ODE and PDE limits for particle systems

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Evolutionary games with weak selection

The investigation is inspired by two papers. The first is for two strategy games. The second for games with \( n \geq 3 \) strategies.

**Q. When is a strategy favored by selection in a spatial games? I.e., in equilibrium its frequency \( > 1/n \).**


Evolutionary games: Homogeneously mixing

Given is a game matrix \( G_{i,j} \geq 0 \) and the frequencies \( x_j \) of strategies in the population, \( F_i = \sum_j G_{i,j} x_j \) is the fitness of strategy \( i \). Moran model like dynamics: each individual dies at rate 1, and is replaced by an individual chosen at random with probability proportional its fitness. Frequencies of strategies follow the replicator equation

\[
\frac{dx_i}{dt} = x_i (F_i - \bar{F})
\]

where \( \bar{F} = \sum_i x_i F_i \), average fitness

Note: If we add a constant to a column of \( G \) then \( F_i - \bar{F} \) is not changed.
Suppose space is the $d$-dimensional integer lattice. **Interaction kernel** $p(x)$ is a probability distribution with $p(x) = p(-x)$, finite range, covariance matrix $\sigma^2 I$. E.g., $p(x) = 1/2d$ for the nearest neighbors $x \pm e_i$, $e_i$ is the $i$th unit vector.

$\xi(x)$ is strategy used by $x$. Fitness is $\Phi(x) = \sum_y p(y - x) G(\xi(x), \xi(y))$.

**Birth-Death dynamics:** Each individual gives birth at rate $\Phi(x)$ and replaces the individual at $y$ with probability $p(y - x)$.

**Death-Birth dynamics:** Each particle dies at rate 1. Is replaced by a copy of $y$ with probability proportional to $p(y - x)\Phi(y)$. When $p(z) = 1/m$ for a set of neighbors $\mathcal{N}$, we pick with a probability proportional to its fitness.
Small selection

We are going to consider games with $\bar{G}_{i,j} = 1 + wG_{i,j}$ where $1$ is a matrix of all $1$'s, and $w$ is small. (Selection is small rather than weak since the population size is infinite.)

If the game matrix is $1$, B-D or D-B dynamics give the voter model. Remove an individual and replace it with a copy of a neighbor chosen at random (according to $p$). The evolutionary game with small selection is a voter model perturbation in the sense of Cox, Durrett, Perkins (2013) *Astérisque* volume 349, or arXiv:1103.1676

Restrict our attention to $d \geq 3$ so that the voter model has a one parameter family of stationary distributions.
**PDE limit**

**Theorem.** Flip rates are those of the voter model $+\epsilon^2 h_{i,j}(0, \xi)$. If we rescale space to $\epsilon \mathbb{Z}^d$ and speed up time by $\epsilon^{-2}$ then in $d \geq 3$

$$u_i^\epsilon(t, x) = P(\xi_{t\epsilon^{-2}}(x) = i)$$

converges to the solution of the system of partial differential equations:

$$\frac{\partial u_i}{\partial t} = \frac{\sigma^2}{2} \Delta u_i + \phi_i(u)$$

where the reaction term

$$\phi_i(u) = \sum_{j \neq i} \langle 1_{(\xi(0) = j)} h_{j,i}(0, \xi) - 1_{(\xi(0) = i)} h_{i,j}(0, \xi) \rangle u$$

and the brackets are expected value with respect to the voter model stationary distribution $\nu_u$ in which the densities are given by the vector $u$. 
Key to proof is duality

Voter model is dual to coalescing random walk. $\zeta_{s,t}^x$ is the individual at time $t - s$ who is responsible for the opinion of $x$ at time $t$. Two lineages that hit coalesce to one.

To handle the perturbation at times of a rate $O(\epsilon^2)$ Poisson process $T_n^x$, a particle at $x$ branches to include $x + y$ for all $y$ with $p(y) > 0$.

The collection of particles $I_s^{x,t}$ is called the influence set. If we know the values in $I_s^{x,t}$ at time $t - s$ then we can compute the value of $x$ at time $t$.

If we run time at rate $\epsilon^{-2}$ the influence set converges to branching Brownian motion. This shows $u(t, x)$ converges. Easy to check it satisfies PDE. See Chapter 2 of CDP.
Birth-Death dynamics

Recall the replicator equation:

\[
\frac{du_i}{dt} = \phi^i_R(u) \equiv u_i \left( \sum_k G_{i,k} u_k - \sum_{j,k} u_j G_{j,k} u_k \right).
\]

Let \( v_1, v_2 \) be independent with distribution \( p \) and define random walk coalescence probabilities

\[
p_1 = p(0|v_1|v_1 + v_2) \quad p_2 = p(0|v_1, v_1 + v_2)
\]

PDE is \( \partial u_i / \partial t = (1/2d)\Delta u + \phi^i_B(u) \) where

\[
\phi^i_B(u) = p_1 \phi^i_R(u) + p_2 \sum_{j \neq i} u_i u_j (G_{i,i} - G_{j,i} + G_{i,j} - G_{j,j})
\]
Two features of the answer

1. Space enters into the answer through the values of two constants.

\[ p_1 = p(0|v_1|v_1 + v_2) \quad p_2 = p(0|v_1, v_1 + v_2) \]

(Also true for Tarnita’s formulas.)

2. \( \phi_B \) is \( p_1 \) times the RHS of the replicator equation for the game matrix \( G + A \) where

\[ A_{i,j} = \frac{p_2}{p_1} (G_{i,i} + G_{i,j} - G_{j,i} - G_{j,j}) \]

“The effect of space is equivalent to changing the game matrix.” (Ohtsuki and Nowak proved this for the pair approximation.)
Evolutionary games on the torus

\[ T_L = (\mathbb{Z} \mod L)^d. \quad N = L^d. \quad \tilde{G} = 1 + wG. \quad w = \epsilon^2 \]

**Regime 1.** \( w \gg N^{-2/d} \).

Run time at rate \( \epsilon^{-2} \). Scale space by multiplying by \( \epsilon \gg L^{-1} \). Scaled torus converges to \( R^d \) and the limit is the PDE we saw on \( \mathbb{Z}^d \).

**Regime 2.** \( N^{-2/d} \gg w \gg N^{-1} \)

In this case the time scale for the perturbation to have an effect, \( \epsilon^{-2} \) is much larger than the time \( O(L^2) \) needed for a random walk to come to equilibrium, but much smaller than the time \( O(L^d) \) it takes for two random walks to hit. Because of this, the particles in the dual will (except for times \( O(L^2 \log L) \) after the initial time or a branching event) be approximately independent and uniformly distributed across the torus.
Regime 2 limit theorem

\[ U_i(t) = \frac{1}{N} \sum_{x \in \mathbb{T}_L} 1(\xi_{t\epsilon - 2}(x) = i) \]

**Theorem** Suppose that \( N^{-2/d} \gg w \gg N^{-1} \). If \( U_i(0) \to u_i(0) \) then \( U_i(t) \) converges uniformly on compact sets to \( u_i(t) \), the solution of

\[ \frac{du_i}{dt} = \phi_i(u) \quad u_i(0) = u_i \]

where \( \phi_i \) is the reaction term in the PDE.

Thus in Regime 2, we have “mean-field” behavior, but the reaction function in the ODE is computed using the voter model equilibrium, not the product measure that is typically used in heuristic calculations.
Tarnita’s formula

Suppose that in addition to the game dynamics each individual switches to a strategy chosen at random from the $n$ possible strategies at rate $\mu$. 

**Theorem.** Suppose that $N^{-2/d} \gg w \gg N^{-1}$. If $\mu \to 0$ and $\mu/w \to \infty$ slowly enough, then in an $n$-strategy game strategy $k$ is favored by selection if and only if

$$\phi_k(1/n, \ldots, 1/n) > 0.$$ 

or

$$(c_1 G_{k,k} - \bar{G}_{k,*} - \bar{G}_{*,k} - c_1 \bar{G}_{*,*}) + c_2 (\bar{G}_{k,*} - \bar{G}) > 0$$

Intuitively, in this regime the change from uniformity will be due to lineages that have one branching event. Our result shows that $c_1$ and $c_2$ can be expressed in terms of coalescence probabilities.
Let $G_n$ be a graph generated by the configuration model. Vertices have degree $k$ with probability $p_k$. We assign i.i.d. degrees $d_i$ to the vertices and condition the sum $d_1 + \cdots + d_n$ to be even. We attach $d_i$ half-edges to vertex $i$ and then pair the half-edges at random. We will assume that

(A0) the graph $G_n$ has no self-loops or parallel edges.

If $\sum_k k^2 p_k < \infty$ then $P(\text{A0})$ is bounded away from 0 as $n \to \infty$.

(A1) $p_m = 0$ for $m > M$, i.e., the degree distribution is bounded.

(A2) $p_k = 0$ for $k \leq 2$, so random walks have good mixing properties.
Latent Voter Model

The Latent Voter Model introduced by Lambiotte, Saramaki, and Blondel in 2009 models the spread of a technology through a social network. If you have just bought a new iPad and see your neighbors Microsoft Surface tablet then you are unlikely to change. We have states 1, 1*, 2, and 2*. The number indicates the technology that the individual owns while * indicates they are in a latent state where they will not change their opinion. Our process takes place on a graph generated by the configuration model. Letting $f_i$ be the fraction of neighbors in state $i$, the transition rates are as follows:

- $1 \rightarrow 2^*$ at rate $f_2$
- $2 \rightarrow 1^*$ at rate $f_1$
- $2^* \rightarrow 2$ at rate $\lambda$
- $1^* \rightarrow 1$ at rate $\lambda$
Construction

Each site $x$ has a Poisson process with rate 1. For each arrival we have a random choice of neighbor $Y^x_n$, $n \geq 1$. At time $T^x_n$, we draw an arrow from $Y^x_n$ to $x$ to indicate that if the individual at $x$ is active (not in state $1^*$ or $2^*$) at time $t$ then they will imitate the opinion at $Y^x_n$.

We introduce for each site $x$, a rate $\lambda$ Poisson process $W^x_n$, $n \geq 1$ of “wake-up dots” that return the voter to the active state.

- If there is only one voter arrow between two wake up dots, the result is an ordinary voter event.
- If between two wake up dots there are voter arrows to $x$ from two different neighbors, an event of probability $O(\lambda^{-2})$, then $x$ will change its opinion if and only at least one of the two neighbors has a different opinion.
LV as a voter model perturbation

There are $O(\lambda)$ wake up dots at a site in time $t$. We run time at rate $\lambda$ so we have some events with two arrows between successive wake-up dots. The probability of three or more arrows between two wake up dots $\to 0$.

If we let $y_1, \ldots y_{d(x)}$ be an enumeration of the nearest neighbors of $x$, the perturbation is

$$h_{1,2}(x, \xi) = 1\{\xi_t(x)=1\} \frac{2}{d(x)^2} \sum_{1 \leq k < \ell \leq d(x)} 1\{\xi(y_k) \text{ or } \xi(y_\ell) \in \{2,2^*\}\}$$

Similar formulas hold when the roles of 1 and 2 are interchanged.

$h_{1*,j} = h_{2*,j} \equiv 0$.

The reaction term is

$$\phi(u) = \langle h_{2,1}(0, \xi) - h_{1,2}(0, \xi) \rangle_u = c_G u(1 - u)(1 - 2u)$$
ODE limit on random graph

Let $\pi(x) = d(x)/D$ be the stationary distribution for the random walk.

$$U^n(t) = \sum_x \pi(x) 1\{\xi_{\lambda t}(x) = 1\}$$

**Theorem.** Suppose that $\log n \ll \lambda n \ll n$. If $U^n(0) \rightarrow u_0$ then $U^n(t)$ converges in probability and uniformly on compact sets to $u(t)$, the solution of

$$\frac{du}{dt} = c_G u(1 - u)(1 - 2u) \quad u(0) = u_0.$$

$\log n \ll \lambda n$ implies that random walks will randomize their positions between non-voter events. $\lambda n \ll n$ since two random walks take time $O(n)$ to hit.
Long time survival on random graph

The latent voter model has two absorbing states $\equiv 1$ and $\equiv 2$ so on a finite graph it will eventually reach one of them. However, by analogy with the contact process on the torus and or on power-law random graphs, we expect survival for time $\exp(\gamma n)$ for some $\gamma > 0$.

**Theorem.** Suppose that $\log n \ll \lambda_n \ll n$. Let $\epsilon > 0$ and $m < \infty$. If $U^n(0) \rightarrow u_0 \in (0, 1)$ there is a $T_0(\epsilon)$ that depends on the initial density so that for any $m < \infty$ if $n$ is large then with high probability

$$|U^n(t) - 1/2| \leq \epsilon \quad \text{for all } t \in [T_0(\epsilon), n^m].$$


Both papers and the slides for this talk are on my web page.