Chapter 1
Erdös-Renyi Random Graphs

Let $V = \{1, 2, \ldots, n\}$. For $1 \leq x < y \leq n$ let $\eta_{x,y}$ be independent $1$ with probability $p$ and $0$ otherwise. Let $\eta_{y,x} = \eta_{x,y}$. If $\eta_{x,y} = 1$ there is an edge from $x$ to $y$. Here we will be primarily concerned with situation $p = \lambda/n$ and in particular with showing that when $\lambda < 1$ most components are small, while for $\lambda > 1$ there is a giant component with $\sim g(\lambda)n$ vertices. The intuition behind this result is that a site has a Binomial($n-1, \lambda/n$) number of neighbors, which has mean $\approx \lambda$. Suppose we start with $I_0 = 1$, and for $n \geq 1$ let $I_n$ be the set of vertices not in $\bigcup_{m=0}^{n-1}I_m$ that are connected to some site in $I_{n-1}$ then when $n$ is not too large the number of points in $I_n$, $Z_n = |I_n|$ is branching process in which each individual in generation $n$ has an average of $\lambda$ children. The first section defines a branching process and gives its basic properties.

1.1 Branching Processes

Let $\xi_n^i, i, n \geq 0$, be i.i.d. nonnegative integer-valued random variables. Define a sequence $Z_n, n \geq 0$ by $Z_0 = 1$ and

$$Z_{n+1} = \left\{ \begin{array}{ll} \xi_1^{n+1} + \cdots + \xi_{Z_n}^{n+1} & \text{if } Z_n > 0 \\ 0 & \text{if } Z_n = 0 \end{array} \right.$$  

$Z_n$ is called a Galton-Watson process. The idea behind the definitions is that $Z_n$ is the number of people in the $n$th generation, and each member of the $n$th generation gives birth independently to an identically distributed number of children. $p_k = P(\xi_i^m = k)$ is called the offspring distribution.

Lemma 1.1.1. Let $\mathcal{F}_n = \sigma(\xi_i^m : i \geq 1, 1 \leq m \leq n)$ and $\mu = E\xi_i^m \in (0, \infty)$. Then $Z_n/\mu^n$ is a martingale w.r.t. $\mathcal{F}_n$.

Proof. Clearly, $Z_n \in \mathcal{F}_n$. Recall, see Exercise 1.1 in Chapter 4 of Durrett (2004), that if $X = Y$ on $B \in \mathcal{F}$ then $E(X|\mathcal{F} = E(Y|\mathcal{F})$ on $B$. On $\{Z_n = k\}$,

$$E(Z_{n+1}|\mathcal{F}_n) = E(\xi_1^{n+1} + \cdots + \xi_k^{n+1}|\mathcal{F}_n) = k\mu = \mu Z_n$$
Theorem 1.1.2. If \( \mu < 1 \) then \( Z_n = 0 \) for all \( n \) sufficiently large.

Proof. \( E(Z_n/\mu^n) = E(Z_0) = 1 \), so \( E(Z_n) = \mu^n \). Now \( Z_n \geq 1 \) on \( \{Z_n > 0\} \) so

\[
P(Z_n > 0) \leq E(Z_n; Z_n > 0) = E(Z_n) = \mu^n \to 0
\]

exponentially fast if \( \mu < 1 \). \( \square \)

The last answer should be intuitive. If each individual on the average gives birth to less than one child, the species will die out. The next result shows that after we exclude the trivial case in which each individual has exactly one child, the same result holds when \( \mu = 1 \).

Theorem 1.1.3. If \( \mu = 1 \) and \( P(\xi_i^m = 1) < 1 \) then \( Z_n = 0 \) for all \( n \) sufficiently large.

Proof. When \( \mu = 1 \), \( Z_n \) is itself a nonnegative martingale, so the martingale convergence theorem, (2.11) in Chapter 4 of Durrett (2004) converges to an a.s. finite limit \( Z_\infty \). Since \( Z_n \) is integer valued , we must have \( Z_n = Z_\infty \) for large \( n \). If \( P(\xi_i^m = 1) < 1 \) and \( k > 0 \) then \( P(Z_n = k \text{ for all } n \geq N) = 0 \) for any \( N \), so we must have \( Z_\infty = 0 \). \( \square \)

Theorem 1.1.4. If \( \mu > 1 \) then \( P(Z_n > 0 \text{ for all } n) > 0 \).

Proof. For \( s \in [0, 1] \), let \( \phi(s) = \sum_{k \geq 0} p_k s^k \) where \( p_k = P(\xi_i^m = k) \). \( \phi \) is the generating function for the offspring distribution \( p_k \). Differentiating gives for \( s < 1 \)

\[
\phi'(s) = \sum_{k=1}^{\infty} kp_k s^{k-1} \geq 0
\]

\[
\phi''(s) = \sum_{k=2}^{\infty} k(k-1)p_k s^{k-2} \geq 0
\]

So \( \phi \) is increasing and convex, and \( \lim_{s \uparrow 1} \phi'(s) = \sum_{k=1}^{\infty} kp_k = \mu \). Our interest in \( \phi \) stems from the following facts.

(a) If \( \theta_m = P(Z_m = 0) \) then \( \theta_m = \sum_{k=0}^{\infty} p_k \theta_{m-1}^k \).

Proof of (a). If \( Z_1 = k \), an event with probability \( p_k \), then \( Z_m = 0 \) if and only if all \( k \) families die out in the remaining \( m - 1 \) units of time, an independent event with probability \( \theta_{m-1}^k \). Summing over the disjoint possibilities for each \( k \) gives the desired result. \( \square \)

(b) If \( \phi'(1) = \mu > 1 \) there is a unique \( \rho < 1 \) so that \( \phi(\rho) = \rho \).

Proof of (b). \( \phi(0) \geq 0, \phi(1) = 1, \) and \( \phi'(1) > 1, \) so \( \phi(1 - \epsilon) < 1 - \epsilon \) for small \( \epsilon \). The last two observations imply the existence of a fixed point. To see it is unique, observe that \( \mu > 1 \) implies \( p_k > 0 \) for some \( k > 1 \), so \( \phi''(\theta) > 0 \) for \( \theta > 0 \). Since \( \phi \) is strictly convex, it follows that if \( \rho < 1 \) is the smallest fixed point, then \( \phi(x) < x \) for \( x \in (\rho, 1) \). \( \square \)
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(c) As \( m \uparrow \infty \), \( \theta_m \uparrow \rho \).

Proof of (c). \( \theta_0 = 0 \), \( \phi(\rho) = \rho \), and \( \phi \) is increasing, so induction implies \( \theta_m \) is increasing and \( \theta_n \leq \rho \). Let \( \theta_\infty = \lim \theta_m \). Taking limits in \( \theta_m = \phi(\theta_{m-1}) \), we see \( \theta_\infty = \phi(\theta_\infty) \). Since \( \theta_\infty \leq \rho \), it follows that \( \theta_\infty = \rho \).

Combining (a)–(c) shows \( P(Z_n = 0 \text{ for some } n) = \lim_{n \to \infty} \theta_n = \rho < 1 \).

Example. Consider the Poisson distribution with mean \( \lambda \), i.e,

\[
P(\xi = k) = e^{-\lambda} \frac{\lambda^k}{k!}
\]

In this case \( \phi(s) = \sum_{k=0}^{\infty} e^{-\lambda s^k} \lambda^k/k! = \exp(\lambda(s - 1)) \) so the fixed point equation is

\[
\rho = \exp(\lambda(\rho - 1)) \tag{1.1.1}
\]

Theorem 1.1.4 shows that when \( \mu > 1 \), the limit of \( Z_n/\mu^n \) has a chance of being nonzero.

The best result on this question is due to Kesten and Stigum:

Theorem 1.1.5. Suppose \( \mu > 1 \). \( W = \lim Z_n/\mu^n \) is not \( \equiv 0 \) if and only if \( \sum p_k k \log k < \infty \).

For a proof, see Athreya and Ney (1972), p. 24–29. We will now show

Theorem 1.1.6. If \( \mu = E(\xi_m^2) > 1 \) and \( \sum k^2 p_k < \infty \) then \( W = \lim Z_n/\mu^n \) is not \( \equiv 0 \).

Proof. Let \( \sigma^2 = \text{var}(\xi_m^2) \). Let \( X_n = Z_n/\mu^n \).

\[
E(X_n^2|\mathcal{F}_{n-1}) = X_{n-1}^2 + 2X_{n-1}E(X_n - X_{n-1}|\mathcal{F}_{n-1}) + E((X_n - X_{n-1})^2|\mathcal{F}_{n-1}) = X_{n-1}^2 + E((X_n - X_{n-1})^2|\mathcal{F}_{n-1}) \tag{1.1.2}
\]

since \( X_n \) is a martingale. To compute the second term, we observe

\[
E((X_n - X_{n-1})^2|\mathcal{F}_{n-1}) = E((Z_n/\mu^n - Z_{n-1}/\mu^{n-1})^2|\mathcal{F}_{n-1}) = \mu^{-2n} E((Z_n - Z_{n-1})^2|\mathcal{F}_{n-1}) \tag{1.1.3}
\]

On \( \{Z_{n-1} = k\} \),

\[
E((Z_n - \mu Z_{n-1})^2|\mathcal{F}_{n-1}) = E\left(\left(\sum_{i=1}^{k} \xi_i^m - \mu k\right)^2|\mathcal{F}_{n-1}\right) = k\sigma^2 = Z_{n-1}\sigma^2
\]

Combining the last three equations gives

\[
EX_n^2 = EX_{n-1}^2 + E(Z_{n-1}\sigma^2/\mu^{2n}) = EX_{n-1}^2 + \sigma^2/\mu^{n+1}
\]
since \( E(Z_{n-1}/\mu^{n-1}) = EZ_0 = 1 \). Now \( EX_0^2 = 1 \), so \( EX_1^2 = 1 + \sigma^2/\mu^2 \), and induction gives

\[
EX_n^2 = 1 + \sigma^2 \sum_{k=2}^{n+1} \mu^{-k} \tag{1.1.4}
\]

This shows \( \sup EX_n^2 < \infty \), so by the \( L^2 \) convergence theorem for martingales, (4.5) in Chapter 4 of Durrett (2004), \( X_n \to W \) in \( L^2 \), and hence \( EX_n \to EW \). \( EX_n = 1 \) for all \( n \), so \( EW = 1 \) and \( W \) is not \( \equiv 0 \).

Our next result shows that when \( W \) is not \( \equiv 0 \) it is positive on the set where the branching process does not die out.

**Theorem 1.1.7.** If \( P(W = 0) < 1 \) then \( \{W > 0\} = \{Z_n > 0 \text{ for all } n\} \).

**Proof.** Let \( \rho = P(W = 0) \). In order for \( Z_n/\mu^n \) to converge to 0 this must also hold for the branching process started by each of the children in the first generation. Breaking things down according to the number of children in the first generation

\[
\rho = \sum_{k=0}^{\infty} p_k \rho^k = \phi(\rho)
\]

so \( \rho < 1 \) is a fixed point of the generating function and hence \( \rho = P(Z_n = 0 \text{ for some } n) \). Clearly, \( \{W > 0\} \subset \{Z_n > 0 \text{ for all } n\} \). Since the two sets have the same probability \( P(\{Z_n > 0 \text{ for all } n\} - \{W > 0\}) = 0 \), which is the desired result.

Our final question: What happens in a supercritical branching process when it dies out?

**Theorem 1.1.8.** A supercritical branching process conditioned to become extinct is a subcritical branching process. If the original offspring distribution is Poisson(\( \lambda \)) with \( \lambda > 1 \) then the conditioned one is Poisson(\( \lambda \rho \)) where \( \rho \) is the extinction probability.

**Proof.** Let \( T_0 = \inf\{n : Z_n = 0\} \) and consider \( \bar{Z}_n = (Z_n | T_0 < \infty) \). To check the Markov property for \( \bar{Z}_n \) note that the Markov property for \( Z_n \) implies:

\[
P(Z_{n+1} = z_{n+1}, T_0 < \infty | Z_n = z_n, \ldots Z_0 = z_0) = P(Z_{n+1} = z_{n+1}, T_0 < \infty | Z_n = z_n, \ldots Z_0 = z_0)
\]

To compute the transition probability for \( \bar{Z}_n \), observe that if \( \rho \) is the extinction probability then \( P_x(T_0 < \infty) = \rho^x \). Let \( p(x, y) \) be the transition probability for \( Z_n \). we note that the Markov property implies

\[
\bar{p}(x, y) = \frac{P_x(Z_1 = y, T_0 < \infty)}{P_x(T_0 < \infty)} = \frac{P_x(Z_1 = y)P_y(T_0 < \infty)}{P_x(T_0 < \infty)} = \frac{p(x, y)\rho^y}{\rho^x}
\]

Taking \( x = 1 \) and computing the generating function

\[
\sum_{y=0}^{\infty} \bar{p}(1, y)s^y = \rho^{-1} \sum_{y=0}^{\infty} p(1, y)(sp)^y = \rho^{-1}\phi(sp)
\]
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where \( p_y = p(1, y) \) is the offspring distribution. In words, we take the graph of \( \phi \) up to the fix point and rescale to make the domain and range \([0, 1]\).

\( \bar{p}_y = \bar{p}(1, y) \) is the distribution of the size of the family of an individual, conditioned on the branching process dying out. If we start with \( x \) individuals then in \( Z_n \) each gives rise to an independent family. In \( \bar{Z}_n \) each family must die out, so \( \bar{Z}_n \) is a branching process with offspring distribution \( \bar{p}(1, y) \). To prove this formally observe that

\[
p(x, y) = \sum_{j_1 \ldots j_x \geq 0, j_1 + \cdots + j_x = y} p_{j_1} \cdots p_{j_x}
\]

Writing \( \sum' \) as shorthand for the sum in the last display

\[
\frac{p(x, y) \rho^y}{\rho^x} = \sum' \frac{p_{j_1} \rho^{j_1}}{\rho} \cdots \frac{p_{j_x} \rho^{j_x}}{\rho} = \sum' \bar{p}_{j_1} \cdots \bar{p}_{j_x}
\]

In the case of the Poisson distribution \( \phi(s) = \exp(\lambda(s - 1)) \) so if \( \lambda > 1 \), using the fixed point equation (1.1.1)

\[
\frac{\phi(s \rho)}{\rho} = \frac{\exp(\lambda(s \rho - 1))}{\exp(\lambda(\rho - 1))} = \exp(\lambda \rho(s - 1))
\]

which completes the proof. \( \square \)