3.3. EIGENVALUES AND EXPANDERS

3.3 Eigenvalues and Expanders

We begin by proving a remarkable result about the distribution of eigenvalues of the adjacency matrix of a random graph due to McCay (1981). This applies not only to random graphs $G(n, r)$ but to a sequence of graphs $G_1, G_2, \ldots$ in which
(a) each vertex has degree $r$,
(b) the number of vertices $|G_i| \to \infty$,
(c) the number cycles of length $k \psi_k(G_i)/|G_i| \to 0$ for all $k$.

Results in section 3.1 show these conditions hold for $G_i$ chosen randomly from $G(i, r)$.

Consider the adjacency matrix $A(G_i)$ for $G$, i.e., $A(G_i)_{i,j} = 1$ if $i, j$ are adjacent in $G$.

If each vertex in $G$ has degree $r$ then rows of $A(G_i)$ sum to $r$, and $r$ is an eigenvalue with eigenvector $(1, 1, \ldots 1)$. Let $F(G, x)$ be the fraction of eigenvalues of $A(G)$ that are $\leq x$.

**Theorem 3.3.1.** Assume (a), (b), and (c). Then for every $x$, $F(G, x) \to F(x)$ where

$$F(x) = \begin{cases} 0 & \text{if } x \leq -2\sqrt{r-1} \\ 1 & \text{if } x \geq 2\sqrt{r-1} \\ \int_{-2\sqrt{r-1}}^{x} \frac{r\sqrt{4(r-1)-y^2}}{2\pi(r^2-y^2)} dy & \text{if } -2\sqrt{r-1} < x < 2\sqrt{r-1} \end{cases}$$

**Proof.** Let $v_0$ be a vertex of $G$. A closed walk of length $m$ is a sequence of adjacent vertices $v_0, v_1, \ldots, v_m$ with $v_m = v_0$.

**Lemma 3.3.2.** Suppose that the subgraph of vertices of $B$ within distance $m/2$ of $v_0$ is acyclic. Then $\theta(m)$, the number of closed walks of length $m$ is 0 if $m$ is odd. If $m = 2s$ is even then

$$\theta(2s) = \sum_{k=1}^{s} \binom{2s-k}{s} \frac{k}{2s-k} r^k (r-1)^{s-k}$$

**Proof.** According to Theorem 4 on page 90 of the third edition of Feller, volume 1, the probability a simple random walk path will its $k$th return to 0 at time $2s$ is

$$\binom{2s-k}{s} \frac{k}{2s-k} \cdot 2^{-2s+k}$$

Multiplying by $2^s$ and then dividing by $2^k$ will give the number of sequences of integers $\delta_i \geq 0$, $0 \leq i \leq 2s$ that have $|\delta_i - \delta_{i-1}| = 1$ for $1 \leq i \leq 2s$ and have $\delta_i = 0$ for $k$ values of $1 \leq i \leq 2s$. Here $\delta_i$ is the distance from $v_0$ after $i$ steps. In any path there will be $s$ steps that increase $\delta_i$, and $s$ that decrease $\delta_i$. Since the graph is a tree near $v_0$ we have 1 choice of step when $\delta_i$ decreases. $k$ of the increasing steps will occur when $\delta_i = 0$ and $0 \leq i < 2s$. In these cases we have $r$ choices. At the other $s-k$ increasing steps we have only $r-1$ choices. \[\square\]
Assumption (c), the number cycles of length $k$ $\psi_k(G_i)/|G_i| \to 0$ for all $k$, implies that if $W_r(G_i)$ is the number of closed walks of length $m$ in $G_i$ then

$$W_m(G_i)/|G_i| \to \theta(r)$$

Recalling some basic linear algebra, we see that

$$W_m(G_i) = \text{trace}(A(G_i)^m) = \sum \lambda_i^m$$

where the $\lambda_i$ are the eigenvalues of $A(G_i)$. This tells us that

$$\int x^m dF(G_i, x) \to \theta(m)$$

The final step is to verify that the $\theta(m)$ are the moments of the distribution given in the theorem. The reader is referred to McKay’s paper for these details. Since the limiting distribution is bounded we know that it is uniquely determined by its moments.

The largest eigenvalue of the adjacency matrix is $\lambda_0 = r$ corresponding to the eigenvector $(1, 1, \ldots, 1)$. MacKay’s result gives an obvious asymptotic lower bound on the second largest eigenvalue, $2\sqrt{r-1}$. However since it assigns mass $1/n$ to each eigenvalue and takes the limit of the distribution, it gives no upper bound. Joel Friedman has studied this problem for a different model. To construct a $2m$ regular graph on $n$ vertices, Friedman picks $m$ random permutations $\pi_i$, $1 \leq i \leq m$ and for $1 \leq j \leq n$ draws edges $(j, \pi_i(j))$ and $(j, \pi_i^{-1}(j))$ (unless $\pi_i(j) = j$ and then we get one self-loop). To see this is not exactly the same consider the case $m = 1$, i.e., $r = 2$, and recall our remark that in $G^*(2, r)$ the asymptotic number of self-loops will be Poisson(1/2) compared to Poisson(1) for Friedman’s model.

Friedman (1991) proved that

$$\lambda_1 \leq 2\sqrt{r-1} + 2 \log r + O(1)$$

More recently, Friedman (2004) has shown that for any $\epsilon > 0$

$$P(\lambda_1 \leq 2\sqrt{r-1} + \epsilon) \to 1$$

The number $2\sqrt{r-1}$ is special. A theorem of Alon and Bopanna (see Lubotzky, Phillips, and Sarnak (1988) for a proof) asserts that for any sequence of $r$-regular graphs $G_n$ with $|G_n| = n$

$$\lim_{n \to \infty} \lambda_1(G_n) \geq 2\sqrt{r-1}$$

A. Nilli (1991), who is also known as N. Alon, proved the following sharper version

**Theorem 3.3.3.** Suppose that $G$ is an $r$-regular graph. Assume that the diameter of $G$ is $\geq 2b + 2 \geq 4$ then

$$\lambda_1(G) \geq 2\sqrt{r-1}\left(1 - \frac{1}{b}\right) - 1/b$$
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An $r$-regular graph $G$ is said to be Ramanujan if $\lambda_1(G) \leq 2\sqrt{r-1}$. Lubotzky, Phillips, and Sarnak (1988) constructed a large explicit family of $r$-regular graphs with this property. The reason for interest in these graphs is that they are expanders. A simple argument (see e.g., Ram Murty (2003) page 16) shows that for an $r$-regular graph $G$ if $|\partial A|$ is the number of edges that connect sites in $A$ to sites in $A^c$ then

$$\frac{|\partial A|}{|A|} \geq (r - \lambda_1(G)) \frac{|A^c|}{|A|}$$

Expanders and their bipartite cousins (see Alon (1986)) have a number of applications to the design of efficient communication networks, construction of codes. See Reingold, Vadhan, and Widgerson (2002) and references therein.


