Calloway-Hopcroft-Kleinberg-Newman-Strogatz
Random Graph Model

At each time a vertex is added to the graph. Number the vertices 1, 2, \ldots n in the order they were added.

For $k \geq 2$ after the $k$th vertex is added we add a number of edges with mean $\delta$. The edges are drawn with replacement from the $\binom{k}{2}$ possible edges.

Original version: number of edges Bernoulli ($\delta$).

We will mainly study the Poisson($\delta$) version.

Phys. Rev. E. Volume 64, Paper number 041902
In the Poisson case if we let $A_{i,j,k}$ be the event no $(i,j)$ edge is added at time $k$ and $i < j \leq k$ then

$$P(A_{i,j,k}) = \exp\left(-\delta/\left(k\choose 2\right)\right)$$

and these events are independent.

$$P(\cap_{k=j}^n A_{i,j,k}) = \prod_{k=j}^n \exp(-2\delta/k(k-1))$$

$\#1 = \exp\left(-2\delta\left(\frac{1}{j-1} - \frac{1}{n}\right)\right)$

$\#2 \approx 1 - 2\delta\left(\frac{1}{j} - \frac{1}{n}\right)$

$\#3 \approx 1 - \frac{2\delta}{j}$

$0 \leq e^{-x} - (1-x) \leq x^2$ for $0 < x < 1$ so

$$E\mathcal{E}_n = \begin{cases} \delta n & \#1 \\ \delta(n + O(\log n)) & \#2 \\ 2\delta n & \#3 \end{cases}$$

$\#3 \approx \#2$ and an independent Erdös-Renyi($2\delta/n$)
Let $N_k(t)$ be the expected number of components of size $k$ at time $t$. Ignoring terms of $O(1/t^2)$

\[
N_1(t + 1) = N_1(t) + 1 - 2\delta \frac{N_1(t)}{t}
\]

\[
N_k(t + 1) = N_k(t) - 2\delta \frac{kN_k(t)}{t} + \delta \sum_{j=1}^{k-1} \frac{jN_j(t)}{t} \cdot \frac{(k - j)N_{k-j}(t)}{t}
\]

**Theorem 0.** For either the Bernoulli or Poisson model, as $t \to \infty$, $N_k(t)/t \to a_k$ where and

\[
a_1 = \frac{1}{1 + 2\delta}
\]

\[
a_k = \frac{\delta}{1 + 2\delta k} \sum_{j=1}^{k-1} ja_j \cdot (k - j)a_{k-j}
\]

**Proof.** If $N_1(t)/t > a_1 + \epsilon$ it decreases by $c\epsilon/(t+1)$. Now use induction.
\[(1 + 2\delta)a_1 = 1\]

\[(1 + 2\delta k)a_k = \delta \sum_{j=1}^{k-1} ja_j \cdot (k - j)a_{k-j}\]

Let \(h(x) = \sum_{k=1}^{\infty} x^k a_k\) and \(g(x) = \sum_{k=1}^{\infty} x^k ka_k\)

\[h(x) + 2\delta g(x) = x + \delta g^2\]

Since \(h'(x) = g(x)/x\) differentiation gives

\[g(x)/x + 2\delta g'(x) = 1 + 2\delta g(x)g'(x)\]

Rearranging we have

\[g'(x) = \frac{1}{2\delta x} \cdot \frac{x - g(x)}{1 - g(x)}\]
\[ g'(x) = \frac{1}{2\delta x} \cdot \frac{x - g(x)}{1 - g(x)} \]

1 − \( g(1) \) gives the fraction of vertices that belong to giant components. Let \( b_k = ka_k \).

(i) If \( g(1) < 1 \) then \( \sum_{k=1}^{\infty} k b_k = g'(1) = 1/2\delta \)

If \( g(1) = 1 \), L’Hôpital’s rule implies

\[
2\delta g'(1) = \lim_{x \to 1} \frac{x - g(x)}{1 - g(x)} = \lim_{x \to 1} \frac{1 - g'(x)}{-g'(x)}
\]

\[
2\delta (g'(1))^2 - g'(1) + 1 = 0
\]

(ii) If \( g(1) = 1 \) then \( g'(1) = (1 - \sqrt{1 - 8\delta})/4\delta \).

This solution tends to 1 as \( \delta \to 0 \). It becomes complex when \( \delta > 1/8 \). Therefore

\[
\delta_c = 1/8
\]
$$\sum_k k b_k = \begin{cases} 
(1 - \sqrt{1 - 8\delta})/4\delta & \delta \leq 1/8 \\
1/2\delta & \delta > 1/8
\end{cases}$$

Note that the mean size of finite clusters is always finite even at $\delta_c$ but is discontinuous at $\delta_c$. 
Theorem 1. For any of the three forms of the inhomogeneous random graph, $\delta_c = 1/8$.

Background. Consider the random graph model on $\{1, 2, 3, \ldots\}$ with $p_{i,j} = \lambda/(i \lor j)$.

Kalikow and Weiss (1988) showed that the probability $G$ is connected (ALL vertices in ONE component) is either 0 or 1, and that

$$\frac{1}{4} \leq \lambda_c \leq 1$$

They conjectured $\lambda_c = 1$ but Shepp (1989) proved

$$\lambda_c = 1/4 \quad \text{(note } \lambda = 2\delta)$$

Durrett and Kesten (1990) proved a general result which includes $\lambda/(i \lor j)$ and implies in particular that if $p_{i,j} = \lambda(i^r + j^r)^{-1/r}$ then

$$\lambda_c(r) = r \Gamma \left(\frac{1}{r}\right) / \Gamma \left(\frac{1}{2r}\right)^2$$
$\delta_c \geq 1/8$. The mean size of the cluster containing a given point $i$ is bounded above by the expected value of the total progeny of a discrete time multi-type branching process in which a particle of type $j$ gives birth to one offspring of type $k$ with probability $p_{j,k}$. (We set $p_{j,j} = 0$.)

**Erdős-Renyi random graph.** $p_{j,k} = \lambda/n$, this is an ordinary branching process with a Poisson mean $\lambda$ offspring distribution so we get $\lambda_c \geq 1$.

**CHKNS random graph.** $p_{j,k} = 2\delta/(j \lor k)$. The mean of the total progeny starting from one of type $i$ is

$$\sum_{m=0}^{\infty} \sum_j p^{m}_{i,j}$$

which will be finite if and only if the spectral radius $\rho(p_{i,j}) < 1$. By Perron-Frobenius $\rho$ is an eigenvalue with positive eigenvector. (Also $p_{i,j} = p_{j,i}$.)
Following Shepp (1989) we made a good guess at the Perron-Frobenius eigenvector. For the upper bound on $\delta_c$ we use the largest model ($p_{i,j} = 2\delta / (i \vee j)$)

$$
\sum_{j=1}^{n} \frac{1}{i \vee j} \cdot \frac{1}{j^{1/2}} = \frac{1}{i} \sum_{j=1}^{i-1} \frac{1}{j^{1/2}} + \sum_{j=i+1}^{n} \frac{1}{j^{3/2}} \\
\leq \frac{1}{i} \left(1 + \int_{1}^{i} \frac{1}{x^{1/2}} \, dx \right) + \int_{i}^{n} \frac{1}{x^{3/2}} \, dx \\
= \frac{1}{i} (1 + 2i^{1/2} - 2) + 2(i^{-1/2} - n^{-1/2}) \\
\leq 4 i^{1/2}
$$

This implies $\sum_j i^{1/2} p_{i,j} j^{-1/2} \leq 8\delta$ so

$$
\sum_k k b_{n,k} = \frac{1}{n} \sum_{i=1}^{n} E|C_i| \leq \frac{1}{n} \sum_{m} \sum_{i,j} p_{i,j}^m \\
\leq \frac{2}{n} \sum_{m=0}^{\infty} \sum_{i \geq j} i^{1/2} p_{i,j}^m j^{-1/2} \leq \frac{2}{1 - 8\delta}
$$
For $\delta_c \leq 1/8$ we consider model 2: $Q(i, j) = \frac{1}{i\sqrt{j}} - \frac{1}{n}$ with $K \leq i, j \leq n$. By the variational characterization of the largest eigenvalue

$$\rho(Q) \geq \left(\sum_{i=1}^{n} v_j^2\right)^{-1} v^T Q v$$

Again we take $v_j = 1/\sqrt{j}$.

$$v^T Q v = 2 \sum_{i=K}^{n} \sum_{j=i+1}^{n} \frac{1}{i^{1/2}} \frac{1}{j^{3/2}} - \frac{1}{n} \left(\sum_{j=K}^{n} \frac{1}{j^{1/2}}\right)^2$$

The second term is $\geq -4$. Bounding sums below by integrals the first is

$$\geq 2 \sum_{i=K}^{n} 2(i + 1)^{-1} - 2i^{-1/2}(n + 1)^{-1/2}$$

$$\geq 4 \sum_{i=K}^{n} (i + 1)^{-1} - 4n^{1/2}(n + 1)^{-1/2}$$

This implies

$$\rho(Q) \geq \frac{4 \sum_{i=K}^{n} (i + 1)^{-1} - 8}{\sum_{i=K}^{n} i^{-1}}$$
Let $q(i, j) = 2\delta \left( \frac{1}{i \vee j} - \frac{1}{n} \right)$, $K \leq i, j < KN \leq n$.

$$\rho(q) \geq 8\delta \frac{\sum_{i=K}^{KN-1} (i + 1)^{-1} - 2}{\sum_{i=K}^{KN-1} i^{-1}} \geq 8\delta \frac{\log N - 3}{\log N}$$

If $8\delta = 1 + 4\epsilon > 1$ and $N = e^{12 + (3/\epsilon)}$ we have

$$\rho(q) \geq 1 + 3\epsilon \quad \text{for all } K \geq 1$$

(2.16) in Durrett and Kesten (1990). Consider the $q$ random graph in $[K, NK)$. There are constants $\gamma$ and $\beta$ so that if $K \geq K_0$ then with probability at least $\beta$, $K$ belongs to a component with at least $\gamma NK$ vertices.

Proof of $\delta_c \leq 1/8$. Apply the lemma to $K = n/N$. 
Proof of (2.16). We subdivide \([K, KN]\) into intervals \(I_m = [K + (m-1)L, K + mL]\) for \(1 \leq m \leq MN\) where \(L = K/M\) and \(M\) is chosen large enough so that \((1+3\epsilon)M/(M+1) \geq 1+2\epsilon\), e.g., \(M = 1 + (1/\epsilon)\).

If \(i \in I_k\) and \(j \in I_\ell\) we let

\[
r(i, j) = \min\{q(x, y) : x \in I_k, y \in I_\ell\}
\]

Until some \(I_m\) has at least \(\epsilon L\) particles, the growth of the cluster dominates a supercritical branching process with \(MN\) types. The mean matrix has for \(1 \leq k \leq \ell \leq MN\) \((M, N\) only depend on \(\epsilon\))

\[
m(k, \ell) = \frac{2\delta L}{K + \ell L} = \frac{2\delta}{M + \ell}
\]

and an offspring distribution that is asymptotically Poisson, so its survival probability is \(\geq \beta(M, N, \epsilon)\) for large \(K\). This proves the result with

\[
\gamma = \frac{\epsilon}{MN} = C\epsilon^2 e^{-3/\epsilon}
\]
Dorogovstev, Mendes, Samukhin.
Phys. Rev. E., 64, paper 066110

\[ S = 1 - g(1) \approx c \exp \left( -\frac{\pi}{\sqrt{8\delta - 1}} \right) \]

To derive this result they change variables
\[ u(\xi) = 1 - g(1 - \xi) \] to get

\[ u'(\xi) = \frac{1}{2\delta(1 - \xi)} \cdot \frac{u(\xi) - \xi}{u(\xi)} \]

They discard the \( 1 - \xi \) in the denominator and note that the solution to the differential equation is the solution of the following transcendental equation

\[
-\frac{1}{\sqrt{8\delta - 1}} \arctan \left( \frac{4\delta [u(\xi)/\xi] - 1}{\sqrt{8\delta - 1}} \right) - \ln \sqrt{\xi^2 - u(\xi)\xi + 2\delta u^2(\xi)}
= -\frac{\pi/2}{\sqrt{8\delta - 1}} - \ln \sqrt{2\delta} - \ln S
\]
Yu Zhang studied \( p_{i,j} = (1/4)/(i \lor j) \) on \{1, 2, \ldots\} in his 1990 Ph.D. thesis at Cornell written under the direction of Harry Kesten.

**Theorem.** If \( i < j \) and \( i \geq \log^{6+\delta} j \) then

\[
\frac{c_1 \log(i+1)}{\sqrt{ij}} \leq P(i \rightarrow j) \leq \frac{c_2 \log(i+1)}{\sqrt{ij}}
\]

By adapting his method to \{1, 2, \ldots n\} we can prove a similar result. The starting point is the fact that the expected number of self-avoiding paths from \( i \) to \( j \) is

\[
EV_{i,j} = \sum_{m=0}^{\infty} \sum_{*} h(i, z_1)h(z_1, z_2) \cdots h(z_m, j)
\]

where \( h(x, y) = (1/4)/(x \lor y) \) and the starred sum is over all self-avoiding paths.
\[ h(x, y) = (1/4)/(x \lor y) \]

The sum restricted to paths with all \( z_i \geq 2 \) has

\[ \Sigma_{i,j}^1 \leq \sum_{m=0}^{\infty} \int_1^n dx_1 \cdots \int_1^n dx_m h(i, x_1)h(x_1, x_2)\cdots h(x_m, j) \]

Introducing

\[ \pi(u, v) = e^{u/2} h(u, v) e^{v/2} = \begin{cases} \frac{1}{4}e^{(u-v)/2} & u \leq v \\ \frac{1}{4}e^{(v-u)/2} & u \geq v \end{cases} \]

and setting \( \log x_i = y_i \), \( dx_i = e^{y_i} dy_i \) we have

\[ \Sigma_{i,j}^1 \leq \frac{1}{\sqrt{ij}} G_{0, \log n}(\log i, \log j) \]

where \( G \) is the green’s function for the bilateral exponential random walk killed when it exits \([0, \log n]\).
Suppose the jump distribution is $(\lambda/2)e^{-\lambda|z|}$. Since boundary overshoots are exponential

$$P_x(T_{(-\infty,u]} < T_{[v,\infty)}) = \frac{(v + 1/\lambda) - x}{(v + 1/\lambda) - (u - 1/\lambda)}$$

the exit probability for Brownian motion from the interval $(u - 1/\lambda, v + 1\lambda)$. Using this formula you can show that for the case $\lambda = 1/2$.

$$G_{K,L}(x, z) = \left\{ \begin{array}{ll}
\frac{1}{4} \cdot \frac{(L-x+2)(z-K+2)}{L-K+4} & z \leq x \\
\frac{1}{4} \cdot \frac{(L-z+2)(y-K+2)}{L-K+4} & z \geq x 
\end{array} \right.$$  

If we discard the +2’s and +4’s this is exactly the formula for $\sqrt{8}B_t$. Taking $x = \log i$, $z = \log j$ and bounding the paths that visit 1

$$P(i \to j) \leq \frac{3}{8\sqrt{ij}} \frac{(\log i + 2)(\log n - \log j + 2)}{(\log n + 4)}$$

$$\frac{1}{n} \sum_{i=1}^{n} E|C_i| \leq 2 \sum_{i<j} RHS \leq 6$$
If \( \log(\kappa - 1) \geq 6 \) then for \( \kappa^2 \leq i < j \leq n \) we have
\[
EV_{i,j} \geq \frac{1}{8\sqrt{ij}} \left[ \frac{(\log i + 2)(\log n - \log j + 2)}{\log n + 4} - \frac{(\log \kappa)^3}{\kappa - 1} \right]
\]

If (i) \((2/\epsilon)^3 \log^6(4/\epsilon) \leq i < j \leq n^{1-\epsilon}, \epsilon \leq \epsilon_0\) or (ii) \((\log n)^3 \leq i < j \leq n, n \geq n_0\) then
\[
EV_{i,j} \geq \frac{1}{16\sqrt{ij}} \frac{(\log i + 2)(\log n - \log j + 2)}{\log n + 4}
\]

The final step is to show that if (i) or (ii) then

\[
EV_{i,j}^2 \leq c EV_{i,j}
\]

Cauchy-Schwarz inequality implies
\[
EV_{i,j} = E(V_{i,j}1\{V_{i,j} > 0\}) \leq \sqrt{E(V_{i,j}^2)P(V_{i,j} > 0)}
\]

and rearranging gives
\[
P(i \rightarrow j) = P(V_{i,j} > 0) \geq \frac{EV_{i,j}}{c}
\]
One can use Zhang’s technique in the subcritical case to get

\[ P(i \to j) \leq \frac{1}{\sqrt{i j}} G^{8 \delta}(i, j) \]

where \( G^{8 \delta} \) is the Green’s function for the bilateral exponent on \( \mathbb{R} \) killed on each step with probability \( 1 - 8 \delta \). Using Fourier transforms one can compute

\[ G^{8 \delta}(x, y) = \frac{2 \delta}{\sqrt{1 - 8 \delta}} e^{-r|x-y|} \]

where \( r = \sqrt{1 - 8 \delta}/2 \) which gives for \( i < j \)

\[ P(i \to j) \leq \frac{c}{i^{1/2-r} j^{1/2+r}} \]

This and previous results imply

\[ E|C_1| \leq \begin{cases} 
  cn^{(1-\sqrt{1-8\delta})/2} & 0 < \delta < 1/8 \\
  cn^{1/2}/\log n & \delta = 1/8
\end{cases} \]

**Problem.** Is \( |C_1| = O(E|C_1|) \)?
DMS studied the preferential attachment model in which one new vertex and an average of $\delta$ edges were added at each time and the probability of an edge from $i$ to $j$ is proportional to $(d_i + \alpha)(d_j + \alpha)$ where $d_k$ is the degree of $k$. The CHKNS model arises as the limit $\alpha \to \infty$. Taking this limit of the DMS results suggests that the probability a randomly chosen vertex belongs to a cluster of size $k$ has

$$b_k \sim \frac{2}{k^2 \ln k} \text{ if } \delta = 1/8$$

In the subcritical regime one has (see their (B16) and (B17) and not (21) which is wrong)

$$b_k \sim C_\delta k^{-2/(1-\sqrt{1-8\delta})} \text{ if } \delta < 1/8$$

**Theorem.** The formulas for $b_k$ hold for the Bernoulli and Poisson models (and model #3).
\[ u(y) = 1 - g(1 - y) \text{ and } u(y) = y(u'(0) - v(y)) \]

\[ u'(y) = \frac{1}{2\delta(1 - y)} \cdot \frac{u(y) - y}{u(y)} \]

\[ v'(y) = \frac{(1 - 4\delta u'(0))v(y) + 2\delta v(y)^2}{2\delta y(1 - y)(u'(0) - v(y))} \]
\[ + \frac{1}{1 - y} (u'(0) - v(y)) \]

\[ \approx \frac{av(y)}{y} + \frac{v(y)^2}{u'(0)y} + u'(0) \]

\[ a = \frac{1 - 4\delta u'(0)}{2\delta u'(0)} = \begin{cases} 0 & \text{if } \delta = 1/8 \\ < 1 & \text{if } \delta > 1/9 \end{cases} \]

\[ v(y) \sim \begin{cases} u'(0)/(-\log y) & \delta = 1/8 \\ y^a & \delta \in (1/9, 1/8) \\ u'(0)y(-\log y) & \delta = 1/9 \\ cy & \delta < 1/9 \end{cases} \]

where \[ c = 2\delta u'(0)^2/(1 - 6\delta u'(0)) \]
\[
\sum_{k} k^i b_k (1 - (1 - y)^{k-1}) = g'(1) - g'(1 - y)
\]

\[
= yv'(y) + v(y)
\]

\[
\sim \begin{cases} 
2 / \log(1/y) & \delta = 1/8 \\
(1 + a) y^a & 1/8 < \delta < 1/9
\end{cases}
\]

To check the guesses we note

\[
\sum_{k > 1/y} \frac{1}{k (\log k)^2} \approx \frac{1}{\log(1/y)}
\]

\[
\sum_{k > 1/y} k^{-\rho+1} \approx (1/y)^{2-\rho}
\]

so \( \rho = a + 2 = 1/2\delta u'(0) = 2/(1 - \sqrt{1 - 8\delta}) \).

If \( k < \frac{1}{2\delta u'(0)} - 1 \leq k + 1 \) write \((u(y) = 1 - g(1 - y))\)

\[
u(y) = -\sum_{i=1}^{k} c_i (-y)^i + (-y)^k v(y)
\]

and apply Flajolet and Odlyzko (1990)