3.4 Mixing Times

The upper bound on the second largest eigenvalue allows us to estimate the mixing time for a random walk on a random $r$-regular graph. The result we use to do this is from Sinclair and Jerrum (1989) but there are many others would also do the job. Consider a Markov chain $p(i, j)$ with reversible stationary distribution $\pi_i$, i.e., $\pi_ip(i, j) = \pi_jp(j, i)$. In the case of an $r$-regular graph $p(i, j) = e(i, j)/r$ where $e(i, j)$ is the number of edges connecting $i$ and $j$. Since $e(i, j) = e(j, i)$, the uniform of distribution $\pi_i = 1/n$ is a reversible stationary distribution.

To measure convergence to equilibrium we will use the relative pointwise distance

$$\Delta(t) = \max_{i,j} \frac{|p^t_{i,j} - \pi_j|}{\pi_j}$$

Here we take advantage of the fact that the stationary distribution is uniform to simplify the proof. In that case

$$\max_{i,j} \frac{|p^t_{i,j} - 1/n|}{1/n} \geq \max_i \sum_j |p^t_{i,j} - 1/n|$$

i.e., it bounds 2 times the maximum total variation distance of $p^t_{i,\cdot}$ from uniform.

Since $p(i, j)$ is symmetric, matrix theory tells us that there are real eigenvalues $1 = \lambda_0 \geq \lambda_1 \geq \ldots \lambda_{n-1} \geq -1$. Let $\lambda_{\max} = \max\{\lambda_1, |\lambda_{n-1}|\}$ be the eigenvalue with largest magnitude.

**Theorem 3.4.1.** Let $p$ be the transition matrix of an irreducible reversible Markov chain with $n$ state and uniform stationary distribution. Then

$$\Delta(t) \leq n\lambda_{\max}^t$$

**Proof.** Since $p$ is symmetric we can select an orthonormal basis $e_m$, $0 \leq m < n$ of right eigenvectors of $p$, and $p$ has spectral decomposition:

$$p = \sum_{m=0}^{n-1} \lambda_m e_m e^T_m$$

The matrix $A_m = e_m e^T_m$ has rank one, $A_m^2 = A_m$, and $A_mA_n = 0$ if $m \neq n$ so using $\mathbf{1}$ for an $n \times n$ matrix of 1’s.

$$p^t = \frac{1}{n} \mathbf{1} + \sum_{m=1}^{n-1} \lambda_m^t A_m$$

Introducing components we have

$$p^t_{i,j} - \frac{1}{n} = \sum_{m=1}^{n-1} \lambda_m^t e_m(i)e_m(j)$$
From this it follows that

\[
\Delta(t) = n \max_{i,j} \left| \sum_{m=1}^{n-1} \lambda_m^t e_m(i)e_m(j) \right| \leq n \lambda_{\max}^t \max_{i,j} \sum_{m=1}^{n-1} |e_m(i)||e_m(j)|
\]

The Cauchy-Schwarz inequality implies

\[
\sum_{m=1}^{n-1} |e_m(i)||e_m(j)| \leq \left( \sum_{m=1}^{n-1} |e_m(i)|^2 \right) \left( \sum_{m=1}^{n-1} |e_m(j)|^2 \right)^{1/2}
\]

To see that \(\sum_{m=1}^{n-1} |e_m(i)|^2 \leq 1\) note that if \(\delta_i\) is the vector with 1 in the \(i\)th place and 0 otherwise then expanding in the orthonormal basis \(\delta_i = \sum_{m=0}^{n-1} e_m(i) e_m\), so the desired result follows by taking the \(L^2\) norm of both sides of the equation.

Using this result with Friedman’s result

\[
\lambda_{\max} \leq \frac{2\sqrt{r-1} + \epsilon}{r}
\]

we see that if \(1 > a > 2\sqrt{r-1}/r\) then

\[
\Delta(b \log n) \leq n \lambda_{\max}^{b \log n} \leq n^{1+(b \log a)}
\]

which \(\to 0\) if \(b > -1/\log a\).

**Problem.** Arguments in the previous paragraph and our analysis of Molloy and Reed graphs, suggest that seen from a given point \(x\) then until we get too far away these graphs look like trees in which each vertex has \(r-1\) children. A random walk that moves +1 with probability \((r-1)/r\) and -1 with probability \(1/r\) has mean \((r-2)/r\). Since the average distance between two points in the graph is \(\log n/(\log(r-1))\), the mixing time cannot possibly be faster than

\[
\frac{r}{r-1} \cdot \frac{\log n}{\log(r-1)}
\]

could this be the right answer?

Cooper, C., and Frieze, A. The cover time of random regular graphs.