Chapter 2

Arbitrary Degree Distributions

2.1 Proof of Phase Transition

In this case the existence of the giant component is actually easier than its nonexistence.

**Theorem 2.1.1.** Suppose \( \sum k(k - 1)p_k/\mu > 1 \). Then there is a giant component of size \( \sim G_0(\rho)n \), and no other clusters of larger than \( \beta \log n \).

**Proof.** To analyze the growth of the cluster, we will expose the cluster one vertex at a time. In the case of Erdős-Renyi random graphs, the main difference between this process and a random walk is that the size of the unexplored set decreases over time. In our new setting there are two addition differences. (i) For fixed \( n \), the empirical sequence of degrees \( d_1^n, \ldots, d_n^n \) is not the same as the i.i.d. sequence. (ii) The set of available degrees changes as choices are made.

The first issue is not a problem. As \( n \to \infty \) the empirical distribution of \( d_m^n \), \( 1 \leq m \leq n \) converges to \( p_k \) in distribution and in total variation norm. The strong law of large numbers implies that the empirical mean of the degree distribution

\[
\bar{\mu}_n = \frac{1}{n} \sum_{m=1}^{n} d_m^n \to \mu = \sum_k k p_k
\]

The empirical size biased distribution has

\[
\bar{q}_{k-1}^n = k|\{1 \leq x \leq n : d_x = k\}|/(n\bar{\mu}_n)
\]

and also converges to \( q_{k-1} \) in distribution and in total variation norm.

To estimate the change that occur when some of the vertices have been exposed, let \( r_k \) be a distribution on the positive integers, and let \( W_r(\omega) \) be an nondecreasing function of \( \omega \in (0, 1) \) so that the Lebesgue measure \( |\{\omega : W_r(\omega) = k\}| = r_k \). We call \( W_r \) the mass function for \( r \). If we remove an amount of mass \( \eta \) from the distribution and renormalize to get a probability distribution then the result will be larger in distribution than \( U = (W(\omega)|\omega < 1 - \eta) \).
From the last two observations, we see that if \( n \) is large and the fraction of vertices exposed is at most \( \eta \), then the cluster growth process dominates a random walk process in which steps (after the first one) have size \(-1\) plus a random variable with distribution \( q^n_k = d(W|W < 1 - 2\eta) \).

When we expose the neighbors of an active vertex, one of them might already be in the active set. We call such an event a collision. If a collision occurs in the Erdos-Renyi growth process we are disappointed not to get another vertex. However in the current situation, we must remove it from the active set, since the collision has reduced its degree to \( d_x - 2 \). To show that this does not slow down the branching process too much, we must bound the number of collisions. Note that \( q^n \) is concentrated on \( \{0, \ldots, L\} \) where \( L = W_q(1 - 2\eta) \). Thus until \( \delta n \) vertices have been exposed, the number of edges with an end in the active set is at most \( \delta n L \). The probability of picking one of these edges in the exposure of an active vertex is at most \( \delta n L / (t - \delta n L) = \gamma_t \), where \( t \) is the total number edges. Let \( Z \) have distribution \( q^n \). The change in the set of active sites correcting for collisions is bounded by 
\[
X = -1 + Z - 2 \cdot \text{Binomial}(\gamma_t, Z).
\]
Therefore if \( \delta < \eta \) is small, \( EX = -1 + a_\eta(1 - 2\gamma) > 0 \).

Define a random walk \( S_t = S_0 + X_1 + \cdots + X_t \) where \( S_0 \) is the number of neighbors of the site chosen and \( X_1, \ldots, X_t \) are independent copies of our lower bounding variable. Since \( EX > 0 \), the random walk has positive probability of not hitting 0, and there is positive probability that the cluster growth persists until there are at least \( \delta n \) vertices. To prove that we will get at least one such cluster with high probability, it is enough to show that with each unsuccessful attempts will with high probability use up at most \( O(\log n) \) vertices. For this guarantees that we will get a large number of independent trails before using a fraction \( \delta \) of the vertices.

The random variable \( X \) is bounded so \( \kappa(\theta) = E e^{\theta x} < \infty \) for all \( \theta \). \( \kappa(\theta) \) is convex, continuous and has \( \kappa'(0) = EX > 0 \), \( \kappa(\theta) \geq P(X = -1)e^{-\theta} \to +\infty \) as \( \theta \to -\infty \), so there is a unique \( \lambda > 0 \) so that \( \kappa(-\lambda) = 1 \). In this case \( E \exp(-\lambda S_k) \) is a nonnegative martingale. Due to the possible removal of active vertices, the random walk may jump down by more than 1, but its jumps are bounded so the optional stopping theorem implies that the probability of reaching 0 from \( S_0 = x \) is \( \leq e^{-\lambda x} \).

The last estimate implies that the probability that the set of active vertices grows to size \( (2/\lambda) \log n \) without generating a large cluster is \( \leq n^{-2} \). Routine large deviations estimates for sums of independent random variables show that if \( \beta \) is large, the probability that the sum of \( \beta \log n \) independent copies of \( X \) is \( \leq (2/\lambda) \log n \) is at most \( n^{-2} \). Thus the probability of exposing more than \( \beta \log n \) vertices and not generating a large cluster is \( \leq 2n^{-2} \).

At this point we can finish up as we did in the proof of 1.3.2. Two clusters that reach size \( \beta \log n \) will with high probability grow to size \( n^{2/3} \) and hence intersect with probability \( 1 - o(n^{-2}) \). The size of the giant component is determined by looking at the size of its complement. The events \( \{|C_8| \leq \beta \log n\} \) are almost independent. When the number of points is \( \leq \beta \log n \) the growth of the cluster can be approximated by the delayed branching process studied in the previous section.

What did Molloy and Reed do?
They used a different model in which one specifies \( v_i(n) \), the number vertices of degree \( i \). They assume

(i) \( v_i(n) \geq 0 \)
(ii) \( \sum_i v_i(n) = n \)
(iii) the degree sequence is feasible, i.e., \( \sum_i iv_i(n) \) is even
(iv) the degree sequence is smooth, i.e., \( \lim_{n \to \infty} v_i(n)/n = p_i \)
(v) the degree sequence is sparse if \( \sum_{i \geq 0} iv_i(n) \to \sum_i ip_i \).

Our random degree model has these properties with probability one. However, Molloy and Reed’s approach is more refined since it deals with individual sequence of degree sequences. Unfortunately it also needs more conditions on maximum degrees.

To state their main result (Theorem 1 on pages 164–165) recall that our condition for criticality is \( 1 = \sum (k-1)kp_k/\mu \). Multiplying each side by \( \mu \), recalling that \( \mu = \sum kp_k \), then subtracting gives

\[
Q \equiv \sum_k k(k-2)p_k = 0
\]

The final detail is that they use almost surely for “that with probability tending to 1 as \( n \to \infty \)”

**Theorem 2.1.2.** Suppose (i)-(v) hold, and that \( v_i(n) = 0 \) for \( i > n^{1/4} - \epsilon \).

a. If \( Q > 0 \) there are constants \( \zeta_1, \zeta_2 > 0 \) so that almost surely \( G \) has a component with at least \( \zeta_1 n \) vertices and \( \zeta_2 n \) cycles.

b. Suppose \( v_i(n) = 0 \) for \( i \geq w_n \) where \( w_n \leq n^{1/8} - \epsilon \). Then there is a \( \beta > 0 \) so that almost surely there is no cluster with \( \beta w_n^2 \log n \) vertices.

Molloy and Reed (1995) did not get the exact size of the giant component since they used their lower bound on the growth starting from one point until the cluster reached size \( \delta n \). In 1998 they found the exact size of the giant component and proved a ”duality result” for small clusters in the supercritical regime.
