k}\lambda} a_{\boldsymbol{k}'\lambda'} >= 0 \tag{C.6}$$

and idem for the creation operators

$$< a_{\boldsymbol{k}\lambda}^\dagger a_{\boldsymbol{k}'\lambda'}^\dagger >= 0. \tag{C.7}$$

On the other hand, this argument does not work for $< a_{\boldsymbol{k}\lambda}^\dagger a_{\boldsymbol{k}'\lambda'} >$ and $< a_{\boldsymbol{k}\lambda} a_{\boldsymbol{k}'\lambda'}^\dagger >$ since the manipulations of (C.5) would merely lead to a trivial equality. We can easily generalize these results to the thermal average of an arbitrary monomial in $a_{\boldsymbol{k}\lambda}$ and $a_{\boldsymbol{k}\lambda}^\dagger$: if the number of creators does not equal the number of annihilators (in particular, if the total number of operators is odd), the average is zero.

We may now ask whether (C.2) is still a symmetry transformation if we couple the radiation field to an ensemble of quantum charges. In order to find the answer, we introduce foremost the Hamiltonian for a charged quantum plasma coupled to one field mode $\omega \neq 0$

$$H_{red}^N = \sum_{i=1}^N \frac{1}{2m_{\alpha_i}} \left(\boldsymbol{p}_i - \frac{e_{\alpha_i}}{c} \left(a e^{\boldsymbol{k}\cdot\boldsymbol{r}_i} + a^\dagger e^{-i\boldsymbol{k}\cdot\boldsymbol{r}_i} \right) \right)^2 + \hbar\omega a^\dagger a + H_C^N, \tag{C.8}$$

where H_C^N is the Coulomb interaction defined at the beginning of section 2.2. We see immediately that U_α is not a symmetry transformation of H_{red}^N because the kinetic term involves monomials comprising an odd number of a and $a^\dagger$ operators. Let us therefore introduce a new transformation $\mathcal{R}_\alpha$ defined on the complete Hilbert space $\mathcal{H}_{tot} = \mathcal{H}^{\otimes N} \otimes \mathcal{F}_{photons}$,

$$\mathcal{R}_\alpha = \mathcal{T}_{\alpha \frac{\hat{\boldsymbol{k}}}{|k|}} \otimes \mathrm{U}_\alpha, \tag{C.9}$$

where $\mathrm{T}_{\boldsymbol{a}}$ is the unitary translation operator in the N-particle Hilbert space

$$\mathcal{T}_{\boldsymbol{a}} = e^{-\frac{i}{\hbar}\mathbf{P}\cdot\boldsymbol{a}}. \tag{C.10}$$

In (C.10), $\mathbf{P} = \sum_i^N \boldsymbol{p}_i$ is the total momentum of the plasma and the action of $\mathrm{T}_{\boldsymbol{a}}$ on the position operators is defined via the mapping

$$\boldsymbol{r}_i \mapsto \mathrm{T}_{\boldsymbol{a}}^\dagger \boldsymbol{r}_i \mathrm{T}_{\boldsymbol{a}} = \boldsymbol{r}_i + \boldsymbol{a}. \tag{C.11}$$

It is then easy to verify that the Hamiltonian (C.8) is invariant under the transformation (C.9)

$$\mathrm{R}_\alpha^\dagger H_{red}^N \mathrm{R}_\alpha = H_{red}^N \tag{C.12}$$

and consequently we find that $< a_{\boldsymbol{k}\lambda} a_{\boldsymbol{k}'\lambda'} >=< a_{\boldsymbol{k}\lambda}^\dagger a_{\boldsymbol{k}'\lambda'}^\dagger >= 0$. Let us remark that, strictly speaking, H_{pot}^N is not invariant under the action of R_α. However, this term is only a representation of the Dirichlet boundary conditions (cf chapter 2) and since in the thermodynamic limit ($L \to \infty$) the confining walls are sent to infinity, the presence of H_{pot}^N does not break the symmetry.

If we add now the full quantum field, i.e. if we consider the full Hamiltonian (2.9), we see that the appearance of different field modes $\boldsymbol{k}\lambda$ breaks the R_α symmetry. Hence it is this very interplay of the different field modes that makes the conclusions (C.6) and (C.7) impossible. Finally, we note that this is consistent with the results of chapter 4 which give non-zero averages for all four second order field moments.

C.2 Equilibrium Values of $\boldsymbol{E}(\boldsymbol{r})$, $\boldsymbol{A}(\boldsymbol{r})$ and $\boldsymbol{B}(\boldsymbol{x})$

Since we are throughout the whole thesis concerned with the truncated field correlations

$$< E^\mu(\boldsymbol{x}) E^\nu(\boldsymbol{y}) >_T = < E^\mu(\boldsymbol{x}) E^\nu(\boldsymbol{y}) > - < E^\mu(\boldsymbol{x}) >< E^\nu(\boldsymbol{y}) > \tag{C.13}$$

and idem for the fields $\boldsymbol{A}$ and $\boldsymbol{B}$, it is interesting to note that the averages $< \boldsymbol{E}(\boldsymbol{x}) >$, $< \boldsymbol{A}(\boldsymbol{x}) >$ and $< \boldsymbol{B}(\boldsymbol{x}) >$ are actually zero. In the case of the electric field, this is to be expected on purely physical grounds: if we had $< \boldsymbol{E}(\boldsymbol{x}) > \neq 0$, the quantum charges would be subjected to a net force. But this is in contradiction to the hypothesis of a thermal equilibrium state of the system and we conclude that $< \boldsymbol{E}(\boldsymbol{x}) >= 0$. Mathematically, this property can be proven by introduction of the unitary spatial inversion operator P whose action on a N-particle wave function is defined as

$$\mathrm{P}\psi(\boldsymbol{r}_1, \ldots, \boldsymbol{r}_N) \equiv \psi(-\boldsymbol{r}_1, \ldots, -\boldsymbol{r}_N). \tag{C.14}$$

First, we verify that P is actually a symmetry transformation for the total Hamiltonian $H_{L,R}^N$ given in (2.9). Considering the definitions of H_C^N, H_0^{rad} and H_{pot}^N we see immediately that they are invariant under the action of P. Regarding the kinetic term

$$H_{kin}^N = \sum_{i=1}^N \frac{\left(\boldsymbol{p}_i - \frac{e_{\alpha_i}}{c}\boldsymbol{A}(\boldsymbol{r}_i)\right)^2}{2m_{\alpha_i}}, \tag{C.15}$$

we note that a spatial inversion yields the mappings $\boldsymbol{p}_i \mapsto -\boldsymbol{p}_i$ and $\boldsymbol{A}(\boldsymbol{r}_i) \mapsto \boldsymbol{A}(-\boldsymbol{r}_i)$. Since the vector potential is a polar vector, i.e. $\boldsymbol{A}(\boldsymbol{r}) = -\boldsymbol{A}(-\boldsymbol{r})$, we conclude that

H_{kin}^N is invariant under P and thus the spatial inversion is a symmetry transformation of our system

$$\mathrm{P}^\dagger H_{L,R}^N \mathrm{P} = H_{L,R}^N. \tag{C.16}$$

This invariance implies then that the density matrix is invariant, too,

$$\mathrm{P}^\dagger \rho \mathrm{P} = \frac{1}{Z_{L,R}^N} \mathrm{P}^\dagger e^{-\beta H_{L,R}^N} \mathrm{P} = \rho \tag{C.17}$$

and consequently

$$\begin{aligned} < \boldsymbol{E}(\boldsymbol{x}) > &= Tr(\rho \boldsymbol{E}(\boldsymbol{x})) = Tr(\mathrm{P}^\dagger \mathrm{P} \rho \boldsymbol{E}(\boldsymbol{x})) = Tr(\mathrm{P}^\dagger \rho \mathrm{P} \mathrm{P}^\dagger \boldsymbol{E}(\boldsymbol{x}) \mathrm{P}) = \\ &= Tr(\rho \boldsymbol{E}(-\boldsymbol{x})) = -Tr(\rho \boldsymbol{E}(\boldsymbol{x})) = - < \boldsymbol{E}(\boldsymbol{x}) >= 0. \end{aligned} \tag{C.18}$$

In (C.18) we have used the unitary property $\mathrm{P}^\dagger \mathrm{P} = \mathbf{1}$ and the fact that $\boldsymbol{E}(\boldsymbol{x}) = -\boldsymbol{E}(-\boldsymbol{x})$ is a polar vector. The same argument shows that $< \boldsymbol{A}(\boldsymbol{x}) >= 0$. On the other hand, this argument does not hold for the magnetic induction because $\boldsymbol{B}(\boldsymbol{x}) = \boldsymbol{B}(-\boldsymbol{x})$ is an axial vector. However, we know that

$$< \boldsymbol{B}(\boldsymbol{x}) >=< \nabla \wedge \boldsymbol{A}(\boldsymbol{x}) >= \nabla \wedge < \boldsymbol{A}(\boldsymbol{x}) > \tag{C.19}$$

and hence the mean magnetic field vanishes, too.

Appendix D

Particle Density Expansions of a and Ω

D.1 The Semi-Classical Coefficient $a(\rho_\alpha, \beta, \hbar)$

The semi-classical field correlation asymptotics calculated in chapter (3) involve the coefficient $a(\rho_\alpha, \beta, \hbar)$ which depends only on internal parameters of the system, namely the temperature and the particle densities

$$a(\rho_\alpha, \beta, \hbar) = \frac{1}{2} \sum_{\alpha,\alpha'} \frac{e_\alpha e_{\alpha'} \lambda_\alpha \lambda_{\alpha'}}{\beta \sqrt{m_\alpha m_{\alpha'}} c^2} \int d\boldsymbol{r} \int D(\xi) \int D(\xi') \times \tag{D.1}$$
$$\times \int_0^1 d\xi^\mu(s) \int_0^1 d\xi'^\nu(s') (\hat{\boldsymbol{k}} \cdot \boldsymbol{\xi}(s))(\hat{\boldsymbol{k}} \cdot \boldsymbol{\xi}'(s')) \delta^{\mu\nu}_{tr}(\hat{\boldsymbol{k}}) n_T(\alpha, \boldsymbol{\xi}, \alpha', \boldsymbol{\xi}', \boldsymbol{r}).$$

As can be seen in equations (3.49)-(3.51), this quantity comprises the entire influence of the quantum plasma on the field correlations. Although it is coordinate-independent and has therefore no influence on the power law decay of the correlations, it is nevertheless instructive to examine it more closely. As we shall see in section D.3, an exact calculation is not feasible and we will content ourselves with a low density expansion.

The crucial object in (D.1) is obviously the truncated loop density correlation function which reads for a translationally invariant system

$$n_T(\alpha_1, \boldsymbol{\xi}_1, \alpha_2, \boldsymbol{\xi}_2, \boldsymbol{r}) = \rho_T^{(2)}(\alpha_1, \boldsymbol{\xi}_1, \alpha_2, \boldsymbol{\xi}_2, \boldsymbol{r}) + \delta(\boldsymbol{r}) \delta(\boldsymbol{\xi}_1, \boldsymbol{\xi}_2) \delta_{\alpha_1 \alpha_2} \rho(\alpha_1, \boldsymbol{\xi}_1). \tag{D.2}$$

The truncated two-loop density $\rho_T^{(2)}$ is defined as

$$\rho_T^{(2)}(\mathcal{F}_1, \mathcal{F}_2) = \rho^{(2)}(\mathcal{F}_1, \mathcal{F}_2) - \rho^{(1)}(\mathcal{F}_1) \rho^{(1)}(\mathcal{F}_2), \tag{D.3}$$

where

$$\rho^{(2)}(\mathcal{F}_1, \mathcal{F}_2) =< \hat{\rho}(\mathcal{F}_1)\hat{\rho}(\mathcal{F}_b) >^{nc}= \sum_{i\neq j}^{N} < \delta(\mathcal{F}_1, \mathcal{F}_i)\delta(\mathcal{F}_2, \mathcal{F}_j) > \tag{D.4}$$

is the two-loop density (excluding the contributions of coincident points) and

$$\rho^{(1)}(\mathcal{F}) =< \hat{\rho}(\mathcal{F}) >= \sum_i \delta(\mathcal{F}_i, \mathcal{F}) \tag{D.5}$$

is the the loop density. Let us note that the loop density $\rho^{(1)}(\mathcal{F})$ is independent of the position of the loop, i.e. $\rho^{(1)}(\alpha, \boldsymbol{\xi}, \boldsymbol{r}) = \rho^{(1)}(\alpha, \boldsymbol{\xi})$ due to translational invariance of the thermal equilibrium state. Furthermore, the loop density should not be confused with the particle density; in order to obtain the physical particle density ρ_α of the chemical species α we have to integrate out the non-physical degrees of freedom, i.e. we have to integrate over the Brownian bridge

$$\rho_\alpha(\boldsymbol{r}) = \rho_\alpha = \int D(\boldsymbol{\xi})\rho(\alpha, \boldsymbol{\xi}). \tag{D.6}$$

Hence, if we wish to obtain an expansion of a in terms of the particle densities, we are urged to reformulate the loop density correlation function (D.2) in terms of the particle densities instead of the loop densities. However, the dependence of $\rho(\alpha, \boldsymbol{\xi})$ on ρ_α is according to (D.6) an implicit one and the inversion procedure is rather involved. A review of the algorithm can be found in [4] and the final result reads (in absence of the radiation field)

$$\begin{aligned}\rho(\alpha, \boldsymbol{\xi}) = \rho_\alpha + \sum_\gamma \rho_\alpha\rho_\gamma \int d\boldsymbol{r} \int \mathcal{D}(\boldsymbol{\xi}_1) \Bigg[\exp\left(-\beta e_\alpha e_\gamma \int_0^1 ds V(|\boldsymbol{r} + \lambda_\gamma \boldsymbol{\xi}_1(s) - \lambda_\alpha \boldsymbol{\xi}(s)|)\right) - \\ - \int \mathcal{D}(\boldsymbol{\xi}_2) \exp\left(-\beta e_\alpha e_\gamma \int_0^1 ds V(|\boldsymbol{r} + \lambda_\gamma \boldsymbol{\xi}_1(s) - \lambda_\alpha \boldsymbol{\xi}_2(s)|)\right)\Bigg] + \mathcal{O}(\rho^{5/2}).\end{aligned} \tag{D.7}$$

We expect that the first order approximation $\rho(\alpha, \boldsymbol{\xi}) \sim \rho_\alpha$ is still valid in the presence of the field. Let us now look for the lowest order contribution to n_T in the particle densities ρ_α. Since the two-loop density $\rho^{(2)}$ is of $\mathcal{O}\left(\rho^2(\alpha, \boldsymbol{\xi})\right)$ and hence of $\mathcal{O}(\rho_\alpha^2)$, the lowest order contribution to an expansion of (D.2) comes from the coincident points contribution. Pluging this term into (D.1) and using the formula [5]

$$\int \mathcal{D}(\boldsymbol{\xi}) \int_0^1 d\xi^\mu(s) \int_0^1 d\xi^\nu(t)\, \xi^\alpha(s)\xi^\beta(t) = \frac{1}{12}\left(\delta^{\mu\nu}\delta^{\alpha\beta} - \delta^{\mu\beta}\delta^{\nu\alpha}\right), \tag{D.8}$$

we find for the coincident points contribution a^{cp}

$$\begin{aligned} a^{cp}(\rho_\alpha,\beta,\hbar) &= \frac{1}{2}\sum_\alpha \frac{e_\alpha^2\lambda_\alpha^2}{\beta m_\alpha c^2}\int \mathcal{D}(\boldsymbol{\xi})\rho_\alpha \int_0^1 d\xi^\mu(s)\int_0^1 d\xi^\nu(s')\xi^\alpha(s)\xi^\beta(s')\hat{k}^\alpha\hat{k}^\beta\delta_{tr}^{\mu\nu}(\hat{\boldsymbol{k}}) + \mathcal{O}(\rho_\alpha^2) \\ &= \frac{1}{2}\sum_\alpha \frac{e_\alpha^2\lambda_\alpha^2}{\beta m_\alpha c^2}\rho_\alpha\frac{1}{12}\left(\delta^{\mu\nu}\delta^{\alpha\beta}-\delta^{\mu\beta}\delta^{\nu\alpha}\right)\hat{k}^\alpha\hat{k}^\beta\delta_{tr}^{\mu\nu}(\hat{\boldsymbol{k}}) + \mathcal{O}(\rho_\alpha^2) \\ &= \frac{1}{12}\sum_\alpha \frac{e_\alpha^2\lambda_\alpha^2}{\beta m_\alpha c^2}\rho_\alpha + \mathcal{O}(\rho_\alpha^2). \end{aligned} \tag{D.9}$$

We deduce that the coefficient a is of $\mathcal{O}(\hbar^2)$.

D.2 The Quantum Coefficient $\Omega(\rho_\alpha, \beta, \hbar)$

In the framework of the full quantum system of chapter 4, the matter-dependence of the long-distance asymptotics of the transverse electric field correlations (6.31) is included in the coefficient Ω (6.32). In order to find its expansion in the particle densities ρ_α, we can follow the procedure of the previous section. Before we calculate the lowest order contribution to $\Omega(\rho_\alpha, \beta, \hbar)$, we perform a little modification which facilitates the succeeding considerations. Let's recall foremost the version we had found in section 6.2

$$\Omega = 2\pi\sum_{\alpha,\alpha'} e_\alpha e_{\alpha'}\int d\boldsymbol{r}\int D(\xi)\int D(\xi')\times \tag{D.10}$$

$$\times\int_0^1 ds\int_0^1 ds'\,\lambda_\alpha\xi^\mu(s)\lambda_{\alpha'}\xi'^\nu(s')\,\delta_{tr}^{\mu\nu}(\hat{\boldsymbol{k}})\,n(\alpha,\boldsymbol{\xi},\alpha',\boldsymbol{\xi}',\boldsymbol{r}). \tag{D.11}$$

In fact, we can use symmetry arguments to show that

$$\int \mathcal{D}(\boldsymbol{\xi})\xi^\mu(s)\rho(\alpha,\boldsymbol{\xi}) = \int \mathcal{D}(-\boldsymbol{\xi})(-\xi^\mu(s))\rho(\alpha,-\boldsymbol{\xi}) = -\int \mathcal{D}(\boldsymbol{\xi})\xi^\mu(s)\rho(\alpha,\boldsymbol{\xi}) = 0 \tag{D.12}$$

and consequently we can replace the loop density correlation function in (D.10) by its truncated version without changing the value of the integral

$$\begin{aligned} \Omega = 2\pi\sum_{\alpha,\alpha'} e_\alpha e_{\alpha'}&\int d\boldsymbol{r}\int D(\xi)\int D(\xi')\times \\ &\times\int_0^1 ds\int_0^1 ds'\,\lambda_\alpha\xi^\mu(s)\lambda_{\alpha'}\xi'^\nu(s')\,\delta_{tr}^{\mu\nu}(\hat{\boldsymbol{k}})\,n_T(\alpha,\boldsymbol{\xi},\alpha',\boldsymbol{\xi}',\boldsymbol{r}). \end{aligned} \tag{D.13}$$

Considering solely the coincident points contribution in (D.2) and using the covariance of the Brownian bridge [2]

$$\int \mathcal{D}(\boldsymbol{\xi})\xi^{\mu}(s)\xi^{\nu}(s') = (min(s,s') - ss')\,, \tag{D.14}$$

we find for the coincident points contribution Ω^{cp}

$$\begin{aligned}\Omega^{cp} &= 2\pi \sum_{\alpha} e_{\alpha}^2 \lambda_{\alpha}^2 \rho_{\alpha} \int \mathcal{D}(\boldsymbol{\xi}) \int_0^1 ds \int_0^1 ds' \xi^{\mu}(s)\xi^{\nu}(s')\delta_{tr}^{\mu\nu}(\hat{\boldsymbol{k}}) + \mathcal{O}(\rho_{\alpha}^2) \\ &= \frac{\pi}{9} \sum_{\alpha} e_{\alpha}^2 \lambda_{\alpha}^2 \rho_{\alpha} + \mathcal{O}(\rho_{\alpha}^2).\end{aligned} \tag{D.15}$$

Furthermore, we conclude that Ω is of $\mathcal{O}(\hbar^2)$.

D.3 Higher Order Expansions of a and Ω

The first order term in the low density expansion of the coefficient a (resp. Ω) has been obtained by means of the coincident points contribution in (D.2). If we would like to find the higher order terms we have to consider both the rest $\mathcal{O}(\rho_{\alpha}^2)$ in (D.9) (resp. (D.15)) as well as the contributions originating from the two-loop density $\rho_T^{(2)}$ in (D.2). Since the particle loops $\mathcal{F}$ belong to a classical phase space, the two-loop density $\rho_T^{(2)}$ is a classical statistical integral and we may use the Mayer graph technique for explicit calculations. Before we reformulate our problem in terms of Mayer diagrams, we introduce the two-loop distribution function

$$g(\mathcal{F}_1, \mathcal{F}_2) = \frac{\rho^{(2)}(\mathcal{F}_1, \mathcal{F}_2)}{\rho^{(1)}(\mathcal{F}_1)\rho^{(1)}(\mathcal{F}_2)} \tag{D.16}$$

and the pair correlation function

$$h(\mathcal{F}_1, \mathcal{F}_2) = g(\mathcal{F}_1, \mathcal{F}_2) - 1. \tag{D.17}$$

The latter is also known as the Ursell function. The two-loop density can now be reformulated in terms of the Ursell function

$$\rho_T^{(2)}(\mathcal{F}_1, \mathcal{F}_2) = \rho^{(1)}(\mathcal{F}_1)\rho^{(1)}(\mathcal{F}_2)\, h(\mathcal{F}_1, \mathcal{F}_2) \tag{D.18}$$

and we may therefore focus our attention on the calculation of the Ursell function. Its well-known diagrammatic expansion is [3]

$$h(\mathcal{F}_a, \mathcal{F}_b) = \sum_{\Gamma} \frac{1}{S_{\Gamma}} \int \prod_{n=1}^{N} d\mathcal{F}_n\, \rho(\mathcal{F}_n) \Big[\prod f \Big]_{\Gamma}, \tag{D.19}$$

where the sum runs over all unlabeled topologically different connected diagrams Γ with two root points $\mathcal{F}_1$ and $\mathcal{F}_2$ and N internal points $N = 0, 1, 2, \ldots$ without articulation points (an articulation point is such that, when removed, the diagram splits into two pieces, at least one of which is disconnected from any root point). $\left[\prod f\right]_\Gamma$ is the product of the f-bonds (Mayer factors) in the Γ diagram and S_Γ is its symmetry factor, i.e. the number of permutations of internal points $\mathcal{F}_n$ that leave this product invariant. The f-bonds are defined as

$$f(\mathcal{F}_1, \mathcal{F}_2) = e^{-\beta_{i,j} U(\mathcal{F}_1, \mathcal{F}_2)} - 1, \tag{D.20}$$

where $U(\mathcal{F}_1, \mathcal{F}_2)$ is the corresponding two-loop interaction potential and $\beta_{i,j} = \beta e_{\alpha_i} e_{\alpha_j}$. According to the Hamiltonian (2.9) and the effective magnetic potential arising in the matter-field interaction, the interaction potential reads in the present case:

$$U(\mathcal{F}_1, \mathcal{F}_2) = V_C(\mathcal{F}_1, \mathcal{F}_2) + V_{sr}(\mathcal{F}_1, \mathcal{F}_2) + W_m(\mathcal{F}_1, \mathcal{F}_2), \tag{D.21}$$

where $V_C(\mathcal{F}_1, \mathcal{F}_2)$ is the Coulomb potential, $V_{sr}(\mathcal{F}_1, \mathcal{F}_2)$ is a short range potential thats regularizes the singular Coulomb term and $W_m(\mathcal{F}_1, \mathcal{F}_2)$ is the effective magnetic potential (quantum or classical). Obviously, the Mayer factor satisfies $\lim_{\boldsymbol{r}\to\infty} f(\boldsymbol{r}) = 0$, but since the Coulomb potential decays only as $\sim \frac{1}{r}$ and W_m is on the border-line of integrability, the f-bonds are not integrable. This problem arises also in the purely classical theory and it can be solved by means of the exact partial resummation of auxiliary diagrams. As a result we obtain a reformulation of the Ursell function in terms of divergence free, so-called prototype diagrams. Unfortunately, it is not possible to calculate the infinite sum of diagrams, but we can nevertheless perform an expansion in the particle densities. Obviously, such an expansion makes only sense if we are in a low density regime. For details we refer to [7] and for a compact review to [4].

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