1.4 CLT for the giant component

Up to this point we have been content to study the growth of clusters while they are $o(n)$. In this section we will use an approach of Martin-Löf (1986) to follow the random walk approach all of the way to the end of the formation of the giant component. To avoid the problem caused by the process dying out, it is convenient to modify the rules so that if $A_t = \emptyset$ we pick $i_t \in U_t$, and rewrite the recursion as

$$ R_{t+1} = R_t \cup \{i_t\} $$
$$ A_{t+1} = A_t - \{i_t\} \cup \{y \in U_t : \eta_{i_t,y} = 1\} $$
$$ U_{t+1} = U_t - (\{i_t\} \cup \{y \in U_t : \eta_{i_t,y} = 1\}) $$

When $A_t = \emptyset$ we subtract $1 + \text{Binomial}(|U_t|, \lambda/n)$ points from $U_t$ versus $\text{Binomial}(|U_t|, \lambda/n)$ points when $A_t \neq \emptyset$. However, we will experience only a geometrically distributed number of failures before finding the giant component, so this difference can be ignored.

Let $\mathcal{F}_t$ be the $\sigma$-field generated by the process up to time $t$. Let $u^n_t = |U_t|$ and $\Delta u^n_t = u^n_{t+1} - u^n_t$. If $A_t \neq \emptyset$ then

$$ E(\Delta u^n_t | \mathcal{F}_t) = -u^n_t \frac{\lambda}{n} $$
$$ \text{var}(\Delta u^n_t | \mathcal{F}_t) = u^n_t \frac{\lambda}{n} \left(1 - \frac{\lambda}{n}\right) $$

If we let $t = [ns]$ for $0 \leq s \leq 1$ and divide by $n$ then

$$ E\left(\frac{\Delta u^n_{[ns]}}{n} \bigg| \mathcal{F}_{[ns]}\right) = -\frac{u^n_{[ns]}}{n} \cdot \lambda \cdot \frac{1}{n} $$
$$ \text{var}\left(\frac{\Delta u^n_{[ns]}}{n} \bigg| \mathcal{F}_{[ns]}\right) = \frac{u^n_{[ns]}}{n} \cdot \lambda \left(1 - \frac{\lambda}{n}\right) \cdot \frac{1}{n^2} $$

Dividing each right-hand side by $1/n$, the time increment in the rescaled process, we see that $\Delta u^n_{[ns]}$ has

infinitesimal mean $= -\frac{u^n_{[ns]}}{n} \lambda$
infinitesimal variance $= \frac{u^n_{[ns]}}{n} \lambda \left(1 - \frac{\lambda}{n}\right) \cdot \frac{1}{n}$

Letting $n \to \infty$, the infinitesimal variance $\to 0$, so using (7.1) in Durrett (1996) it follows that $u^n_{[ns]}/n$ converges in distribution $u_s$ the solution of

$$ \frac{du_s}{ds} = -\lambda u_s \quad u_0 = 1 $$

and hence $u_s = \exp(-\lambda s)$. 

Scaling \( r^n_t = |R_t| \) as we did \( u^n_t, r^n_{[ns]} / n \to s \). [Here and in what follows we will use \( t \) for the original integer time scale and \( s \in [0,1] \) for rescaled time.] When \( u^n_t + r^n_t = n \), \( A_t = \emptyset \). Setting \( e^{-\lambda s} + s = 1 \) and solving we have \( 1 - s = \exp(\lambda((1 - s) - 1)) \), which is the fixed point equation for the extinction probability. This calculation tells us that we are interested only in \( u^n_{[ns]} / n \) for \( 0 \leq s \leq 1 - \rho \). In this part of the process we first generate a geometrically distributed number of small clusters and then expose the giant component.

Consider now \( y^n_{[ns]} = (u^n_{[ns]} - n \exp(-\lambda s)) / \sqrt{n} \) for \( 0 \leq s \leq 1 - \rho \). If \( A_{[ns]} \neq \emptyset \) then

\[
E(\Delta y^n_{[ns]} | F_{[ns]}) = -\frac{1}{\sqrt{n}} \left( -u^n_{[ns]} \cdot \frac{\lambda}{n} - n \exp(-\lambda s)(\exp(-\lambda/n) - 1) \right) \\
\sim -\frac{\lambda}{n} \left( \frac{u^n_{[ns]} - n \exp(-\lambda s)}{\sqrt{n}} \right) = -\lambda y^n_{[ns]} \cdot \frac{1}{n}
\]

\[
\var(\Delta y^n_{[ns]} | F_{[ns]}) = \var \left( \frac{\Delta u^n_{[ns]}}{\sqrt{n}} \middle| F_{[ns]} \right) = \frac{1}{n} \cdot u^n_{[ns]} \cdot \frac{\lambda}{n} \left( 1 - \frac{\lambda}{n} \right) \sim \lambda e^{-\lambda s} \cdot \frac{1}{n}
\]

Using (7.1) in Durrett (1996) again \( y^n_{[ns]} \) converges in distribution to the solution of the following stochastic differential equation.

\[
dy_s = -\lambda y_s \, ds + \sqrt{\lambda e^{-\lambda s}} \, dB_s \quad y_0 = 0
\]

The solution to this equation is

\[
y_s = \int_0^s \exp(-\lambda(s - r)) \sqrt{\lambda e^{-\lambda r}} \, dB_r \quad (1.4.1)
\]
To guess this, recall that if one invests an amount $g_s$ in an exponentially decaying stock market then your net wealth $x_s$ satisfies

$$\frac{dx_s}{ds} = -\lambda x_s + g_s$$

but since computation of interest is linear, each amount decays exponentially from its date of investment so this differential equation has solution

$$x_s = \int_0^s \exp(-\lambda(s - r))g_r dr$$

Alternatively one can use stochastic calculus to check this or note that since $\sqrt{\lambda e^{-\lambda t}}$ is deterministic then $y_t$ is a Gaussian process and you can check that the proposed answer has the right mean and covariance.

As just observed, the integrand is deterministic so $y_s$ has a normal distribution with mean 0 and variance

$$\int_0^t \exp(-2\lambda(s - r))\lambda e^{-\lambda r} dr = e^{-2\lambda s} - e^{\lambda s}$$

Taking $s = 1 - \rho$ and letting $Z$ denote a normal with variance $e^{-\lambda(1-\rho)} - e^{-2\lambda(1-\rho)}$ we have

$$u_{n[1-\rho]}^n \approx n \exp(-\lambda(1 - \rho)) + \sqrt{n}Z$$

$A_{[ns]} = \emptyset$ when $u_{[ns]}^n = n - [ns]$. To find out when this occurs, we take $s_0 = (1 - \rho) + \mathcal{Y}/\sqrt{n}$ in the previous equation. Using (1.4.2) and noting $s_0 \to (1 - \rho)$ as $n \to \infty$

$$n \exp(-\lambda[(1 - \rho) + \mathcal{Y}/\sqrt{n}]) + \sqrt{n}Z = u_{[ns_0]}^n = n - [ns_0] = n - n(1 - \rho) - \sqrt{n}\mathcal{Y}$$

or rearranging

$$\exp(-\lambda s_0) - 1 + s_0 = -\sqrt{Z}/\sqrt{n}$$

Let $h(t) = e^{-\lambda t} - 1 + t$ which is $= 0$ at $t = 1 - \rho$. $h'(t) = -\lambda e^{-\lambda t} + 1$ so we can write the above as

$$h'(1 - \rho)\mathcal{Y}/\sqrt{n} \approx -\sqrt{Z}/\sqrt{n}$$

or $\mathcal{Y} \approx Z/h'(1 - \rho)$. $h'(1 - \rho) = 1 - \lambda\rho$ so putting the pieces together.

**Theorem 1.4.1.** Suppose $\lambda > 1$. The size of the largest component $\mathcal{C}^{(1)}$ satisfies

$$\frac{|\mathcal{C}^{(1)}| - n(1 - \rho)}{\sqrt{n}} \Rightarrow \chi$$

where $\Rightarrow$ means convergence in distribution and $\chi$ has a normal distribution with mean 0 and variance $(\rho - \rho^2)/(1 - \lambda\rho)$. 
For other approaches to this result see Pittel (1990) and Barraez, Boucherno, and Fernandez de la Vega (2000). To compare variances note that Pittel’s \( c = \lambda \) and \( T = \rho / \lambda \).

