2.2 Chung-Lu model

This model is specified by a collection of weights $\mathbf{w} = (w_1, \ldots, w_n)$ that represent the expected degree sequence. The probability of an edge between i to j is $w_i w_j / \sum_k w_k$. They allow loops from i to i so that the expected degree at i is

$$\sum_j \frac{w_i w_j}{\sum_k w_k} = w_i$$

Of course for this to make sense $\max_i w_i^2 < \sum_k w_k$.

In the power law model the probability of having degree k is $p_k = k^{-\beta}/\zeta(\beta)$ where $\zeta(\beta) = \sum_{k=1}^{\infty} k^{-\beta}$. The probability of having degree $\geq K$ is $\sim BK^{-\beta+1}$ where $1/B = (\beta-1)\zeta(\beta)$. Thinking of the wieghts as being decreaasing we have

$$w_i = K \quad \text{when} \quad i/n = BK^{-\beta+1}$$

Solving gives

$$w_i = (i/nB)^{-1/(\beta-1)}$$

Setting $i_0 = 1$ on page 15881, changing their c to κ, and correcting a typo, Chung and Lu have for $1 \leq i \leq n$

$$w_i = \kappa i^{-1/(\beta-1)} \quad \text{where } \kappa = \frac{\beta-2}{\beta-1} d n^{1/(\beta-1)}$$

and $d = (1/n)\sum_i w_i$ is the average degree. When $\beta > 2$, the distribution of the degrees has finite mean, d will converge to a limit, so apart from constants our two models are the same.

Let $\bar{d} = \sum_i w_i^2 / \sum_k w_k$ be the second order average degree. If vertices are chosen proportional to their weights, i.e., i is chosen with probability $v_i = w_i / \sum_k w_k$ then the chosen vertex will have mean size $\bar{d}$ so that quantity is the analogue of ν for Molloy-Reed graphs. When $\beta > 3$, using Chung and Lu's definitions of the weights

$$\begin{aligned} \sum_i w_i &\sim \kappa \sum_{i=1}^n i^{-1/(\beta-1)} \sim \kappa \frac{\beta-1}{\beta-2} n^{(\beta-2)/(\beta-1)} \\ \sum_i w_i^2 &\sim \kappa^2 \sum_{i=1}^n i^{-2/(\beta-1)} \sim \kappa^2 \frac{\beta-1}{\beta-3} n^{(\beta-3)/(\beta-1)} \end{aligned} \tag{2.2.1}$$

so we have

$$\frac{\sum_i w_i^2}{\sum_i w_i} \sim \kappa \frac{\beta-2}{\beta-3} n^{-1/(\beta-1)} = d \frac{(\beta-2)^2}{(\beta-1)(\beta-3)}$$

When $\beta = 3$ the asymptotics for $\sum_i w_i$ are once again the same but $\sum_i w_i^2 \sim \kappa^2 \log n$ so

$$\frac{\sum_i w_i^2}{\sum_i w_i} \sim \kappa \log n \cdot \frac{1}{2} n^{-1/2} = \frac{1}{4} d \log n \tag{2.2.2}$$

When $2 < \beta < 3$ the asymptotics for $\sum_i w_i$ stay the same but $\sum_i w_i^2 \sim \kappa^2 \sum_{i=1}^\infty i^{-2/(\beta-1)}$ so we have

$$\frac{\sum_i w_i^2}{\sum_i w_i} \sim \kappa \sum_{i=1}^\infty i^{-2/(\beta-1)} \cdot \frac{\beta-2}{\beta-1} n^{-(\beta-2)/(\beta-1)} = \zeta(2/(\beta-1)) \frac{\beta-2}{\beta-1} n^{(3-\beta)/(\beta-1)} \tag{2.2.3}$$

The formulas in last two cases look different from those in Chung and Lu (2002)since they write the answer in terms of the maximum degree $m = \kappa$ rather than in terms of n. However the qualitative behavior is the same. $\bar{d}$ grows like $\log n$ when $\beta = 3$ and like a positive power of n when $2 < \beta < 3$.

Chung and Lu prove results first under abstract conditions and then apply them to the power law example. The expected degree sequence $\mathbf{w}$ is said to be *strongly sparse* if

(i) the second order average degree satisifes $0 < (\log \bar{d}) \ll (\log n)$

(ii) for some constant $c > 0$, all but $o(n)$ vertices have $w_i \leq c$

(iii) there is an $\epsilon > 0$ so that the average expected degree sequence $d \geq 1 + \epsilon$

To state the final condition we need some more notation. Given a set of vertices $S \subset G$

$$\text{vol}_k(S) = \sum_{i \in S} w_i^k$$

When $k = 1$ we drop the subcript and call $\text{vol}(S)$ the volume. The final condition is

(iv) We say that $G(w)$ is *admissible* if there is a subset U satisfying

$$\text{vol}_2(U) = (1 + o(1))\text{vol}_2(G) \gg \text{vol}_3(U) \frac{\log \bar{d}(\log \log n)}{\bar{d} \log n}$$

Theorem 2.2.1. *For a strongly sparse random graph with admissible expected degree sequence the average distance is almost surely* $(1 + o(1)) \log n/(\log \bar{d})$

It is easy to see that our power law graphs with $\beta > 3$ satisfy (i) and (ii). Chung and Lu sketch a proof that it also satisfies (iv) so we have

Theorem 2.2.2. *For a power law graph with exponent* $\beta > 3$ *and average degree* $d > 1$ *then the average distance is almost surely* $(1 + o(1)) \log n/(\log \bar{d})$

Note that the condition is on d rather than $\bar{d}$.

For smaller values of β they have the following.

Theorem 2.2.3. *Suppose a power law graph with exponent* $2 < \beta < 3$ *has average degree* $d > 1$ *and maximum degree*

$$\log m \gg (\log n)/ \log \log n$$

Then the average distance is at most

$$(2 + o(1)) \frac{\log \log n}{\log(1/(\beta - 2))}$$

while the diameter is $\Theta(\log n)$.

To explain the intuition behind this we return to Molloy and Reed model. Recall $p_k = k^{-\beta}/\zeta(\beta)$ while $q_{k-1} = kp_k/\mu = k^{1-\beta}/\zeta(\beta-1)$. The tail of the distribution

$$Q_K = \sum_{k=K}^{\infty} q_k \sim \frac{1}{\zeta(\beta-1)(\beta-2)} k^{2-\beta} \tag{2.2.4}$$

The power $0 < \beta - 2 < 1$, and q_k is concentrated on the nonnegative integers so q_k is in the domain of attraction of a one sided stable law with index $\alpha = \beta - 2$. To explain this let $X_1, X_2, \ldots$ be i.i.d. with distribution q_k and let $S_n = X_1 + \cdots + X_n$.

To understand how S_n behaves, for $0 < a < b < \infty$, let

$$N_n(a,b) = |\{m \le n : X_m/n^{1/\alpha} \in (a,b)\}|$$

Let $B_\alpha = 1/\{\zeta(\beta-1)(2-\beta)\}$. For each m the probability $X_m \in (an^{1/\alpha}, bn^{1/\alpha})$ is

$$\sim \frac{1}{n} B_\alpha (a^{-\alpha} - b^{-\alpha})$$

Since the X_m are independent, $N_n(a,b) \Rightarrow N(a,b)$ has a Poisson distribution with mean

$$B_\alpha (a^{-\alpha} - b^{-\alpha}) = \int_a^b \frac{\alpha B_\alpha}{x^{\alpha+1}}\, dx \tag{2.2.5}$$

If we interpret $N(a,b)$ as the number of points in (a,b) the limit is a Poisson process on $(0,\infty)$ with intensity $\alpha B_\alpha x^{-(\alpha+1)}$. There are finitely many points in (a,∞) for $a > 0$ but infinitely many in $(0,\infty)$.

The last paragraph describes the limiting behavior of the random set

$$\mathcal{X}_n = \{X_m/n^{1/\alpha} : 1 \le m \le n\}$$

To describe the limit of $S_n/n^{1/\alpha}$, we will "sum up the points." Let $\epsilon > 0$ and

$$I_n(\epsilon) = \{m \le n : X_m > \epsilon n^{1/\alpha}\}$$
$$\hat{S}_n(\epsilon) = \sum_{m \in I_n(\epsilon)} X_m \qquad \bar{S}_n(\epsilon) = S_n - \hat{S}_n(\epsilon)$$

$I_n(\epsilon) =$ the indices of the "big terms," i.e., those $> \epsilon n^{1/\alpha}$ in magnitude. $\hat{S}_n(\epsilon)$ is the sum of the big terms, and $\bar{S}_n(\epsilon)$ is the rest of the sum.

The first thing we will do is show that the contribution of $\bar{S}_n(\epsilon)$ is small if ϵ is. To do this we note that

$$E\left(\frac{X_m}{n^{1/\alpha}}; X_m \le \epsilon n^{1/\alpha}\right) = \sum_{k=1}^{\epsilon n^{1/\alpha}} \frac{k^{2-\beta}}{n^{1\alpha}} \sim \frac{\epsilon^{3-\beta}(n^{1/\alpha})^{2-\beta}}{3-\beta}$$

Since $\beta - 2 = \alpha$ multiplying on each side by n gives

$$E(\bar{S}_n(\epsilon)/n^{1/\alpha}) \to \epsilon^{3-\beta}/(3-\beta) \tag{2.2.6}$$

If $Z = Poisson(\lambda)$ then

$$E(\exp(itaZ)) = \sum_{k=0}^{\infty} e^{-\lambda} \frac{e^{itak}\lambda^k}{k!} = \exp(\lambda(e^{ita} - 1))$$

Dividing (ϵ, ∞) into small strips, using independence of the number of points in different strips, and passing to the limit gives

$$E \exp(it\hat{S}_n(\epsilon)/n^{1/\alpha}) \to \exp\left(\int_\epsilon^\infty (e^{itx} - 1)\frac{\alpha B_\alpha}{x^{\alpha+1}}\, dx\right) \tag{2.2.7}$$

Now $e^{itx} - 1 \sim itx$ as $t \to 0$ and $\alpha < 1$ so combining (2.2.6) and (2.2.7) and letting $\epsilon \to 0$ slowly (see (7.6) in Chapter 2 of Durrett (2004) for more details) we have

$$E(\exp(it\hat{S}_n/n^{1/\alpha}) \to \exp\left(\int_\epsilon^\infty (e^{itx} - 1)\frac{\alpha B_\alpha}{x^{\alpha+1}}\, dx\right)$$

This shows $S_n/n^{1/\alpha}$ has a limit. The limit is the one-sided stable law with index α, which we will denote by Γ_α

Branching process. This proof comes from

Davies, P.L. (1978) The simple branching process: a note on convergence when the mean is infinite. *J. Appl. Prob.* **15**, 466–480

Theorem 2.2.4. *Consider a branching process with offspring distribution ξ with $P(\xi > k) \sim B_\alpha k^{-\alpha}$ where $\alpha = \beta - 2 \in (0,1)$. As $n \to \infty$, $\alpha^n(\log Z_n + 1) \to W$ with $P(W = 0) = 0$ the extinction probability for the branching process.*

Proof. Now if $Z_n > 0$ then

$$Z_{n+1} = \sum_{i=1}^{Z_n} \xi_{n,i}$$

where the $\xi_{n,i}$ are indepndent and have thse same distribution as ξ. We can write

$$\log(Z_{n+1} + 1) = \frac{1}{\alpha}\log(Z_n + 1) + \log Y_n$$
$$\text{where} \quad Y_n = \left(1 + \sum_{i=1}^{Z_n} \xi_{n,i}\right) \Bigg/ (Z_n + 1)^{1/\alpha}$$

Multiplying each side by α^n and iterating we have

$$\alpha^n \log(Z_{n+1} + 1) = \log(Z_1 + 1) + \alpha^n \log Y_{n-1} + \cdots + \alpha \log(Y_1)$$

As $n \to \infty$, Y_n converges to Γ_α. Straightforward but somewhat tedious estimates on the tail of the distribution of Y_n show that, see pages 474–477 of Davies (1978),

$$E\left(\sum_{m=1}^{\infty} \alpha^n \log^+ Y_n < \infty\right) \quad \text{and} \quad E\left(\sum_{m=1}^{\infty} \alpha^n \log^- Y_n < \infty\right)$$

This shows that $\lim_{n\to\infty} \alpha^n \log(Z_{n+1}+1) = W$ exists.

It remains to show that the limit W is nontrivial. Davies has a complicated proof that involves getting upper and lower bounds on $1 - G_n(x)$ where G_n is the distribution of Z_n which allows him to conclude that if $J(x) = P(W \le x)$ then

$$\lim_{x\to\infty} \frac{-\log(1 - J(x))}{x} = 1$$

Problem. Find a simple proof that $P(W > 0) > 0$.

Once this is done it is reasonably straightforward to up grade the conclusion to $J(0) = q$, where q is the extinction probability. To do this we begin with the observation that

Lemma 2.2.5. *Consider a supercritical branching process with offspring distribution p_k and generating function ϕ. If we condition on nonextinction and look only at the individuals that have an infinite line of descent then the number of individuals in generation n, $\tilde{Z}_n$ is a branching process with offspring generating function*

$$\tilde{\phi}(z) = \frac{\phi((1-q)z+q)}{1-q}$$

where q is the extinction probability, i.e., the smallest solution of $\phi(q) = q$ in $[0,1]$.

Proof. There is nothing to prove if $q = 0$ so suppose $0 < q < 1$. If $Z_0 = 1$ and we condition on survival of the branching process then the number of individuals in the first generation who have an infinite line of descent has distribution

$$\tilde{p}_j = \frac{1}{1-q} \sum_{k=j}^{\infty} p_k \binom{k}{j} (1-q)^j q^{k-j}$$

Thus multiplying by z^j, summing, and interchanging the order of summation

$$\begin{aligned}
\sum_{j=1}^{\infty} \tilde{p}_j &= \frac{1}{1-q} \sum_{j=1}^{\infty} \sum_{k=j}^{\infty} p_k \binom{k}{j} (1-q)^j q^{k-j} z^j \\
&= \frac{1}{1-q} \sum_{k=1}^{\infty} p_k \sum_{j=1}^{k} \binom{k}{j} (1-q)^j z^j q^{k-j}
\end{aligned}$$

Using the binomial theorem and noticing that the $j = 0$ term is missing the above

$$= \frac{1}{1-q} \sum_{k=1}^{\infty} p_k \{((1-q)z + q)^k - q^k\}$$

We can add the $k = 0$ term to the sum since its value is 0. Having done this the result is

$$\frac{\phi((1-q)z + \phi(q))}{1-q}$$

Since $\phi(q) = q$ the result follows. □

It is easy to check that the new law is also in the domain of atttraction of the stable α. On $S = \{\omega : Z_n(\omega) > 0 \text{for all} n\}$. $Z_n^{(1)}$. By the definition of the process $n \to Z_n^{(1)}$ is nondecreasing. Wait until the time $N = \min\{n : Z_n^{(1)} > M\}$. In order for $\alpha^n \log(Z_n^{(1)}+1) \to 0$ this must occur for each of the M families at time N. However we have already shown that the probability of a positive limit is $\delta > 0$, so the probability all M fail is $(1-\delta)^M \to 0$ as $M \to \infty$. □

The double exponenetial growth of the branching process associated with the degree distribution $p_k = k^{-\beta}/\zeta(\beta)$ where $2 < \beta < 3$ suggests that the average distance between two members of the giant component will be $O(\log\log n)$. To get a constant we note that our limit theorem says

$$\log(Z_t + 1) \approx \alpha^{-t} W$$

so $Z_t + 1 \approx \exp(\alpha^{-t} W)$. Replacing $Z_t + 1$ by n and solving gives $\log n = \alpha^{-t} W$. Discarding the W and writing $\alpha^{-t} = \exp(-t \log \alpha)$ we get

$$\sim \frac{\log\log n}{\log(1/\alpha)} \tag{2.2.8}$$

This agrees with the Chung and Lu bound in Theorem 2.2.3 except for a factor of 2, but that factor may be necessary. Recall that in our analysis of the Erdös-Renyi case in order to connect two points x and y we grew their clusters until size $n^{2/3}$ and each of the growth processes will take the time given in (2.2.8).

Quest. Find a real proof of the folk theorem about the $\log\log n$ behavior for $2 < \beta < 3$.

A curious aspect of the problem that Gena pointed out is that in the independent power law model the largest degree is $O(k^{1/(\beta-1)})$ which is $n^{1-\epsilon}$ when $\beta = 1 + 1/(1-\epsilon)$.

In addition to the Chung and Lu PNAS paper are Cohen, R., and Havlin, S. (2003) Scale-free networks are ultra-small. *Phys. Rev. Letters.* **90**, paper 058701.

Reitu, H., and Norros, I. (2004) On the power-law random graph model of masssive data networks. *Performance Evaluation.* **55**, 3–23

Two new references that have recently came to my attention are:

Chung, Fan and Lu, Linyuan (2004) The average distance in a random graph with given expected degrees. *Internet Mathematics.* **1**, 91–114

Lu, L. (2002) Probabilistic graphs in Massive Graphs and Internet Computing." Ph.D. dissertation http://math.ucsd.edu/~llu/thesis.pdf

* * * * * * * * * *

Chung, F. and Lu, L. (2002) Connected components in random graphs with a given degree expected sequence . *Annals of Combinatorics* **6**, 125-145

has a nice result about the subcritical phase. Recall vol$S = \sum_{i \in S} w_i$.

Theorem 2.2.6. *If $\bar{d} < 1$ then all components have volume at most $C\sqrt{n}$ with probability at least*

$$1 - \frac{d\bar{d}^2}{C^2(1-\bar{d})}$$

Proof. Let x be the probability that there is a component with volme $> C\sqrt{n}$. Pick two vertices at random with probabilities proportional to their weights. If $\gamma = 1/\sum_i w_i$ then for each vertex, the probability it is in the component is $\geq C\sqrt{n}\gamma$. Therefore the probability a randomly chosen pair of vertices is in the same component is at least

$$x(C\sqrt{n}\gamma)^2 = C^2 x n \gamma^2 \tag{2.2.9}$$

On the other hand for a fixed pair of vertices u and v the probability $p_k(u,v)$ of u and v being connected by a path of length $k+1$ is a most

$$p_k(u,v) \leq \sum_{i_1,i_2,\ldots i_k} (w_u w_{i_1}\gamma)(w_{i_1} w_{i_2}\gamma)\cdots(w_{i_k} w_v \gamma) \leq w_u w_v \gamma (\bar{d})^k$$

Summing over $k \geq 0$ the probability u and v belong to the same component is at most

$$\frac{w_u w_v \gamma}{1-\bar{d}}$$

The probabilities of u and v being selected are $w_u\gamma$ and $w_v\gamma$. Summing over u and v the probability a randomly chosen pair of verties belong to the same component is at most

$$\frac{(\bar{d})^2\gamma}{1-\bar{d}}$$

Using this with (2.2.9)

$$C^2 x n \gamma^2 \leq \frac{(\bar{d})^2\gamma}{1-\bar{d}}$$

which implies

$$x \leq \frac{(\bar{d})^2}{C^2 n(1-\bar{d})\gamma} = \frac{d(\bar{d})^2}{C^2(1-\bar{d})}$$

since $\gamma = 1/\sum_i w_i$ and $d = (1/n)\sum_i w_i$ implies $n\gamma = 1/d$. □