Evolving voter model

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A result of the 2010-2011 SAMSI program on complex networks

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Holme and Newman (2006)

They begin with a network of \(N\) nodes and \(M\) edges, where each node \(x\) has an opinion \(\xi(x)\) from a set of \(G\) possible opinions and the number of people per opinion \(\gamma_N = N/G\) stays bounded as \(N\) gets large.

On each step a vertex \(x\) is picked at random. If its degree \(d(x) = 0\), nothing happens. If \(d(x) > 0\),

(i) with probability \(\alpha\) an edge attached to vertex \(x\) is selected and the other end of that edge is moved to a vertex chosen at random from those with opinion \(\xi(x)\).

(ii) otherwise (i.e., with probability \(1 - \alpha\)) a random neighbor \(y\) of \(x\) is selected and the opinion of \(x\) is set to \(\xi(y)\).

Eventually there are no edges that connect different opinions and the system freezes at time \(\tau_N\).

Extreme Cases

When \(\alpha = 1\) only rewiring steps occur, so once all of the \(M\) edges have been touched the graph has been disconnected into \(G\) components, each of which is small. By results for the coupon collector’s problem, \(\tau_N \sim M \log M\) updates.

When \(\phi = 0\) this is a voter model on a static graph. If we use an Erdős-Rényi random graph in which each vertex has average degree \(\lambda > 1\) then there is a giant component with a positive fraction of the vertices and a large number of small components with size \(O(\log N)\). The giant component will reach consensus after \(\tau_N = O(N^2)\) updates, so the end result is one opinion with a large number of followers while all of the other populations are small.

Community sizes \(N = 3200, M = 6400, \gamma = 10\).

Finite size scaling

Events happen on each oriented edge \((x, y)\) at times of a rate one Poisson process. (Isothermal voter model.) \(N^2\) updates \(\rightarrow\) time \(N\).

If the voters at the two ends of the edge agree then we do nothing.

If they disagree, then with probability \(1 - \alpha\) the voter at \(x\) adopts the opinion of the voter at \(y\).

With probability \(\alpha\), \(x\) breaks its connection to \(y\) and makes a new connection to a voter chosen at random:

(i) from all of the vertices in the graph “rewire to random”,

(ii) from those that share its opinion “rewire to same.”

Opinions \(\{0, 1\}\). Initial state product measure with density \(u\).
Rewire to random with \( u = 1/2 \)

Figure: Erdos-Renyi, \( \lambda = 4 \), \( N = 100,000 \)

Rewire to same: discontinuous transition

The simulation that showed us the answer

Holley and Liggett (1975)

Consider the voter model on the \( d \)-dimensional integer lattice \( \mathbb{Z}^d \) in which each vertex decides to change its opinion at rate 1, and when it does, it adopts the opinion of one of its \( 2d \) nearest neighbors chosen at random.

In \( d \leq 2 \), the system approaches complete consensus. That is if \( x \neq y \) then \( P(\xi_t(x) \neq \xi_t(y)) \rightarrow 0 \).

In \( d \geq 3 \) if we start from \( \xi_0^d \) product measure with density \( \rho \), i.e., \( \xi_0^d(x) \) are independent and equal to 1 with probability then \( \xi_t^d \) converges in distribution to a limit \( \xi_\rho^p \), which is a stationary distribution for the voter model.

Cox and Greven (1990)

The voter model on the torus in \( d \geq 3 \) at time \( Nt \) then it locally looks like \( \nu_\theta(t) \) where the density changes according to the Wright-Fisher diffusion:

\[
d\theta_t = \sqrt{2\theta_t(1-\theta_t)} dB_t
\]

The quantity under the square root is the fraction of discordant edges under \( \nu_\theta(t) \).

There is a one parameter family of quasi-stationary distributions, and the parameter changes according to a diffusion.

quasi-stationary since in the finite voter model all 0’s and all 1’s are absorbing.
Finite dim. distr. on a random graph

A definition from the theory of graph limits of Lovasz et al. $N_{ijk}$ is the number of homomorphisms from the labeled graph

\[
\begin{array}{ccc}
i & j & k \\
a & b & c
\end{array}
\]

into our labeled graph $(G, \xi)$. When $i = 0$, $j = 1$, $k = 0$ every triple is counted twice but this seems like the natural definition.

$N_{ijk}$ are polynomials?

Bill Shi’s simulations for $\lambda = 4$, $\alpha = 0.5$

$N_01 = -3.42x^2 + 3.42x - 0.38$

$N_{110} = -13.53x_3 + 10.87x^2 + 1.19x - 0.30$

$N_{001} = 13.54x^3 - 29.74x^2 + 17.67x - 1.77$

$N_{101} = -10.14x^3 + 10.93x^2 - 1.89x + 0.08$

$N_{010} = 10.15x^3 - 19.51x^2 + 10.46x - 1.02$

$N_{110}(x) = N_{001}(1 - x)$, $N_{101}(x) = N_{010}(1 - x)$

Evolution Equations

\[
\begin{align*}
\frac{dN_{10}}{dt} &= -(2 - \alpha)N_{10} + (1 - \alpha)[N_{100} - N_{010} + N_{110} - N_{101}] \\
\frac{1}{2} \frac{dN_{11}}{dt} &= (1 - \alpha(1 - u))N_{11} + (1 - \alpha)[N_{011} - N_{101}] \\
\frac{1}{2} \frac{dN_{00}}{dt} &= (1 - \alpha u)N_{00} + (1 - \alpha)[N_{010} - N_{100}]
\end{align*}
\]

Of course $N_{11} + 2N_{10} + N_{00} = M$, the number of edges.

\[
\sum_{ijk} N_{ijk} = \sum_y d(y)(d(y) - 1)
\]

\[
\frac{d}{dt} \sum_{ijk} N_{ijk} = -2\alpha[N_{101} + N_{010} + N_{100} + N_{110}] + 4\alpha N_{10} \frac{M}{N}
\]
One equation short

When $\lambda = 4$, $\alpha = 0.5$

From equations:

- $N_{01}/N_{01}$: $(2a_3 + 2b_1) - 2b_2u = -0.23 + 2.96u$
- $N_{00}/N_{01}$: $2a_3 + 2b_2u = 2.73 - 2.96u$
- $N_{10}/N_{01}$: $(2a_3 + 2b_2 + 1) - (2b_3 - 1)u = 0.77 + 3.96u$
- $N_{10}/N_{01}$: $(2a_3 + 2) + (2b_3 - 1)u = 4.73 - 3.96u$

From $(d/dt)\sum_{ijk} N_{ijk} = 0$ we get

$$2\lambda(1 - \alpha) = 4a_3 + 2 + 2b_3 - \alpha$$

Equations and simulation agree if $2a_3 = 2.73$ and $2b_3 = -2.96$.

Arches for rewire to random

Why do the arches behave differently in the two versions of the model?

Extensions

We get the same result if we start with a random 4-regular graph

OR

if we designate $uN$ vertices as 1 and $(1 - u)N$ as 0 and connect an $i$ node to a $j$ node with probability $p_{ij}/N$.

In the second case by choosing the $p_{ij}$ correctly we can achieve any possible value of $N_0/N$ and $N_{00}/M$ where $M = \lambda N/2$.

For these initial conditions we quickly move to the arch of quasistationary distributions.

Degree distribution Poisson?

Figure: Note that constant term $\approx 0$, explaining discontinuous distribution.
Open Problems

Prove quasi-stationary distributions exist when \( \alpha \) is small.

When \( \alpha = 0 \) we know the quasi-stationary distributions for the voter model, so it is natural to try a perturbation argument. However when we consider \((G, \xi)\) for the voter model the \( G \) does not change.

Singular perturbation problem.
Work with Jonathan Mattingly and David Sivakoff.

First step: understand \( \lim_{\alpha \to 0} \) which is \( \neq \) system with \( \alpha = 0 \).

Answer: rewire, voter model equilibrates, rewire again.

Harder second step: There is a unique stationary distribution on the space of graphs with \( N \) vertices and \( M \) edges.

Conjecture. In the rewire to random model if \( \alpha < \alpha_c(1/2) \) and \( \nu(\alpha) < u \leq 1/2 \) then starting from product measure with a density \( u \) of 1’s, the evolving voter model converges rapidly to a quasi-stationary distribution \( \nu_{\alpha, u} \).

At time \( tN \) the evolving voter model looks locally like \( \nu_{\alpha, \theta(t)} \) where the density changes according to a generalized Wright-Fisher diffusion process

\[
d\theta_t = \sqrt{(1 - \alpha)}[c_\alpha \theta_t (1 - \theta_t) - b_\alpha] dB_t
\]

until \( \theta_t \) reaches \( \nu(\alpha) \) or \( 1 - \nu(\alpha) \).

Rewire to same is similar but \( b_\alpha = 0 \).

What happens with more than two initial types?

Arch is \( c_\alpha (1 - \sum_i u_i^2)/2 - b_\alpha \) for same \( c_\alpha, b_\alpha \)?

\( c_\alpha \) and \( b_\alpha \) as a function of \( \beta = \alpha/(1 - \alpha) \)

Generator \( L = V + \beta R \), Voter, Rewire

Figure: Conjecture. \( c_\alpha = a + 2b\beta \), \( b_\alpha = -b\beta \)

Linear in \( \beta \), not a power series.
Infinitely many phase transitions

Suppose \( c_\alpha = a + 2b\beta \), \( b_\alpha = -b\beta \), \( \beta = \frac{\alpha}{1+\alpha} \)

If there are \( k \) opinions the smallest value of \( 1 - \sum_i u_i^2 \) is \( 1 - 1/k \)

\( N_k = \frac{1}{2}(1 - 1/k) - b_\alpha \)

This is 0 when \( \beta_k = \frac{a(k-1)}{2b} \). (needs 2b and \( -b \)).

\( a = 1.3, b = 0.25 \)
\( k = 2, \beta_2 = 2.6, \alpha_2 = \frac{\beta_2}{1 + \beta_2} = 0.72 \)
\( k = 3, \beta_3 = 5.2, \alpha_3 = 0.84 \)

\( \alpha_k \) is the critical value for starting with \( k \) types.

If we start with a large number types then for \( \alpha < \alpha_2 \) we may end up with two or more types at the end, but if \( \alpha_2 < \alpha < \alpha_3 \) we will always end up with three or more, etc.

Higher order statistics

\[\alpha \quad y = \frac{N_{010}}{N} \text{ versus } x = \frac{N_1}{N}\]

\[0.1 \quad y = 8.3256x^3 - 16.8145x^2 + 8.6220x - 0.14048\]
\[0.2 \quad y = 8.3574x^3 - 17.8626x^2 + 8.6622x - 0.24319\]
\[0.3 \quad y = 8.6960x^3 - 17.2870x^2 + 9.0065x - 0.41525\]
\[0.4 \quad y = 8.9222x^3 - 17.6819x^2 + 9.3873x - 0.63602\]
\[0.5 \quad y = 9.9584x^3 - 19.2445x^2 + 10.3545x - 1.0078\]
\[0.6 \quad y = 11.7247x^3 - 21.7348x^2 + 11.8134x - 1.5414\]
\[0.7 \quad y = 16.9464x^3 - 29.4114x^2 + 16.2660x - 2.7904\]

\( \alpha = 0 \)
\[ax(1-x)^2 + bx(1-x) = ax^3 - (2a + b)x^2 + (a + b)x\]

\( a = c_\lambda \bar{p}(x|y) \quad b = c_\lambda \bar{p}(xz|y) \)

\( c_\lambda = \sum d_\lambda(d_\lambda - 1)/N \), \( \bar{p} \) coalesce probs averaged over triples

\( N_{010} \) Coefficients Quadratic: Constant Term

Graph statistics \( N_{01}, \ N_{010}, \ N_{100}, \ etc. \) are polynomials in \( u \) and \( \beta \)

Unfortunately there does not seem to be a dual for the evolving voter model.