A Personal Perspective

Harry Kesten's Publications
stand Kesten's prowess until you have seen him demolish a problem that you have worked on.

The first two decades of Kesten's work, while containing important contributions, are difficult for me to properly put in context since the work was already complete when I started learning probability. Because of this I have decided to tell the story of his work as I experienced it. Thus, following the style of some paperback novelists, I will begin in the middle of the story with some exciting events to grab the reader's attention. Then after the story line is established, I will go back and fill in earlier developments.

I spent the 1980–81 academic year at Cornell. At this time, Dynkin’s Russian style seminar, held Wednesdays 7–9 PM, was a lively affair, with Avi Mandelbaum, Bob Vanderbei and Patrick Sheppard as students. Kesten had recently proved that "The critical probability for bond percolation on the square lattice equals \(1/2\)" [67] and followed this up with "power estimates of functions in percolation theory" [71].

The title of the last paper is a double entendre. The results concern the power law behavior of functions near the critical value but introduced powerful new rigorous renormalization arguments. To facilitate writing his book [72], Kesten taught a graduate seminar on percolation and first passage percolation. Harry prepared for class while swimming laps in the pool at Teagle Hall, so in his lectures you got to see how he thought. He would start with the main idea of the proof, but then often would have to go back and insert a technicality at an angle on the margin of the board. This made it difficult for the students to get good notes, but for me it provided valuable insights about why things are true and how he went about solving problems.

In this brief article there is not enough space to discuss why things are true, so we will only discuss what Kesten (and others) have done. Our first two topics, percolation and first passage percolation (which we will interpret very broadly) are those of Harry's course in 1980 and of his Wald Lectures in 1986 (see [95]). In each case we will take the subject from the 80's up to today. For the third section of the paper we will go back to Kesten's Ph.D. thesis and follow his work on random walks and related topics up to the present. In these three forays we will touch on much, but by no means all, of Kesten's best work. Like a one week bus tour of Europe, there is only time to drive past the outside of some of the most important landmarks. We apologize in advance for the fact that in order to say things quickly, we will not always be able to say things carefully. We will never intentionally lie about what is true, but sometimes we will not take the time to sort out all the details of who did what when.

### 1.1 Percolation

Broadbent and Hammersley (1957), and Hammersley (1959) introduced percolation as a model for the spread of a fluid or gas through a random medium. To formulate the bond percolation model in \(d\) dimensions, we make the \(d\)-dimensional integer lattice \(\mathbb{Z}^d\) into a graph by drawing edges connecting adjacent sites. We imagine that the edges are channels and that fluid will move through a channel if and only if the channel is wide enough. We declare that the edges are independently designated as open (wide enough) or closed with probabilities \(p\) and \(1 - p\) respectively, and let \(P_p\) denote the resulting probability measure on the configurations of open and closed edges. We will also sometimes consider site percolation in which the sites are independently open with probability \(p\) or closed with probability \(1 - p\), but for this article the default process is bond percolation.

With that set-up it is natural to ask about the set of sites \(C_0\) that can be reached from the origin by a path of open edges. The first papers mentioned above showed that if \(p\) is small, then the number of points in \(C_0, |C_0|\), is always finite, while if \(p\) is close enough to \(1\), then

\[
\theta(p) = P_p(|C_0| = \infty) > 0. \tag{1.1.1}
\]

This and an obvious monotonicity establishes the existence of a critical value \(p_c = \inf\{p : \theta(p) > 0\}\) but does not give much information about its value. The first step in that direction for two dimensional bond percolation was taken by Harris (1960). He noticed that when \(p = 1/2\), symmetry dictates that the probability of a left to right crossing of a "square," an \(n \times (n + 1)\) piece of the square lattice, is \(1/2\), and used this observation to show that at \(p = 1/2\) the origin is surrounded by infinitely many cut sets of vacant edges, so \(p_c \geq 1/2\).

The next step in this direction was taken by Sykes and Essam (1954) who introduced a quantity they called the free energy:

\[
\Delta(p) = E_p(1/|C_0|; |C_0| > 0). \tag{1.1.2}
\]

Probabilistically, this is the limiting value of the number of clusters per unit volume. By analogy with the Ising model, Sykes and Essam argued that the phase transition in percolation must be manifest in a "singularity" at \(p_c\). Their calculations showed that the square lattice, \(\Delta(p) - \Delta(1 - p)\) is a polynomial in \(p\), so assuming that such a singularity was unique, they arrived at \(p_c = 1/2\) for the square lattice. In addition, Sykes and Essam used variations of this argument to show that the critical value of site percolation on the triangular lattice is \(1/2\) and supplemented this with the star-triangle transformation to show that the critical values for bond percolation on the triangular and hexagonal lattice are \(p = 1/2\) and \(1 - p = 2 \sin(\pi/18)\). The unique root of \(3p - \rho^3 = 1\) in \((0, 1)\).

Not much beyond Harris' result was rigorously proved about percolation until 1978, when Russo (1978) and Seymour and Welsh (1978) provided two valuable steps. The first is that if sponge crossing probabilities are large enough, then percolation occurs. To state the second, we need to define the dual of a planar graph, which is constructed by putting sites in each component of the complement of the graph and connecting two sites by an edge if the boundaries of their associated components share an edge. Denoting the dual by a star and defining

\[
p_T = \sup\{p : E_p(|C_0|) = \infty\}, \tag{1.1.3}
\]
The theoretical framework developed in the previous section provides a solid foundation for analyzing the performance of the proposed algorithm. In particular, the analysis focuses on the convergence properties of the iterative update process. The convergence rate is determined by the choice of the step size, which is shown to be crucial for achieving optimal performance.

The theoretical results are validated through extensive simulations, which demonstrate the effectiveness of the proposed approach in various scenarios. The simulations further highlight the robustness of the algorithm against variations in the input data, thereby confirming its practical applicability.

In summary, the theoretical analysis and empirical validation together provide strong evidence for the efficacy of the proposed method in solving the optimization problem. This work opens up new avenues for research in the field of signal processing and machine learning, particularly in scenarios where high-dimensional data is involved.

Acknowledgments

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References


First, consider the case where $p = 0$. Then the left-hand side of (3.1) becomes 0, and the right-hand side is 0 as well. Therefore, (3.1) holds in this case.

Next, consider the case where $p > 0$. Then the left-hand side of (3.1) is positive, and the right-hand side is positive as well (since $\alpha > 0$). Therefore, (3.1) holds in this case too.

Finally, consider the case where $p < 0$. Then the left-hand side of (3.1) is negative (since $\alpha > 0$), but the right-hand side is positive. Therefore, (3.1) does not hold in this case.

In summary, (3.1) holds for all $p \neq 0$. This shows that the function $f_p$ defined in (1.1) satisfies the condition (3.1).
First Passage Percolation

The problem of finding the shortest path in a random environment is central to many problems in probability and statistical physics. In this section, we will discuss a model of first passage percolation, which is a particular case of this problem. The model is defined on a graph, where the edges are assigned random weights. The goal is to find the shortest path between two given vertices.

The model is defined on a graph $G = (V,E)$, where $V$ is the set of vertices and $E$ is the set of edges. Each edge $e = (u,v) \in E$ is assigned a random weight $w_e = w(u,v)$, which is i.i.d. across edges.

The problem of finding the shortest path between two given vertices $s$ and $t$ is equivalent to finding the minimal flow from $s$ to $t$ in a network where each edge has a capacity equal to its weight. This is known as the max-flow min-cut theorem.

The shortest path problem is NP-hard in general, but there are efficient algorithms for some special cases, such as the case where all edge weights are non-negative. In the general case, the problem is solved using random algorithms, such as the algorithm of Dijkstra or Bellman.

In the next section, we will discuss some of the known results and open problems in the area of first passage percolation.
where $y_i$ is a component that only depends on $y_{i-1}$ and $y_{i+1}$.

The function $f$ is given by

$$f(y_i) = \frac{1}{2\pi} \exp \left( -\frac{y_i^2}{2} \right)$$

The distribution $f$ is the normal distribution. Theorem 1.4 (966) shows that if we take a certain function $g$ and reparameterize $y_i$ in $g(y_i)$, then the distribution of $y_i$ is also normal.

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In the above and in these definitions, we have an expression

\[ \frac{\alpha_x}{\beta} \theta + \frac{\beta_x}{\gamma} \phi = \psi \]

which is the basis of the following definitions.

\[ (x \in X) \quad \text{and} \quad (y \in Y) \]

The results in (2.1) imply that if we can solve the problem on \( X \) by solving a similar problem on \( Y \), then we have

\[ u_{x-y} = \frac{\alpha_x}{\beta} \theta + \frac{\beta_x}{\gamma} \phi = \psi \]

showing that \( u_{x-y} \) is the same as \( \frac{\alpha_x}{\beta} \theta + \frac{\beta_x}{\gamma} \phi \) on both sides.

If the inverse conjugated (or equivalent) relations (1961) and conditions (10) are satisfied

\[ u_{x-y} = \frac{\alpha_x}{\beta} \theta + \frac{\beta_x}{\gamma} \phi \]

we have

\[ \psi = \psi \]

This is a necessary condition for the conjunction being true.

For instance, in the case of the conjunction or for the conjunction,

\[ (x \in X) \quad \text{and} \quad (y \in Y) \]

the problem is reduced to the conjunction

\[ \psi = \psi \]

and only if \( \psi \) is valid in the conjunction.

Moreover, in the case of the conjunction or for the conjunction,

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Consider the following scenario: To model the distribution of the

Theorem 1: For a given function $f(x)$, there exists a unique

Proof: Assume that there are two functions $g(x)$ and $h(x)$

where

$$f(x) = g(x) \text{ and } f(x) = h(x)$$

for all $x$ in the domain of $f$. Then, by the uniqueness of solutions

$$g(x) = h(x)$$

for all $x$ in the domain of $f$. Thus, $g(x)$ and $h(x)$ are different.

This result is a consequence of the non-existence of two different

functions that can be written in a similar form.
REFERENCES

A review of the most recent papers on the topic of discussion.

1.4 Development

In [1] and [2], the authors proposed a new method for solving the problem of...

1.5 Conclusion

In conclusion, the proposed method in this paper...

The experimental results show that...

ACKNOWLEDGMENTS

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APPENDIX

A.1 Supplementary Material

A.2 Additional Information

APPENDIX B

B.1 Additional Data