Gauge-string duality and Elizabeth Meckes's infinitesimal Stein's method

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Quantum Yang–Mills theories

- Quantum gauge theories, also known as quantum Yang–Mills theories, are components of the Standard Model of quantum mechanics.
- In spite of many decades of research, physically relevant quantum gauge theories have not yet been constructed in a rigorous mathematical sense.
- The most popular approach to solving this problem is via the program of constructive field theory.
- In this approach, one starts with a statistical mechanical model on the lattice; the next step is to pass to a continuum limit of this model; the third step is to show that the continuum limit satisfies certain 'axioms'; if these axioms are satisfied, then there is a standard machinery which allows the construction of a quantum field theory.
- Taking this program to its completion is one of the Clay millennium prize problems.

Lattice gauge theories

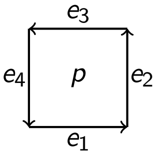
- The statistical mechanical models considered in the first step of the above program are known as lattice gauge theories.
- A lattice gauge theory may be coupled with a matter field (such as a Higgs field), or it may be a pure lattice gauge theory.
- We will only deal with pure lattice gauge theories in this talk.
- A pure lattice gauge theory is characterized by its gauge group (usually a compact matrix Lie group), the dimension of spacetime, and a parameter known as the coupling strength.

Notation

- We will now define lattice gauge theories.
- Let $N \geq 1$ and $d \geq 2$ be two integers.
- Let G be a closed connected subgroup of $U(N)$.
- Let E be the set of positively oriented nearest-neighbor edges of $\mathbb{Z}^d$.
- Let Ω be the set of all functions from E into G. That is, an element $\omega \in \Omega$ assigns a matrix $\omega_e \in G$ to each edge $e \in E$.
- If $\omega \in \Omega$ and e is a negatively oriented edge, we define $\omega_e := \omega_{e^{-1}}^{-1}$, where e^{-1} is the positively oriented version of e.

Plaquettes

- A plaquette in $\mathbb{Z}^d$ is a sequence of four positively oriented edges that form the boundary of a square.
- Let P be the set of all plaquettes in $\mathbb{Z}^d$.
- Given some $p \in P$ and $\omega \in \Omega$, we define ω_p as follows.
- Write p as a sequence of directed edges e_1, e_2, e_3, e_4, each one followed by the next.



- Let $\omega_p := \omega_{e_1}\omega_{e_2}\omega_{e_3}\omega_{e_4}$.
- Although there is an ambiguity in this definition about the choice of e_1, that is not problematic because we will only use the quantity $\Re(\mathrm{Tr}(\omega_p))$, which is not affected by this ambiguity.

Lattice gauge theory

- Endow the product space $\Omega = G^E$ with the product σ-algebra and let λ denote the normalized product Haar measure on Ω.
- Pure lattice gauge theory on $\mathbb{Z}^d$ with gauge group G and coupling parameter β (equal to the inverse of the squared coupling strength) is formally defined as the probability measure

  $$d\mu(\omega) = Z^{-1} e^{-\beta H(\omega)} d\lambda(\omega)$$

  on Ω, where H is the formal Hamiltonian

  $$H(\omega) := -\sum_{p \in P} \Re(\mathrm{Tr}(\omega_p))$$

  and Z is the normalizing constant.
- Note that this does not make sense as stated, since the infinite series defining H is not convergent for most $\omega \in \Omega$.

Precise definition

- Although the definition of the probability measure μ as stated above does not make sense since the series defining H may not be convergent, the conditional distribution of any finite set of ω_e's given all other ω_e's, under such a hypothetical probability measure μ, is well-defined.
- In the language of mathematical physics, this defines is a specification (of conditional distributions).
- Any actual probability measure μ on Ω which has these specified conditional distributions is called a Gibbs measure for this specification.
- It is not obvious that Gibbs measures exist. In the case of lattice gauge theories with compact gauge groups, the existence of at least one Gibbs measure follows from standard results.
- Uniqueness is generally an open question unless β is small.

Wilson loop observables

- Let us fix a lattice gauge theory on $\mathbb{Z}^d$ with gauge group G and coupling parameter β.
- Let π be a finite-dimensional irreducible unitary representation of the group G, and let χ_π be the character of π.
- Let ℓ be a closed loop in $\mathbb{Z}^d$, with directed edges $e_1, \ldots, e_k$.
- Given a configuration ω, the Wilson loop variable $W_\ell(\omega)$ is defined as $W_\ell(\omega) := \chi_\pi(\omega_{e_1}\omega_{e_2}\cdots\omega_{e_k})$.
- Let $\langle W_\ell \rangle$ denote the expected value of $W_\ell(\omega)$ under a given Gibbs measure of our theory. This is known as a Wilson loop expectation.
- Calculating Wilson loop expectations is one of the main problems in lattice gauge theories, for a variety of reasons (which I do not have the time to go into).

Gauge-string duality

- Gauge theories (i.e., Yang–Mills theories) are theories of the quantum world. String theories are theories of gravity.
- It is a major goal of theoretical physics to make a connection between the above two.
- Physicists have been aware of a duality between gauge theories and string theories since the 1970s. A concrete duality relation found by Maldacena (1997) kicked off a vast field of research, now known as gauge-string duality or gauge-gravity duality or AdS-CFT duality.

Large N gauge theories: 't Hooft's approach

- Gauge groups such as $SU(5)$, $SU(3)$ and $SU(2) \times U(1)$ are the ones that are relevant for physical theories.
- However, theoretical understanding is difficult to achieve.
- 't Hooft (1974) suggested a simplification of the problem by considering groups such as $SU(N)$ where N is large.
- The $N \to \infty$ limit, after replacing β by $N\beta$, simplifies many theoretical problems. This is known as the 't Hooft limit.
- The result that I am going to present gives a duality between a lattice gauge theory in the 't Hooft limit and a string theory on the lattice.

A string theory on the lattice

- Basic objects: Collections of finitely many loops in $\mathbb{Z}^d$, called 'strings'. Analogous to strings in the continuum.
- Strings can evolve in time according to certain rules.
- At each time step, only one loop in a string is allowed to be modified.
- Four possible modifications:
  - Positive deformation. (Addition of a plaquette without erasing edges.)
  - Negative deformation. (Addition or deletion of a plaquette that involves erasing at least one edge.)
  - Positive splitting. (Splitting a loop into two loops without erasing edges.)
  - Negative splitting. (Splitting a loop into two loops in a way that erases at least one edge.)

Trajectories

- The evolution of a string is called a trajectory.
- A trajectory is called vanishing if it ends in nothing in a finite number of steps.

Action of a vanishing trajectory

- The lattice string theory defined here has a parameter β.
- Depending on the value of β, each step of a trajectory is given a weight, as follows.
- Let m be the total number of edges in the string before the step is taken.
- The weight of the step is defined to be:

$$\begin{aligned}
-\beta/m &\text{ if the step is a positive deformation;} \\
\beta/m &\text{ if the step is a negative deformation;} \\
-2/m &\text{ if the step is a positive splitting;} \\
2/m &\text{ if the step is a negative splitting.}
\end{aligned}$$

- The weight or action of a vanishing trajectory X is defined to be the product of the weights of the steps in the trajectory. Denoted by $w_\beta(X)$.

Gauge-string duality in the 't Hooft limit

Theorem (C., 2019)

There exists $\beta_0 > 0$ such that the following is true. Let ℓ be a fixed loop in $\mathbb{Z}^d$. Let $\langle W_\ell \rangle$ denote the expectation of the Wilson loop variable W_ℓ (for the defining representation of $SO(N)$) with respect to any Gibbs measure of $SO(N)$ lattice gauge theory on $\mathbb{Z}^d$ with coupling parameter $N\beta$. If $|\beta| \le \beta_0$, then

$$\lim_{N\to\infty} \frac{\langle W_\ell \rangle}{N} = \sum_{X \in \mathcal{X}(\ell)} w_\beta(X)\,,$$

where $\mathcal{X}(\ell)$ is the set of all vanishing trajectories starting at ℓ and $w_\beta(X)$ is the action of X defined earlier. Moreover, the infinite sum on the right is absolutely convergent.

Neighborhood of a string

- In the next few slides, I will try to give an outline of the proof of this theorem.
- In addition to deformations and splittings, two additional operations on strings are needed in the proof: mergers and twistings.
- Let

$$\begin{aligned}
\mathbb{D}^+(s) &:= \{s' : s' \text{ is a positive deformation of } s\}\,,\\
\mathbb{D}^-(s) &:= \{s' : s' \text{ is a negative deformation of } s\}\,,\\
\mathbb{S}^+(s) &:= \{s' : s' \text{ is a positive splitting of } s\}\,,\\
\mathbb{S}^-(s) &:= \{s' : s' \text{ is a negative splitting of } s\}\,,\\
\mathbb{M}^+(s) &:= \{s' : s' \text{ is a positive merger of } s\}\,,\\
\mathbb{M}^-(s) &:= \{s' : s' \text{ is a negative merger of } s\}\,,\\
\mathbb{T}^+(s) &:= \{s' : s' \text{ is a positive twisting of } s\}\,,\\
\mathbb{T}^-(s) &:= \{s' : s' \text{ is a negative twisting of } s\}\,.
\end{aligned}$$

The master loop equation

Theorem (C., 2019)

For a string $s = (\ell_1, \ldots, \ell_n)$, *define*

$$\phi(s) := \frac{\langle W_{\ell_1} W_{\ell_2} \cdots W_{\ell_n} \rangle}{N^n}.$$

Let $|s|$ *be the total number of edges in* s. *Then*

$$\begin{aligned}(N-1)|s|\phi(s) &= \sum_{s' \in \mathbb{T}^-(s)} \phi(s') - \sum_{s' \in \mathbb{T}^+(s)} \phi(s') + N \sum_{s' \in \mathbb{S}^-(s)} \phi(s') \\ &- N \sum_{s' \in \mathbb{S}^+(s)} \phi(s') + \frac{1}{N} \sum_{s' \in \mathbb{M}^-(s)} \phi(s') - \frac{1}{N} \sum_{s' \in \mathbb{M}^+(s)} \phi(s') \\ &+ N\beta \sum_{s' \in \mathbb{D}^-(s)} \phi(s') - N\beta \sum_{s' \in \mathbb{D}^+(s)} \phi(s').\end{aligned}$$

Proof sketch, given the loop equation

- The loop equation relates the expectation of the Wilson variable for one string with the expectations of a set of 'neighboring strings'.
- The recursion naturally leads to a formal expression in terms of a sum over trajectories of strings.
- Main challenge is to prove convergence. This is the part that needs β to be small. I will not talk about this part.
- The proof of the loop equation is obtained via a version of Stein's method that I learned from Elizabeth Meckes when I was a student at Stanford. This will be explained in the next few slides.

Meckes's infinitesimal Stein's method

- Let G be a compact Lie group and let Q be G-valued random variable distributed according to the Haar measure.
- Suppose that for each $\varepsilon > 0$, we have a G-valued random variable Q_ε such that (Q, Q_ε) is an exchangeable pair of random variables.
- Then for any bounded measurable $f, g : G \to \mathbb{R}$,

$$\mathbb{E}[(f(Q_\varepsilon) - f(Q))g(Q)] = -\frac{1}{2}\mathbb{E}[(f(Q_\varepsilon) - f(Q))(g(Q_\varepsilon) - g(Q))].$$

- Define an operator T as

$$Tf(x) := \lim_{\varepsilon \to 0} \frac{\mathbb{E}(f(Q_\varepsilon)|Q = x) - f(x)}{\varepsilon^2},$$

  assuming that the limit exists.
- Then, taking $g \equiv 1$ gives $\mathbb{E}(Tf(Q)) = 0$ for any f.
- Such a T is called a Stein operator for the Haar measure.
- Elizabeth was the first to systematically investigate Stein operators for Haar measures and other similar objects.

Exchangeable pair for $SO(N)$

- Following constructions in Elizabeth's thesis, we construct an exchangeable pair for $SO(N)$ as follows.
- Choose (I, J) uniformly at random from $\{(i,j) : 1 \le i \neq j \le N\}$.
- Let η be uniformly distributed in $\{-1, 1\}$.
- Let R_ε be the $N \times N$ matrix whose $(i,j)^{\text{th}}$ entry is

$$\begin{cases} \sqrt{1-\varepsilon^2} & \text{if } i = j = I \text{ or } i = j = J, \\ \eta\varepsilon & \text{if } i = I \text{ and } j = J, \\ -\eta\varepsilon & \text{if } i = J \text{ and } j = I, \\ 1 & \text{if } i = j \text{ and } i \notin \{I, J\}, \\ 0 & \text{in all other cases.} \end{cases}$$

- Finally, let $Q_\varepsilon := R_\varepsilon Q$. It turns out that (Q, Q_ε) is an exchangeable pair.

Stein equation for $SO(N)$

- Recall the equation

$$\mathbb{E}[(f(Q_\varepsilon)-f(Q))g(Q)] = -\frac{1}{2}\mathbb{E}[(f(Q_\varepsilon)-f(Q))(g(Q_\varepsilon)-g(Q))].$$

- Dividing both sides by ε^2 and sending $\varepsilon \to 0$, we get the following.

Theorem (C., 2019)

Let f and g be C^2 functions in a neighborhood of $SO(N) \subseteq \mathbb{R}^{N^2}$, and let $\mathbb{E}(\cdot)$ denote expectation with respect to the Haar measure. Then

$$\mathbb{E}\biggl(\sum_{i,k} x_{ik}\frac{\partial f}{\partial x_{ik}} g\biggr) = \frac{1}{N-1}\mathbb{E}\biggl(\sum_{i,k}\frac{\partial^2 f}{\partial x_{ik}^2} g - \sum_{i,j,k,k'} x_{jk}x_{ik'}\frac{\partial^2 f}{\partial x_{ik}\partial x_{jk'}} g + \sum_{i,k}\frac{\partial f}{\partial x_{ik}}\frac{\partial g}{\partial x_{ik}} - \sum_{i,j,k,k'} x_{jk}x_{ik'}\frac{\partial f}{\partial x_{ik}}\frac{\partial g}{\partial x_{jk'}}\biggr).$$

How to prove the loop equation

- Fix some edge $e \in \ell$.
- Let $Q = (q_{ij})_{1 \le i,j \le N}$ be the element of $SO(N)$ attached to e.
- Fact: If m is the number of occurrences of e and e^{-1} in ℓ, then
$$W_\ell = m \sum_{i,j} q_{ij} \frac{\partial W_\ell}{\partial q_{ij}} .$$
- Let g be the density of the lattice gauge theory (restricted to a finite cube) with respect to the product Haar measure.
- If $\langle \cdot \rangle$ is expectation in the lattice gauge theory and $\mathbb{E}(\cdot)$ is expectation with respect to the product Haar measure, then
$$\langle W_\ell \rangle = \mathbb{E}(W_\ell g) = m\, \mathbb{E}\biggl(\sum_{i,j} q_{ij} \frac{\partial W_\ell}{\partial q_{ij}} g \biggr) .$$
- One can now apply the Stein equation to the right-hand side. It turns out that the resulting identity is the master loop equation that was written down earlier.