The Contact Process with Fast Voting

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Abstract

Consider a combination of the contact process and the voter model in which deaths occur at rate 1 per site, and across each edge between nearest neighbors births occur at rate λ and voting events occur at rate θ. We are interested in the asymptotics as $\theta \to \infty$ of the critical value $\lambda_c(\theta)$ for the existence of a nontrivial stationary distribution. In $d \geq 3$, $\lambda_c(\theta) \to 1/(2d\rho_d)$ where ρ_d is the probability a d dimensional simple random walk does not return to its starting point. In $d = 2$, $\lambda_c(\theta)/\log(\theta) \to 1/4\pi$, while in $d = 1$, $\lambda_c(\theta)/\theta^{1/2}$ has $\liminf \geq 1/\sqrt{2}$ and $\limsup < \infty$. The lower bound might be the right answer, but proving this, or even getting a reasonable upper bound, seems to be a difficult problem.

1 Introduction

In this paper we consider a particle system on $\mathbb{Z}^d$ that is a contact process plus a voter model run at a fast rate θ. To define the process precisely, if $n_i(x)$ is the number of nearest neighbors of x in state i then x changes from $0 \to 1$ at rate $(\lambda + \theta)n_1(x)$ and from $1 \to 0$ at rate $\delta + \theta n_0(x)$. The behavior of the contact process is essentially the same in all dimensions. There is a critical value λ_c (depending on dimension) of the infection parameter λ with the property that the system has a nontrivial stationary distribution for $\lambda > \lambda_c$ and does not for smaller λ. The voter model, on the other hand, displays a strong dimension dependence. For $d \leq 2$, the only extremal invariant measures are the pointmasses on "all zeros" and "all ones". If $d \geq 3$, however, the extremal invariant measures ν_ρ^{vm} are indexed by the density ρ of ones. For precise statements and proofs, see Chapters V and VI of [11] and Part I of [13].

Now suppose, informally, that the two processes are superimposed on one another in $d \geq 3$, with the voter model being run at infinite rate. The effect is to make the distribution μ_t at time t be an invariant measure for the voter model at all times t. Since the density of

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ones $\rho(t) = \mu_t(1)$ is not changed by the voter part, the evolution of the density is determined by the contact part:

$$\frac{d}{dt}\rho(t) = 2d\lambda\mu_t(10) - \mu_t(1) = 2d\lambda\nu^{\rm vm}_{\rho(t)}(10) - \rho(t). \tag{1}$$

cvODE

Here $\mu_t(1)$ and $\mu_t(10)$ represent the probabilities that a site takes the value 1, and that two adjacent sites take the values 1,0 at time t respectively. In the voter equilibirium, $\nu^{\rm vm}_\rho(10) = \rho(1-\rho)\rho_d$, where ρ_d is the probability that a simple random walk on $\mathbb{Z}^d$ does not return to the origin. (See page 242 of [11].) Therefore, in the ODE (1)

$$\lim_{t\to\infty}\rho(t) = \begin{cases} 1-(2d\lambda\rho_d)^{-1} & \text{if } \lambda \geq 1/(2d\rho_d); \\ 0 & \text{if } \lambda \leq 1/(2d\rho_d). \end{cases} \tag{2}$$

cvrho

This suggests that $\lambda_c(\theta) \to 1/(2d\rho_d)$ as $\theta \to \infty$. The asymptotics for the critical value and the equilibrium density will be proved in Theorem 1 below.

In $d \leq 2$ this heuristic does not work well, since running the voter model infinitely fast would drive the system to consensus. However, the fact that clustering occurs in the voter model means that an occupied site is usually completely surrounded by other occupied sites preventing births from occurring, and suggests that $\lambda_c(\theta) \to \infty$. Theorems 2–5 prove this and identify the order of magnitude of $\lambda_c(\theta)$.

If we let $\theta = \epsilon^{-2}$ then our model beomes a voter model perturbation in the sense of Cox, Durrett, and Perkins [2]. From their results we get

d3asy **Theorem 1.** *In $d \geq 3$, $2d\rho_d\lambda_c(\theta) \to 1$.*

Proof. The key is the fact proved in Theorem 1.2 (all the results we cite here are from [2]): if $\theta \to \infty$, space is scaled to $\theta^{-1/2}\mathbb{Z}^d$, and the initial profile converges to a continuous function $v(x)$ (in a sense made precise in [2]), then the particle system at each time t converges (in the same sense) to $u(t,x)$, the solution of

$$\frac{\partial u}{\partial t} = \Delta u - u + 2d\lambda\rho_d u(1-u).$$

If $\beta = 2d\lambda\rho_d > 1$ results of Aronson and Weinberger (cited and used in a similar way in [2]) imply that Assumption 1 is satisfied, so we can apply Theorem 1.4 to conclude that there is a stationary distribution that concentrates on $\Omega_{0,1}$, the configurations with infinitely many 1's and 0's, and that in any such stationary distribution sites are occupied with probability close to $(\beta-1)/\beta$, confirming the heuristic calculation in (2). Using $-u + \beta u(1-u) < (\beta-1)u$, we see that if $\beta < 1$, then for any initial condition the PDE converges exponentially fast to 0, which checks Assumption 2, so Theorem 1.5 implies that that there is no nontrivial stationary distribution. □

To prove results in dimensions $d \leq 2$ we will use the fact that the contact plus voter model, written as $\xi_t : \mathbb{Z}^d \to \{0,1\}$, is dual to a set-valued branching coalescing random walk, η_t, in which each particle dies at rate 1, jumps to each of its $2d$ nearest neighbors at rate θ, and gives birth onto each neighbor at rate λ. If the transitions in either of the last two possibilities produce two particles on one site, they immediately coalesce to one. Due to work of Harris [9] and Griffeath [8], this is often called the "additive" dual. The precise relationship between the two processes is that for any finite set A

$$P(\xi_t(x) = 0 \text{ for all } x \in A) = P(\xi_0(y) = 0 \text{ for all } x \in \eta_t^A)$$

where η_t^A denotes the dual process starting from $\eta_0^A = A$.

To study the dual process, we will adapt arguments in Durrett and Zähle [5] to show that if we let $\theta \to \infty$, scale space to $\theta^{-1/2}\mathbb{Z}^2$, let $\lambda/\log(\theta) \to \gamma$, and remove particles that quickly coalesce with their parents, then on any finite time interval the dual process converges to a branching Brownian motion in which the Brownian motions are run at rate 2, particles give birth at rate $4\pi\gamma$ and die at rate 1. Once this it is done, it is routine to use a block construction to show that if $4\pi\gamma > 1$ the limiting Brownian motion dominates a supercritical one-dependent oriented percolation. A weak convergence argument then allows us to extend the comparison with oriented percolation to the dual process with large θ, and results from Durrett's St. Flour Notes [4] give the following:

d2surv **Theorem 2.** *In $d = 2$,*

$$\limsup_{\theta\to\infty} \lambda_c(\theta)/\log(\theta) \le \frac{1}{4\pi}.$$

In the other direction we have

d2die **Theorem 3.** *Let $\epsilon = 1/(1 + 2\lambda + 4\theta)$. If $d = 2$, the process dies out if*

$$E^{(0,0)}(1-\epsilon)^{\tau-1} > 1 - \frac{1}{4\lambda},$$

where τ is the hitting time of the origin for the lazy simple symmetric random walk on Z^2, that stays put with probability 1/2. From this it follows that

$$\liminf_{\theta\to\infty} \lambda_c(\theta)/\log(\theta) \ge \frac{1}{4\pi}.$$

To get the second conclusion from the first, integrate by parts (or use Fubini's theorem) to conclude that if F is the distribution function of a nonnegative random variable and ϕ is its Laplace transform

$$\int_0^\infty e^{-\lambda x}(1 - F(x))\, dx = \frac{1 - \phi(\lambda)}{\lambda}.$$

Using the well-known fact that

$$P(\tau > n) \sim \pi/\log n \tag{3}$$ 2dSRW

(see page 167 of [15] or Lemma 3.1 in [3]) and applying a Tauberian theorem (e.g., Theorem 4 from Section XIII.5 of [6]), we see that

$$1 - \phi(\epsilon) \sim \pi/\log(1/\epsilon).$$

If $\lambda \sim c\log(\theta)$, then from the definition of ϵ it follows that $1/\epsilon \sim 4\theta$, and the inequality will hold for large θ if $c < 1/4\pi$.

In $d = 1$, the best technique known to obtain an upper bound on the critical value for the contact process is the one of Holley and Liggett [10]. It gives $\lambda_c \le 2$. This was later improved in [12] to $\lambda_c \le 1.942$ using an extension of that technique. It is natural to ask whether this approach can be applied to the contact+voter process, and if so, what the resulting bound is. In fact, it does apply, and shows that the process survives if

$$\theta = 2\lambda^{3/2} \frac{\sqrt{\lambda} - \sqrt{2}}{2\sqrt{2\lambda} - 1}.$$

Since the fraction on the right converges to $1/2\sqrt{2}$ as $\lambda \to \infty$, it follows that

$$\limsup_{\theta\to\infty} \lambda_c(\theta)/\theta^{2/3} \le 2^{1/3}.$$

The details of the argument can be found in [14]. Even though the Holley-Liggett bound for the contact process is very close to the right answer $\lambda_c \approx 1.65$, here it fails to give the correct order of magnitude. As the next two results show, the correct rate of growth is $\theta^{1/2}$.

To prove an upper bound on the critical value it is convenient to rescale time so that particles die at rate $\delta = 1/\lambda$, give birth on each nearest neighbor at rate $\nu = \theta/\lambda$ and jump across each edge at rate 1.

d1surv **Theorem 4.** *In $d = 1$, $\liminf_{\nu\to\infty} \nu\delta_c(\nu) > 0$ and hence*

$$\limsup_{\theta\to\infty} \lambda_c(\theta)/\theta^{1/2} < \infty.$$

To get the second conclusion from the first note that $\delta \ge c/\nu$ translates into $\lambda \le (\theta/c)^{1/2}$.

The key to the proof is a block construction to compare the process with $\delta = 0$ with a supercritical one-dependent oriented percolation. To do this, we use an idea from Bramson and Griffeath [1]. We follow two tagged particles in the dual: a white particle that only follows random walk steps and a red particle that follows random walk steps and in addition branchings that take it to the right. The two particles may come together several times and follow the same trajectory for a while, but the drift in the red particle will eventually take it to the right and away from the white particle.

The existence of a comparison with oriented percolation for the process with $\delta = 0$ immediately implies that $\delta_c > 0$. To get the result given in Theorem 4, we use the fact that our construction takes place in a space time box that is $8L\nu$ wide (in space) and $10L\nu$ high (in time) with L is large, to conclude that the dual process survives with positive probability if $10L\nu\delta$ is small.

Since the proof of Theorem 4 is based on a block construction, the constant is very large. This problem was avoided in the proof of Theorem 2 by using a large number of particles in the block construction. This is possible because two dimensional Brownian motions do not hit points or each other. This strategy does not seem to be practical in $d = 1$.

The final result gives a good lower bound on the critical value. The proof is similar to the one for Theorem 3, so the result might also be sharp.

d1die **Theorem 5.** *In $d = 1$, the contact process dies out if $\theta > 2\lambda(\lambda - 1)$ and hence*

$$\liminf_{\theta\to\infty} \lambda_c(\theta)/\theta^{1/2} \ge 1/\sqrt{2}.$$

The remainder of the paper is devoted to proofs. Theorem 5 is proved in Section 2 and Theorem 3 in Section 3. This follows the order in which the result were first proved, but the second proof is somewhat simpler, so if the reader gets bogged down in the details of the first proof, he might have more success in understanding the second one. Theorem 2 is proved in Section 4, and Theorem 4 in Section 5. Section 3 depends heavily on the ideas used in Section 2. However, these sections are otherwise independent.

2 Extinction in one dimension

Proof of Theorem 5. Let μ be the upper invariant measure for the process, i.e., the limit starting from all 1's, and for finite $A \subset \mathbb{Z}$, put

$$\psi(A) = \mu\{\eta \equiv 0 \text{ on } A\}.$$

(This limit exists by monotonicity – see Theorem 2.3 of Chapter III of [11].) From the definition of ψ it is clear that

$$\begin{aligned}\psi(A) - \psi(A \cup B) &= \mu\{\eta \equiv 0 \text{ on } A, \eta \not\equiv 0 \text{ on } B \backslash A\} \\ &\leq \mu\{\eta \equiv 0 \text{ on } A \cap B, \eta \not\equiv 0 \text{ on } B \backslash A\} = \psi(A \cap B) - \psi(B).\end{aligned}$$

That is, ψ is *supermodular*:

$$\psi(A \cup B) + \psi(A \cap B) \geq \psi(A) + \psi(B).$$

It is a long known and frequently used fact that invariant measures for a Markov process translate into harmonic functions for the dual process. In the present case, the duality function is $H(\eta, A) = 1_{\{\eta \equiv 0 \text{ on } A\}}$. Early examples that show the usefulness of this observation are found in the analysis of the voter model (Section 1 of Chapter V of [11]) and symmetric exclusion process (Section 1 of Chapter VIII of [11]). See Corollary 1.3 of Chapter VIII for an explicit statement of this connection. The discussion there applies equally well to our process and its dual.

At the level of generators, the duality relation asserts that

$$(LH)(\eta, A) = \sum_B q(A, B)[H(\eta, B) - H(\eta, A)],$$

where L is the generator of the particle system acting on the first coordinate of H, and $q(A, B)$ is the $Q-$matrix of the dual Markov chain. Integrating both sides with respect to μ gives 0 on the left, since μ is invariant, so it follows that

$$0 = \sum_B q(A, B)[\psi(B) - \psi(A)]. \qquad (4) \qquad \boxed{\texttt{harmonic}}$$

So, this function ψ is harmonic for the dual chain. Using (4) and the shift invariance of μ,

$$\begin{aligned}(1 + 2\lambda)\psi(\{0\}) &= 1 + 2\lambda\psi(\{0, 1\}), \\ (1 + 2\theta + \lambda)\psi(\{0, 1\}) &= (1 + \theta)\psi(\{0\}) + \theta\psi(\{0, 2\}) + \lambda\psi(\{0, 1, 2\}),\end{aligned}$$

To check the second equation, note that while events happen to the pair $\{0,1\}$ at rate $2+4\theta+2\lambda$, it suffices by symmetry to consider only those that affect the site on the right. Similarly, for $n \geq 2$,

$$\begin{aligned}(1+2\theta+2\lambda)\psi(\{0,n\}) = \psi(\{0\}) + \theta\psi(\{0,n+1\}) + \theta\psi(\{0,n-1\}) \\ + \lambda\psi(\{0,n,n+1\}) + \lambda\psi(\{0,n-1,n\}).\end{aligned}$$

Let $f(0) = \psi(\{0\})$ and $f(n) = \psi(\{0,n\})$ for $n \geq 1$. Though it is not needed below, we note that $f(n) \downarrow$ by Theorem 1.9 of Chapter VI of [11]. The proof given there for the contact process applies to our process as well. Changing notation, we have from above

$$(1+2\lambda)f(0) = 1 + 2\lambda f(1). \tag{5}$$ $\boxed{\texttt{ineq0}}$

To simplify the other equations, we use the supermodularity equation

$$\psi(\{0,m,m+1\}) \geq \psi(\{0,m\}) + \psi(\{m,m+1\}) - \psi(\{m\})$$

to eliminate $\psi(A)$ for $|A| = 3$ in the above harmonicity equations. This gives

$$(1+2\theta-\lambda)f(1) \geq (1+\theta-\lambda)f(0) + \theta f(2), \tag{6}$$ $\boxed{\texttt{ineq1}}$

and for $n \geq 2$,

$$(1+2\theta+\lambda)f(n) + (2\lambda-1)f(0) \geq 2\lambda f(1) + (\theta+\lambda)f(n-1) + \theta f(n+1). \tag{7}$$ $\boxed{\texttt{ineq2}}$

Now let $g(n) = f(n) - f(n+1)$ for $n \geq 0$. Using $\sum_{k=0}^{n-1} g(k) = f(0) - f(n)$, we see that

$$\theta g(n) \geq (\theta+\lambda)g(n-1) - 2\lambda g(0) + \sum_{k=0}^{n-1} g(k)$$

for $n \geq 1$. In fact, this is (6) if $n = 1$ and (7) if $n \geq 2$. Furthermore, (5) becomes $g(0) = (1-f(0))/(2\lambda)$.

Define $h(n)$ recursively by $h(0) = 1, \theta h(1) = 1 + \theta - \lambda$, and

$$\theta h(n) + 2\lambda = (\theta+\lambda)h(n-1) + \sum_{k=0}^{n-1} h(k) \tag{8}$$ $\boxed{\texttt{recursion}}$

for $n \geq 2$. For the future, note that (8) also holds when $n = 1$. Then

$$g(n) \geq \frac{1-f(0)}{2\lambda}h(n), \quad n \geq 0. \tag{9}$$ $\boxed{\texttt{ineq}}$

To see this, define $g^*(0) = g(0)$, and then $g^*(n)$ by using the recursion satisfied by $g(n)$ with inequalities replaced by equalities. By induction, $g(n) \geq g^*(n)$. Since the recursion is linear, $g^*(n) = g(0)h(n)$.

Lemma 2.1. *The solution to the recursion (8) for h can be given explicitly: $h(n) = h^*(n)$, where*

$$h^*(n) = 1 + \sum_{k=1}^{n}\sum_{j=0}^{k}\binom{n+k-j}{2k-j}\binom{k}{j}\frac{k-2j}{k}\frac{\lambda^j}{\theta^k}.$$

Proof. To check this, it suffices to show that h^* satisfies (8), together with the initial conditions. The initial conditions are immediate. For the recursion, it suffices to check that after replacing $h(n)$ in (8) by the above expression for $h^*(n)$, the coefficient of λ^j/θ^k is the same on both sides of the resulting equation. To simplify matters, we will use the convention that $\frac{0}{0}=1$ and $\binom{k}{j}=0$ unless $0 \le j \le k$. The coefficient of λ^j/θ^k is

$$\binom{n+k-j+1}{2k-j+2}\binom{k+1}{j}\frac{k-2j+1}{k+1}+2\times 1_{\{j=1,k=0\}} \qquad (10) \quad \boxed{\texttt{leftside}}$$

for the expression on the left, and

$$\binom{n+k-j}{2k-j+2}\binom{k+1}{j}\frac{k-2j+1}{k+1}+\binom{n+k-j}{2k-j+1}\binom{k}{j-1}\frac{k-2j+2}{k} + \binom{k}{j}\frac{k-2j}{k}\sum_{l=k}^{n-1}\binom{l+k-j}{2k-j} \qquad (11) \quad \boxed{\texttt{rightside}}$$

for the expression on the right. The sum on the right above can be expressed as

$$\sum_{l=k}^{n-1}\binom{l+k-j}{2k-j}=\binom{n+k-j}{2k-j}\frac{n-k}{2k-j+1}. \qquad (12) \quad \boxed{\texttt{binomsum}}$$

To see this, note that the two expressions above agree when $n=k$. The differences between the two sides at $n+1$ and n are

$$\binom{n+k-j}{2k-j} \quad \text{and} \quad \binom{n+k-j+1}{2k-j}\frac{n+1-k}{2k-j+1}-\binom{n+k-j}{2k-j}\frac{n-k}{2k-j+1} \qquad (13) \quad \boxed{\texttt{binomid}}$$

respectively. Writing

$$\binom{n+k-j+1}{2k-j}=\binom{n+k-j}{2k-j}\frac{n+k-j+1}{n-k+1}, \qquad (14) \quad \boxed{\texttt{binomred}}$$

we see that the two expressions in (13) agree. Now replace the sum in (11) with the right side of (12), and combine the resulting binomial coefficients (using identities similar to (14)) to check that (10) and (11) agree. $\square$

In order to determine the behavior of $h(n)$ for large n, it is simplest to compute the generating function. To avoid convergence issues in changing of order of summation, one can consider the positive and negative terms separately, but the result is the same. To be precise, one would write $(k-2j)/k = 1-2/jk$ and sum the resulting two series separately, adding the results at the end. Using the Taylor series

$$\sum_{m=0}^{\infty}\binom{m+l}{l}u^m=\frac{1}{(1-u)^{l+1}}, \quad |u|<1,$$

$$\begin{aligned}\sum_{n=0}^{\infty} u^n h(n) &= 1 + \sum_{k=1}^{\infty}\sum_{j=0}^{k}\binom{k}{j}\frac{k-2j}{k}\frac{\lambda^j}{\theta^k}\sum_{n=k}^{\infty}\binom{n+k-j}{2k-j}u^n \\ &= 1 + \sum_{k=1}^{\infty}\sum_{j=0}^{k}\binom{k}{j}\frac{k-2j}{k}\frac{\lambda^j}{\theta^k}u^k(1-u)^{j-2k-1} \\ &= 1 + \frac{u(1-\lambda(1-u))}{\theta(1-u)^3}\sum_{k=1}^{\infty}\left[\frac{u(1+\lambda(1-u))}{\theta(1-u)^2}\right]^{k-1},\end{aligned}$$

where the final step comes from the binomial theorem, provided that

$$\left|\frac{u(1+\lambda(1-u))}{\theta(1-u)^2}\right| < 1.$$

Let $0 < v < 1$ be the solution of $v(1+\lambda(1-v)) = \theta(1-v)^2$. (Note that the ratio of the left side to the right side increases from 0 to ∞ as v goes from 0 to 1, so the solution exists and is unique.) As $u \uparrow v$, the final sum above tends to ∞. Therefore

$$\lim_{u\uparrow v}\sum_{n=0}^{\infty} u^n h(n) = \begin{cases} +\infty & \text{if } \lambda(1-v) < 1; \\ -\infty & \text{if } \lambda(1-v) > 1. \end{cases}$$

Solving the quadratic for v gives

$$v = \frac{1+\lambda+2\theta-\sqrt{(1+\lambda)^2+4\theta}}{2(\lambda+\theta)},$$

and therefore

$$1-\lambda(1-v) = \frac{2\theta+\lambda(3-\lambda)-\lambda\sqrt{(1+\lambda)^2+4\theta}}{2(\lambda+\theta)}.$$

It follows that $\lambda(1-v) < 1$ is equivalent to

$$2\theta + \lambda(3-\lambda) > \lambda\sqrt{(1+\lambda)^2+4\theta}.$$

For this to be true, the left side must be positive, and the inequality obtained by squaring both sides must hold, i.e.,

$$2\theta > \lambda(\lambda-3) \quad \text{and} \quad \theta > 2\lambda(\lambda-1).$$

Note that once $\lambda > \frac{1}{3}$, the second condition is more restrictive than the first.

To complete the proof, note that if $\theta > 2\lambda(\lambda-1)$, then

$$\lim_{u\uparrow v}\sum_{n=0}^{\infty} u^n h(n) = +\infty.$$

Since $0 < v < 1$, this implies that

$$\limsup_{n\to\infty} h(n) = +\infty,$$

Since $g(n)$ is bounded, this together with (9) implies $f(0) = 1$, so the process dies out. $\square$

Alternative proof. For readers who are less adept at computations with Binomial coefficients, we now sketch another approach. We want to solve (8),

$$\theta h(n) + 2\lambda = (\theta+\lambda)h(n-1) + \sum_{k=0}^{n-1} h(k) \qquad \text{for } n \geq 1,$$

with $h(0) = 1$. If we let $j(n) = \sum_{k=0}^{n-1} h(k)$ then we have the system that can be written as

$$\begin{pmatrix} h(n) \\ j(n) \end{pmatrix} = \begin{pmatrix} 1+(\lambda+1)/\theta & 1/\theta \\ 1 & 1 \end{pmatrix} \begin{pmatrix} h(n-1) \\ j(n-1) \end{pmatrix} + \begin{pmatrix} -2\lambda/\theta \\ 0 \end{pmatrix} \qquad \text{for } n \geq 1,$$

where $j(0) = 0$. Writing A for the matrix and taking into account the initial condition:

$$\begin{pmatrix} h(n) \\ j(n) \end{pmatrix} = A^n \begin{pmatrix} 1 \\ 0 \end{pmatrix} + \sum_{m=0}^{n-1} A^m \begin{pmatrix} -2\lambda/\theta \\ 0 \end{pmatrix}.$$

Being a positive matrix, the Perron-Frobenius theorem implies $A^n_{ij} \sim v_i \gamma_1^n w_j$ where λ_1 is the largest eigenvalue and w and v are associated left and right eigenvectors, which have strictly positive entries and are normalized so that $\sum_i w_i v_i = 1$. Here $a_n \sim b_n$ means that $a_n/b_n \to 1$. From the asymptotics for A^n we see that

$$h(n) \sim v_1 w_1 \left[\gamma_1^n (1 + (1-\lambda)/\theta) - \frac{\gamma_1^{n-1}}{1-1/\gamma_1}(2\lambda/\theta) \right]. \qquad (15) \quad \boxed{\texttt{hasy}}$$

To find the eigenvalues we set $(1+(\lambda+1)/\theta - x)(1-x) - 1/\theta = 0$ which becomes

$$x^2 - (2+(\lambda+1)/\theta)x + 1 + \lambda/\theta = 0.$$

Solving we get

$$\gamma_i = \frac{2+(\lambda+1)/\theta \pm \sqrt{(2+(\lambda+1)/\theta)^2 - 4[1+\lambda/\theta]}}{2}.$$

Underneath the square root we have

$$4 + \frac{4(\lambda+1)}{\theta} + \frac{(\lambda+1)^2}{\theta^2} - 4 - 4\frac{\lambda}{\theta} = \frac{(\lambda+1)^2}{\theta^2} + \frac{4}{\theta}.$$

Multiplying top and bottom by θ, the largest eigenvalue is

$$\gamma_1 = \frac{1+\lambda+2\theta + \sqrt{(\lambda+1)^2 + 4\theta}}{2\theta} > 1.$$

From (15) we see that $h(n) \to \infty$ if $\gamma_1 - 1 > 2\lambda/\theta$. Subtracting 1 from γ_1 removes the 2θ from the numerator. When $\theta = c\lambda^2$,

$$\gamma_1 - 1 \sim \frac{\lambda}{2\theta}(1 + \sqrt{1+4c}),$$

so again we find $\liminf \lambda_c(\theta)/\theta^{1/2} \geq 1/\sqrt{2}$.

3 Extinction in two dimensions

Proof of Theorem 3. Now let $f(0,0) = \psi(\{(0,0)\})$ and $f(m,n) = \psi(\{(0,0),(m,n)\})$ for $m,n \geq 0, m+n \geq 1$. Imitating the proof in the previous section, we begin by observing that

$$\begin{aligned}(1+4\lambda)\psi(\{(0,0)\}) &= 1 + 2\lambda\psi(\{(0,0),(1,0)\}) + 2\lambda\psi(\{(0,0),(0,1)\} \\ (1+4\lambda)f(0,0) &= 1 + 2\lambda f(0,1) + 2\lambda f(1,0). \qquad (16) \quad \boxed{\texttt{f00}}\end{aligned}$$

Note that we have used reflection symmetry to keep the points in the first quadrant. Next,

$$\begin{aligned}(1+3\lambda+3\theta)\psi(\{(0,0),(1,0)\}) &= \psi(\{(0,0)\}(1+\theta) + 2\theta\psi(\{(0,0),(1,1)\} \\ &+ \theta\psi(\{(0,0),(2,0)\}) + 2\lambda\psi(\{(0,0),(1,0),(1,1)\}) + \lambda\psi(\{(0,0),(1,0),(2,1)\}).\end{aligned}$$

Supermodularity implies

$$\psi(\{(0,0),(1,0),(1,1)\}) \geq \psi(\{(0,0),(1,0)\}) + \psi(\{(1,0),(1,1)\}) - \psi(\{1,0\}),$$

so switching to the f notation

$$\begin{aligned}(1+3\lambda+3\theta)f(1,0) \geq{}& f(0,0)(1+\theta) + 2\theta f(1,1) + \theta f(2,0) \\ &+ 2\lambda[f(1,0) + f(0,1) - f(0,0)] + \lambda[2f(1,0) - f(0,0)],\end{aligned}$$

and rearranging gives

$$(1-\lambda+4\theta)f(1,0) + (3\lambda-\theta-1)f(0,0) \geq 2\lambda f(0,1) + 2\theta f(1,1) + \theta f(2,0). \qquad (17) \quad \boxed{\texttt{f10}}$$

When $m \geq 2$ there are terms $\theta\psi(\{(0,0),(m-1,0)\})$ and $\lambda\psi(\{(0,0),(m-1,0),(m,0)\}$, so switching to the f notation

$$\begin{aligned}(1+4\lambda+4\theta)f(m,0) \geq{}& f(0,0) + \theta f(m-1,0) + 2\theta f(m,1) + \theta f(m+1,0) \\ &+ \lambda[f(m-1,0) + f(1,0) - f(0,0)] + 2\lambda[f(m,0) + f(0,1) - f(0,0)] \\ &+ \lambda[f(m,0) + f(1,0) - f(0,0)],\end{aligned}$$

and rearranging gives

$$\begin{aligned}(1+\lambda+4\theta)f(m,0) + (4\lambda-1)f(0,0) \geq{}& 2\lambda f(1,0) + 2\lambda f(0,1) \qquad (18) \quad \boxed{\texttt{fm0}} \\ &+ (\lambda+\theta)f(m-1,0) + \theta f(m+1,0) + 2\theta f(m,1).\end{aligned}$$

Similarly, when $m,n \geq 1$

$$\begin{aligned}(1+2\lambda+4\theta)f(m,n) + (4\lambda-1)f(0,0) \geq{}& 2\lambda f(1,0) + 2\lambda f(0,1) \\ &+ (\lambda+\theta)[f(m-1,n) + f(m,n-1)] + \theta[f(m+1,n) + f(m,n+1)].\end{aligned}$$

Letting $g(m,n) = 1 - f(m,n)$, (16) becomes

$$(1+4\lambda)g(0,0) = 2\lambda[g(0,1) + g(1,0)], \qquad (19) \quad \boxed{\texttt{g00}}$$

Multiplying each side of (18) by -1, noting that the coefficients on each side sum to $5\lambda + 4\theta$ and using the last identity we have

$$\begin{aligned}(1+4\theta+\lambda)g(m,0) \leq\, & 2g(0,0) + (\lambda+\theta)g(m-1,0) \\ & + \theta g(m+1,0) + 2\theta g(m,1).\end{aligned} \tag{20}$$ gm0

Performing these manipulations on (17) gives the same result so this equation holds for $m \geq 1$. A similar argument shows that for $m, n \geq 1$

$$\begin{aligned}(1+2\lambda+4\theta)g(m,n) \leq\, & 2g(0,0) + (\lambda+\theta)[g(m-1,n) + g(m,n-1)] \\ & + \theta[g(m,n+1) + g(m+1,n)].\end{aligned} \tag{21}$$ gmn

As a warmup, consider formally the case $\theta = \infty$. Then (19), (20) and (21) imply that if $\bar{S}_k$ is simple random walk on the positive quadrant in $\mathbb{Z}^2$ with reflection at the boundaries then

$$g(m,n) \leq E^{(m,n)} g(\bar{S}_1). \tag{22}$$ warmup

Since S_k is recurrent, the bounded subharmonic function g is constant. By (19), this constant is zero, so the process dies out.

Suppose now that the process survives, $g(0,0) > 0$, and let

$$h(m,n) = g(m,n)/2g(0,0).$$

Then $h(0,0) = 1/2$ and (19), (20) and (21) imply that for $m, n \geq 1$

$$h(0,1) + h(1,0) = 1 + 1/4\lambda, \tag{23}$$ ineq9

$$(1+\lambda+4\theta)h(m,0) \leq 1 + (\lambda+\theta)h(m-1,0) + \theta h(m+1,0) + 2\theta h(m,1), \tag{24}$$ ineq7

$$\begin{aligned}(1+2\lambda+4\theta)h(m,n) \leq\, & 1 + (\lambda+\theta)[h(m-1,n) + h(m,n-1)] \\ & + \theta[h(m,n+1) + h(m+1,n)].\end{aligned} \tag{25}$$ ineq8

Clearly, $f(m,n) \leq f(0,0)$. Since the contact process has positive correlations (see e.g., Theorem 2.13 of Chapter III of [11]), $f(m,n) \geq f^2(0,0)$, so for all m, n

$$1/2 \leq \frac{1 - f(m,n)}{2(1-f(0,0))} \leq \frac{1 - f(0,0)^2}{2(1-f(0,0))} = \frac{1 + f(0,0)}{2} \leq 1.$$

i.e., $1/2 \leq h(m,n) \leq 1$. Let $\epsilon = 1/(1+2\lambda+4\theta)$. To motivate the next definition divide each side of (25) by $(1+2\lambda+4\theta)$ to get

$$\begin{aligned}h(m,n) \leq \epsilon + & (1-\epsilon)\frac{(\lambda+\theta)}{2\lambda+4\theta}[h(m-1,n) + h(m,n-1)] \\ & + (1-\epsilon)\frac{\theta}{2\lambda+4\theta}[h(m,n+1) + h(m+1,n)],\end{aligned} \tag{26}$$ hmn2

and note that if $\lambda \ll \theta$ the two fractions on the right are $\approx 1/4$. For a future comparison, note that $(1-\epsilon)/(2\lambda+4\theta) = \epsilon$.

Define $h_k(m,n)$ for $k \geq 0$ and $m,n \in Z$ by $h_k(0,0) = 1/2$ for all k. For $(m,n) \neq (0,0)$, let $h_0(m,n) = 1$ and

$$h_{k+1}(m,n) = \epsilon + (1-\epsilon)E^{(m,n)} h_k(\tilde{S}_1), \tag{27}$$ rec

where $\tilde{S}_k$ is the lazy version of the reflecting random walk $\bar{S}_k$, defined before (22), that stays put with probability 1/2. It is easy to see $1/2 \leq h_k(m,n) \leq 1$ for all k,m,n. Let τ be the hitting time of the origin for $\tilde{S}_k$. One can think of $h_k(m,n)$ of a reward earned by our process $\tilde{S}_k$ starting from (m,n) and stopped at time τ.

- A reward of 1/2 is earned at time τ.
- A reward of 1 is earned at time k if $\tau > k$.
- A reward of ϵ is earned at time $j < k$ if $\tau > j$.
- Rewards earned at time j are discounted by $(1-\epsilon)^j$.

This interpretation leads easily to the following formula:

$$h_k(x) = \frac{1}{2}\sum_{i=1}^{k}(1-\epsilon)^i P^x(\tau = i) + \epsilon \sum_{j=0}^{k-1}(1-\epsilon)^j P^x(\tau > j) + (1-\epsilon)^k P^x(\tau > k).$$

Writing $P^x(\tau = i) = P^x(\tau > i-1) - P^x(\tau > i)$ and rearranging gives

$$h_k(x) = \frac{1}{2} + \frac{\epsilon}{2}\sum_{j=0}^{k-1}(1-\epsilon)^j P^x(\tau > j) + \frac{1}{2}(1-\epsilon)^k P^x(\tau > k), \quad x \neq 0. \tag{28}$$ hkx

We have used the lazy random walk so that $x \to P^x(\tau > j)$ is increasing (and hence $x \to h_k(x)$ is also) in the usual partial order on the positive quadrant. To prove this we note that two random walks started at (m,n) and (m',n') with $m' \geq m$ and $n' \geq n$ can be coupled to preserve the order. The ordering is trivial to maintain while the walk is in the interior of the quadrant or on the left side. To handle the situation when one walker is at $(\ell,1)$ and the other is at $(\ell,0)$ note that:

from/to	$(\ell,2)$	$(\ell,1)$	$(\ell,0)$
$(\ell,1)$	1/8	1/2	1/8
$(\ell,0)$	0	2/8	1/2

To compare h_k with h note that if $m,n \geq 1$

$$\begin{aligned} h_{k+1}(m,n) &= \frac{1}{2}[\epsilon + (1-\epsilon)h_k(m,n)] \\ &\quad + \frac{1}{2}\left(\epsilon + \frac{1-\epsilon}{4}[h_k(m-1,n) + h_k(m,n-1) + h_k(m+1,n) + h_k(m,n+1)]\right). \end{aligned}$$

Since $h_k(m,n) \le 1$ and $(1-\epsilon)/4 = (\lambda/2+\theta)/(1+2\lambda+4\theta) = \epsilon(\lambda/2+\theta)$

$$\begin{aligned} h_{k+1}(m,n) \le {} & \frac{1}{2} h_k(m,n) + \frac{\epsilon}{2} \\ & + \frac{1}{2}\epsilon \Big((\lambda+\theta)[h_k(m-1,n) + h_k(m,n-1)] + \theta[h_k(m+1,n) + h_k(m,n+1)] \\ & \qquad + \frac{\lambda}{2}[h_k(m+1,n) - h_k(m-1,n)] + \frac{\lambda}{2}[h_k(m,n+1) - h_k(m,n-1)] \Big). \end{aligned} \tag{29}$$

comp1

To handle the equation on the boundary we add $\lambda h(m,0)$ to each side of (24), and divide by $1+2\lambda+4\theta$ to rewrite it as

$$h(m,0) \le \epsilon + \epsilon[(\lambda+\theta)h(m-1,0) + \lambda h(m,0) + \theta h(m+1,\theta) + 2\theta h(m,1)]. \tag{30}$$

hm02

The corresponding equation for h_{k+1} is

$$\begin{aligned} h_{k+1}(m,0) = {} & \frac{1}{2}[\epsilon + (1-\epsilon)h_k(m,0)] \\ & + \frac{1}{2}\left(\epsilon + \frac{1-\epsilon}{4}[h_k(m-1,0) + h_k(m+1,0) + 2h_k(m,1)] \right). \end{aligned}$$

Arguing as before we can convert this into

$$\begin{aligned} h_{k+1}(m,0) \le {} & \frac{1}{2} h_k(m,0) + \frac{\epsilon}{2} \\ & + \frac{1}{2}\epsilon \Big((\lambda+\theta)h_k(m-1,0) + \lambda h_k(m,0) + \theta h_k(m+1,0) + 2\theta h_k(m,n+1) \\ & \qquad + \frac{\lambda}{2}[h_k(m+1,0) - h_k(m-1,0)] + \lambda[h_k(m,1) - h_k(m,0)] \Big). \end{aligned} \tag{31}$$

comp2

The last lines in (29) and (31) are ≥ 0 by the monotonicity of h_k, so comparing with (26) and (30), we see that h_{k+1} and h satisfy similar iterations with $\le$ and $\ge$ respectively. Since $h_0 \ge h$, it follows by induction that

$$h_k(m,n) \ge h(m,n). \tag{32}$$

hbound

Taking the limit in (28) and using (32) gives

$$h(m,n) \le \frac{1}{2} + \frac{\epsilon}{2} \sum_{j=0}^{\infty} (1-\epsilon)^j P^{(m,n)}(\tau > j).$$

Since $2h(1,0) = 1 + 1/4\lambda$ by (23),

$$1 \le 4\lambda\epsilon \sum_{j=0}^{\infty} (1-\epsilon)^j P^{(1,0)}(\tau > j).$$

Since this argument was based on the assumption that the process survives, it follows that the process dies out whenever the opposite (strict) inequality holds. $\square$

4 Survival in two dimensions

The proof is simple since most of the work has already been done by Durrett and Zähle [5]. It is only necessary to make some minor modifications to their proof presented on pages 1758–1761. However, since we have to give the details, we have taken the opportunity to clarify the proof and sharpen some of the bounds. As in [5], we work with the dual coalescing random walks η_t, but now jumps occur at rate θ across each edge between nearest neighbors, births occur at rate $\lambda = \gamma \log(\theta)$ across each edge, and deaths occur at rate 1 per site. Thus, in contrast to [5], we have no need to speed up the process, only to scale space to $\theta^{-1/2}\mathbb{Z}^2$.

Most of the particles that are born in η_t coalesce with their parents. These short intervals in which the size of the duals increases and then decreases are not compatible with weak convergence, so we define a **pruned dual process** $\bar{\eta}_t$ using the rule that a newly born particle has mass 0 until it has avoided coalescence with its parent for $1/\log^a(\theta)$ units of time, where a is the first of several positive constants whose values will be specified later. In the proof we will show that it is very unlikely for a particle to collide with a particle that is not its parent.

Let Z_t^1 and Z_t^2 be independent random walks that make jumps to nearest neighbors at rate 1 and let $S_t = Z_t^1 - Z_t^2$. Suppose S_0 is a neighbor of 0. Let $T_0 = \inf\{t : S_t = 0\}$. From (3), $P(T_0 > t) \sim \pi/(\log t)$. Changing to the fast time scale let $s_t = \theta^{-1/2} S(\theta t)$ and $t_0 = \inf\{t : s_t = 0\}$. We use lower case letters to remind us that this process is smaller in space and time. Therefore

$$P(t_0 > 1/\log^a(\theta)) = P(T_0 > \theta/\log^a(\theta)) \sim \frac{\pi}{\log\theta - a\log\log\theta}. \tag{33}$$

tauasy

Let $R = \theta/\log^a\theta$ and $r = 1/\log^a\theta$. Using a trivial inequality, then (33), and Chebyshev's inequality we have

$$\begin{aligned} P(|S_R| \geq \theta^{1/2}/\log^b(\theta) | T_0 > R) &\leq \frac{1}{P(T_0 > R)} P(|S_R| \geq \theta^{1/2}/\log^b(\theta)) \\ &\leq C \log\theta \cdot \frac{\theta}{\log^a\theta} \cdot \frac{\log^{2b}(\theta)}{\theta} = C(\log\theta)^{1+2b-a}. \end{aligned} \tag{34}$$

close

which goes to 0 if $a > 1 + 2b$. Here, and in what follows, we have collected all the constants into one placed at the front.

Note that when $|S_R| \leq \theta^{1/2}/\log^b(\theta)$, we have $|s_r| \leq 1/\log^b(\theta)$, i.e., the new particle is close to its parent. The next result, which follows from using the local central limit theorem, shows that they are not too close

$$\begin{aligned} P(|S_R| \leq \theta^{1/2}/\log^c(\theta) | T_0 > R) &\leq \frac{1}{P(T_0 > R)} P(|S_R| \leq \theta^{1/2}/\log^c(\theta)) \\ \leq C \log(\theta) \cdot \left(\frac{\theta^{1/2}}{\log^c(\theta)}\right)^2 \cdot \frac{\log^a(\theta)}{\theta} &= C(\log\theta)^{1+a-2c}. \end{aligned} \tag{35}$$

far

Here the second term on the last line gives the order of magnitude of the number of points x with $|x| \leq \theta^{1/2}/\log^c(\theta)$, and the third is a bound on the order of magnitude of $P(S_R = x)$. If $1 + a < 2c$ this tends to 0.

To prove results about the asymptotic behavior of the pruned dual $\bar\eta_t$ we will consider the **pruned branching random walk** $\hat\eta_t$ in which particles are assigned mass 0 until they have avoided coalescing with their parent for time $1/\log^a\theta$. After this time, we ignore collisions with any other particle (including its parent) until time L^2, which again is a constant whose value will be specified later. By (33) particles in $\hat\eta_t$ are born at rate

$$\sim 4\gamma\log\theta\cdot\frac{\pi}{\log\theta}=4\gamma\pi,$$

and die at rate 1. From this and (34) it is easy to see that

pbrwconv **Lemma 4.1.** *As $\theta\to\infty$ the pruned branching random walk $\hat\eta_t$, $0\le t\le L^2$, converges to a branching Brownian motion ζ_t, $0\le t\le L^2$ in which births occur at rate $4\gamma\pi$, deaths at rate 1, and the Brownian motions run at rate 2.*

To estimate the difference between $\bar\eta_t$ and $\hat\eta_t$, we need to estimate the probability that a newly born particle collides with another particle before time L^2. Using (33), one can easily show that

$$P(t_0\le L^2|t_0>1/\log^a(\theta))=P(T_0\le L^2\theta|T_0>\theta/\log^a(\theta))\to 0. \tag{36}$$ nocmom

To do this, we note that if $\epsilon>0$ then for large θ

$$\begin{aligned}P(\theta/\log^a(\theta)<T_0\le L^2\theta)&\le\frac{\pi(1+\epsilon)}{\log\theta-a\log\log\theta}-\frac{\pi(1-\epsilon)}{\log\theta+\log\log(L^2)}\\&\le\frac{\pi\epsilon\log\theta+O(1)}{\log\theta-a\log\log\theta)^2}.\end{aligned}$$

The result in (36) implies that after the initial separation we don't have to worry about a particle colliding with its parent before time L^2. To show that it avoids the other particles, we will first show that if two particles are sufficiently separated then they will not collide by time L^2. Then we will use induction to prove the desired amount of separation is maintained. The first step is to suppose $S_0=x$ with $|x|\ge\theta^{1/2}/\log^\Delta(\theta)$, with the capital letter Δ to suggest this is a large constant, and then estimate the probability $\alpha(x)$, that S_t will hit the ball of radius K before it exits the ball of radius $M\theta^{1/2}-1$. Our goal is to conclude that

$$\bar\alpha=\sup_{|x|\ge\theta^{1/2}/\log^\Delta(\theta)}\alpha(x)\to 0.$$

If S_t were a two-dimensional Brownian motion, then we would use the harmonic function $\log|z|$ to calculate $\alpha(x)$. For the random walk, we use the recurrent potential kernel defined by $A(x)=\sum_{k=0}^\infty[q^k(0)-q^k(x)]$ where q is uniform on the four nearest neighbors and q^k denotes the k-fold convolution. It is immediate from the definition that

$$\sum_y q(y-x)A(x)-A(x)=1_{(x=0)}$$

i.e., the difference is 1 if $x=0$ and 0 otherwise. From this we see that $A(S_t)$ is a martingale until time T_0. By Theorem 2 of [7] or P3 of Section 12 of [15],

$$A(x)=c_A\log|x|+O(1). \tag{37}$$ pkasy

If K is large and $|x| \geq \theta^{1/2}/\log^{\Delta}(\theta)$, then applying the optimal stopping theorem to $A(S_t)/c_A$

$$\begin{aligned}(1/2)\log\theta - &\Delta \log\log\theta + O(1) \leq A(x) \\ &\leq \alpha(x)[\log K + O(1)] + (1-\alpha(x))[(1/2)\log\theta + \log M + O(1)],\end{aligned}$$

where the second inequality comes from the fact that upon exit from the annulus the random walk will jump over the boundary by a distance ≤ 1. Rearranging we have

$$\alpha(x)[(1/2)\log\theta - \log K] \leq \Delta \log\log\theta + \log M + O(1), \tag{38}$$ nocoal

and we see that for any $\Delta < \infty$, $\bar{\alpha} \to 0$ as $\theta \to \infty$.

To do the induction argument to bound the spacings between the particles, we can ignore the death of particles. If we do this then, for large θ, the growth of the number of particles is bounded by a branching process with births at rate $5\gamma\pi$. This implies that the expected number of particles $N(L^2)$ in the system at time L^2 is finite, so if $\epsilon > 0$ then we can pick N_ϵ so that $P(N(L^2) > N_\epsilon) \leq \epsilon$.

We now choose the values of our parameters:

$$b = 1/3, \qquad a = 2, \qquad c = 2,$$

which satisfy $a > 1 + 2b$ and $2c > 1 + a$. With these choices, (35) implies that when a new particle is added to $\bar{\eta}_t$ the newly born particle is with high probability at distance $\geq \theta^{1/2}/\log^2\theta$ from its parent. Suppose that at time 0 we have k particles that are separated by $\geq \theta^{1/2}/\log^2\theta$. The result in (38) implies that none of the k particles will coalesce by time L^2. Since, for large θ, births in $\bar{\eta}_t$ occur at rate $\leq 5\gamma\pi$, it follows that for any $\delta > 0$ the first successful birth will with high probability occur after $R' = \theta/\log^{\delta}\theta$. If $R' \leq t \leq L^2$ then the argument for (35) can be repeated to show

$$P(|S_t| \leq \theta^{1/2}/\log^2(\theta)) \leq C\left(\frac{\theta^{1/2}}{\log^2(\theta)}\right)^2 \cdot \frac{\log^{\delta}(\theta)}{\theta} = C(\log\theta)^{2-\delta}.$$

This implies that when the new particle is added to $\bar{\eta}_t$ the $k+1$ particles are with high probability separated by $\theta^{1/2}/\log^2(\theta)$. Since this conclusion holds with probability $\geq 1-\epsilon$ and the argument is only repeated N_ϵ times, we have shown that the desired separation is maintained with high probability, and it follows from (38) that there is no coalescence. Since this shows that $\bar{\eta}_t = \hat{\eta}_t$ with high probability, we have:

pdconv **Lemma 4.2.** *As $\theta \to \infty$ the pruned dual $\bar{\eta}_t$, $0 \leq t \leq L^2$, converges to a branching Brownian motion ζ_t, $0 \leq t \leq L^2$, in which births occur at rate $4\gamma\pi$, deaths at rate 1, and the Brownian motions run at rate 2.*

With Lemma 4.2 established, the rest of the proof is almost identical to that in [5]. Let $\bar{\zeta}_t$ be a modification of ζ_t in which particles are killed when they leave $[-4L, 4L]^2$. In [5] this is $[0, 4L]^2$ but that is because they need to keep their construction in one half-space. We will use the more natural symmetric geometry in the construction here.

If we use $\bar{\zeta}^x_t(A)$ to denote the number of points in A when the killed branching Brownian motion $\bar{\zeta}_t$ starts with one particle at x and let $\mu = 4\gamma\pi - 1$ then

$$E[\bar{\zeta}^x_t(A)] = e^{\mu t} P(\bar{B}^x_t \in A),$$

where B_t^x is a Brownian motion run starting from x, run at rate 2, and killed when it exits $[-4L,4L]^2$. Let $I_k = (2kL^2, 0) + [-L,L]^2$. From the last estimate it follows easily (see page 1760 of [5] for more details) that if $\mu > 0$ (an assumption that will be in force until we state Lemma 4.3)

$$\inf_{x \in I_0} E[\bar\zeta^x_{L^2}(I_i)] \geq 2 \qquad \text{for } i = 1, -1,$$

and hence if we let $\bar\zeta^A_t = \sum_{x \in A} \bar\zeta^x_t$, then for $A \subset I_0$

$$E[\bar\zeta^A_{L^2}(I_i)] \geq 2|A| \qquad \text{for } i = 1, -1.$$

where $|A|$ is the number of particles in A.

Since $E[\bar\zeta^x_{L^2}(I_i)^2] \leq E[\zeta^x_{L^2}(\mathbb{R}^2)^2] \equiv c_L$, and the $\bar\zeta^x_t$ are independent, it follows that for $A \subset [-L,L]^2$

$$\text{var}\,[\bar\zeta^A_{L^2}(I_i)] \geq |A| c_L \qquad \text{for } i = 1, -1.$$

Using by Chebyshev's inequality that if $A \subset [-L,L]^2$ has $|A| \geq K$ then

$$P(\bar\zeta^A_{L^2}(I_i) < K) \leq c_L/K \qquad \text{for } i = 1, -1.$$

The last conclusion gives us the block event for the branching Brownian motion $\bar\zeta_t$:

bbmbc **Lemma 4.3.** *Suppose $4\gamma\pi > 1$. Let $\epsilon > 0$. If $L \geq L_0(\epsilon)$ and $\bar\zeta_0(I_0) \geq K$ then we will with probability $\geq 1 - 2\epsilon$ have $\bar\zeta_{L^2}(I_i) \geq K$ for $i = -1, 1$. We do this in a system in which particles are killed when they leave $[-4L,4L]^2$ so the events for I_0 and I_m are independent when $m \geq 4$.*

Combining this with Lemma 4.2 it follows that

pdbc **Lemma 4.4.** *Suppose $4\gamma\pi > 1$. Let $\epsilon > 0$ and pick $L \geq L_0(\epsilon)$. If $\theta \geq \theta_0(\epsilon, L)$ and we start with $\geq K$ particles in I_0 in $\bar\eta_t$ that are separated by $\geq 1/\log^2\theta$ at time 0 then we will with probability $\geq 1 - 2\epsilon$ have $\geq K$ particles in I_{-1} and I_1 that are separated by $\geq 1/\log^2\theta$ at time L^2. We do this in a system in which particles are killed when they leave $[-4L,4L]^2$ so the events for I_0 and I_m are independent when $m \geq 4$.*

With Lemma 4.4 established the existence of a stationary distribution follows from Theorem 4.2 in [4].

5 Survival in one dimension

In this section we consider the version of the contact plus voter process in which births occur at rate 1. The dual process has deaths at rate δ, births across each edge at rate 1, and random walks that jump to each nearest neighbor at rate ν, with coalescence when two walks hit. Let L be a large constant that will be chosen later. Let $I_m = 4mL\nu + [-0.2L\nu, 0.2L\nu]$. We will show for the dual process with $\delta = 0$ that if we start with one particle in I_0 at time 0 then with high probability we will have one in I_0 and one in I_1 at time $10L\nu$. We will do this by following the behavior of two tagged particles in the dual. In order to have the events in our construction one dependent, we will also show that with high probability the particles do not leave $[-2L\nu, 6L\nu]$.

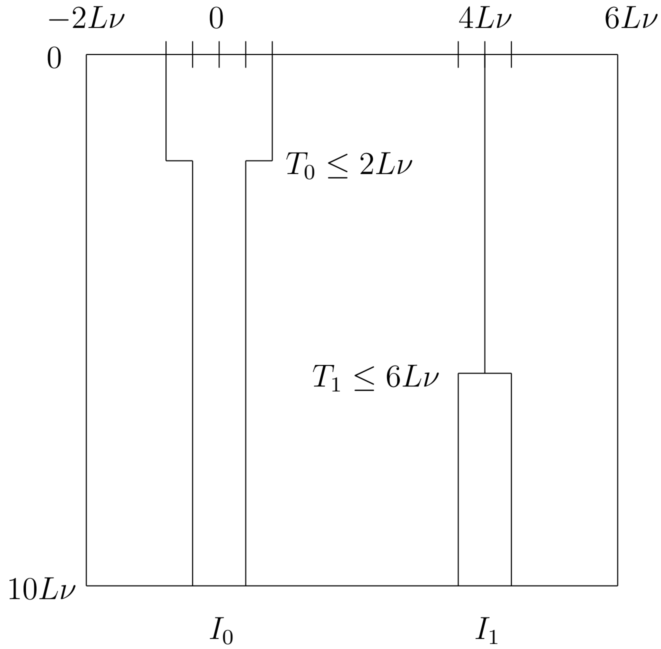

Figure 1: Picture of the block construction in $d = 1$. Time runs down the page. The white particle stays in the region on the left. Once the red particle hits $L\nu$ at time T_1 it stays within the region on the right.

The Key to the proof is a trick used by Bramson and Griffeath [1]. We follow a tagged particle in the dual that moves according to the following rules: when it is affected by a random walk event it must jump, but when there is a branching event, it follows the birth if and only if it takes it to the right. This process, which we call $X_r(t)$, makes jumps

$$\begin{aligned} x &\to x+1 \quad \text{at rate } \nu+1, \\ x &\to x-1 \quad \text{at rate } \nu. \end{aligned}$$

It is easy to check that $X_r(2\nu t)/2\nu \Rightarrow B_r(t) = B(t)+t$. Let X_ℓ denote the analogous process that only follows births to the left and has limit $B_\ell(t) = B(t) - t$, and let X_c denote the process the ignores branching arrows and has limit $B_c(t) = B(t)$.

The Construction. Suppose we have a particle at $x \in I_0$. There are three things to prove.

(i) Suppose the particle in I_0 is at x. We use X_r if $x < 0$ or X_ℓ if $x > 0$ to bring the particle to 0, which happens at time T_0. We will declare this part of the construction a success if the particle never leaves $[-0.4L\nu, 0.4L\nu]$ and $T_0 \leq 2L\nu$. Having brought our particle to 0, we change our rule so that the particle we are watching, which we call the white particle, only follows random walk arrows.

(ii) At time T_0 we begin to follow a red particle that moves according to $X_r(t)$. The red particle may coalesce with the white particle a large number of times, but will keep separating from it becuase it follows births to the right, while the white particle does not. Since the red particle is approximately a Brownian motion with drift, it will reach $4L\nu$ at some time T_1. We declare this part of the construction a success if $T_1 \le T_0 + 6L\nu$. At time T_1, we change our rule so that the red particle, only follows random walk arrows.

(iii) The last thing we need is for the white particle not to leave $[-0.2L\nu, 0.2L\nu]$ in $[T_0, 10L\nu]$ and for the red particle not to leave $4L\nu + [-0.2L\nu, 0.2L\nu]$ in $[T_1, 10L\nu]$. Note that between time T_0 and time T_1, the red particle is always to the right of the white particle, and always to the left of $4L\nu$ so if the two events in the first sentence happen, the red particle will not leave the interval $[-0.2L\nu, 4.2L\nu]$.

Rescaled Construction. The processes B_r, B_ℓ, and B_c arise from X_r, X_ℓ, and X_c by scaling space and time by 2ν. Using the weak convergence of these processes it is enough to show that if L is large, then the correspondingly scaled events hold for the $B's$ with high probability.

(i) The initial particle starting in $[-0.1L, 0.1L]$ can be brought to 0 without exiting $[-0.2L, 0.2L]$ and gets there at time $\bar{T}_0 \le L$.

(ii) The red particle born at time $\bar{T}_0$ gets to $2L$ at time $\bar{T}_1 \le \bar{T}_0 + 3L$.

(iii) The white particle does not leave $[-0.1L, 0.1L]$ during $[\bar{T}_0, 5L]$ and the red particle does not leave $2L + [-0.1L, 0.1L]$ during $[\bar{T}_1, 5L]$.

Lemma 5.1. *Suppose $B_r(0) = x < 0$ and let $\bar{T}_0 = \inf\{t : B_r(t) = 0\}$. In this case, as $L \to \infty$*

$$\sup_{x\in[-0.1L,0]} P_x(\bar{T}_0 \ge L) \to 0 \quad \textit{and} \quad \sup_{x\in[-0.1L,0]} P_x\left(\inf_{t\le L} B_r(t) \le -0.2L\right) \to 0.$$

Proof. It suffices to prove the result when $x = -0.1L$. As $L \to \infty$, $B_r(tL)/L \Rightarrow -0.1 + t$ where $\Rightarrow$ indicates weak convergence in $C([0,1])$. The two results follow from this. $\square$

Let $\tau_1 = \inf\{t : B_r(t) = L\}$. Repeating the previous proof we see that $P_0(\tau_1 \ge 2L) \to 0$. To check (iii) now, use scaling to conclude that for any $0 < \gamma, M < \infty$

$$\lim_{L\to\infty} P\left(\sup_{t\le ML} |B_c(t)| \le \gamma L\right) = \lim_{L\to\infty} P\left(\sup_{0\le t\le 1} (ML)^{1/2} B(t) \le \gamma L\right) = 1.$$

At this point we have shown that if $L \ge L_0$ then all three events in the rescaled construction occur with probability $\ge 1-\epsilon$. If we take $L = L_0$ then the weak convergence of the X's to the B's implies that if $\nu \ge \nu_0$ all three events in the construction occur with probability $\ge 1 - 2\epsilon$. Up to this point the calculations are for the process with no death. If we pick $\delta = \eta/L\nu$ with η small then the probability of a death affecting either of our two tagged particles is $\le \epsilon$. The events in the block construction are one dependent, so applying Theorem 4.1 from Durrett's St. Flour Notes [4], we see that the dual process starting from a single particle at 0 survives with positive probability.

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