Eigenvalue is not a dirty word
My mathematical collaborations with Elizabeth Meckes

Mark Meckes
Case Western Reserve University
Currently visiting Oxford University

Southeastern Probability Conference
17 May 2021
From the preface of Elizabeth’s book:

If my writing helps to illuminate the ideas I have tried to describe, it is because I got to talk it out first at the breakfast table.
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If my writing helps to illuminate the ideas I have tried to describe, it is because I got to talk it out first at the breakfast table.
We weren’t the only ones at that breakfast table:

Peter and Juliette
You may have met those two before
You may have met those two before at a conference,
You may have met those two before at a conference, or an AMS meeting,
You may have met those two before

at a conference,

or when we went somewhere to give seminar talks.

or an AMS meeting,
Meckes family fun fact

Juliette got her first passport as a baby to go with us to a conference in Banff...
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and so did Peter.
Meckes family fun fact

Juliette got her first passport as a baby to go with us to a conference in Banff... and so did Peter.

(This wasn’t the first conference for either of them.)
Elizabeth grew up around academic shop talk:

Intestinal Uptake and Metabolism of Auranofin, A New Oral Gold-Based Antiarthritides Drug

Katherine Tepperman*
Richard Finer
Stuart Donovan

Department of Biological Sciences,
University of Cincinnati,
Cincinnati, Ohio 45221

R. C. Elder, J. Doi
David Ratliff, Kin Ng

Department of Chemistry,
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Katherine Tepperman = “Mom”
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Katherine Tepperman = “Mom”
R.C. Elder = “Dad”
It goes back farther than that:

FOURIER EXPANSIONS OF DOUBLY PERIODIC FUNCTIONS OF THE THIRD KIND.*

By John D. Elder.


John D. Elder = “Grandpa John”

METABOLIC AND ENDOCRINE PHYSIOLOGY

Jay Tepperman = “Grandpa Jay”

Helen M. Tepperman = “Grandma”
The dark side to this kind of background...

Young Katherine Tepperman automatically stopped listening to the dinner table conversation whenever she heard “glucose-6-phosphate”.

Role of hormones in glucose-6-phosphate dehydrogenase adaptation of rat liver

HELEN M. TEPPERM AN AND JAY TEPPERM AN (With the Technical Assistance of Janet M. DeWitt and Andrew Branch)
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She had to unlearn this habit on her way to becoming a biologist!
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Biophysical Journal Volume 79 August 2000 853–862

Combined Allosteric and Competitive Interaction between Extracellular Na\(^+\) and K\(^+\) During Ion Transport by the \(\alpha_1\), \(\alpha_2\), and \(\alpha_3\) Isoforms of the Na, K-ATPase

David M. Balshaw,* Lauren A. Millette,† Katherine Tepperman,† and Earl T. Wallick*
The dark side to this kind of background...

Young Elizabeth Elder (later Meckes) tuned out when she heard “Na, K-ATPase”.

She sidestepped having this as a potential problem by going into pure mathematics.
The next generation

Juliette says she tunes out when she hears “eigenvalue”.

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2004: Elizabeth was a PhD student at Stanford and I was a lecturer.
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Are marginals of the uniform measure on a high-dimensional convex body typically almost Gaussian?
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and quickly got stuck.
I asked Elizabeth for help and the result was:

“Theorem” (E. and M. Meckes, 2007)

If $n$ is big and a probability measure $\mu$ on $\mathbb{R}^n$ has certain kinds of symmetries, and satisfies some other hypotheses, then certain (one-dimensional) marginals of $\mu$ are quantifiably almost Gaussian.
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**Remarks:**

- Think of this as a Berry–Esseen-type theorem with stochastic independence replaced by a geometric symmetry condition.
- The “other hypotheses” are now known to hold for uniform measures on convex bodies.
- Elizabeth and I each wrote follow-ups to this work but didn’t work together on this again.
Some time (and one child) later...

“Theorem” (Chatterjee–Ledoux, 2009)

If $1 \ll k \leq n$ and $A$ is an $n \times n$ Hermitian matrix, then the eigenvalues of almost all $k \times k$ principal submatrices of $A$ are almost the same.
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Elizabeth and I realized we could prove a coordinate-free version of this by combining some basic matrix analysis with tools she was using to study random marginals:

"Theorem" (E. and M. Meckes, 2011)

If $1 \ll k \leq n$ and $A$ is an $n \times n$ Hermitian matrix, then the eigenvalues of almost all $k$-dimensional compressions of $A$ are almost the same.
A little while after that...

Elizabeth improved her methods and thus her results for random marginals (measure-theoretic Dvoretzky theorem).
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We applied the same improvements to eigenvalues of random compressions, then extended the same methods to eigenvalues of other random matrix models.
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For example:

**Theorem (E. and M. Meckes, 2013)**

Let $U$ be an $n \times n$ random matrix uniformly chosen from any of the classical compact matrix groups. Then

$$\mathbb{P} \left[ W_1(\mu_U, \nu) \geq Cn^{-2/3} + t \right] \leq e^{-cn^2t^2},$$

where $t > 0$, $\mu_U$ is the spectral measure of $U$, $\nu$ is the uniform measure on the circle, and $W_1$ denotes Wasserstein/Kantorovich distance.
Summer 2011

The work for that paper was done during the summer of 2011.

Peter was born on July 20.
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Elizabeth wrote on her web page:

*I think of this paper as being unofficially dedicated to our children: Peter, who stubbornly refused to be born while most of the work in this paper was done; and Juliette, who told me one morning that it would make her happy if I proved a theorem that day (I’m pretty sure it was what became Theorem 3.5).*
Then the next year...

Sandrine Dallaporta used determinantal point process techniques to improve some of our results for eigenvalues of Gaussian random matrices.
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Then the next year...

Sandrine Dallaporta used determinantal point process techniques to improve some of our results for eigenvalues of Gaussian random matrices.

We adapted her techniques to the classical compact groups, added in some cool results of Eric Rains, and mixed with log-Sobolev inequalities to prove:

**Theorem (E. and M. Meckes, 2013)**

Let $U$ be an $n \times n$ random matrix uniformly chosen from any of the classical compact matrix groups and let $1 \leq m \leq n$. Then

$$
P \left[ W_1(\mu_{U^m}, \nu) \geq C \frac{\sqrt{m(\log(n/m) + 1)}}{n} + t \right] \leq e^{-cn^2t^2/m}.
$$
An observation about lemmas

A useful lemma often gets more citations than a big theorem.
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But most of the recent citations go here:
Sabbatical in Toulouse, 2013–14

By then it got to be habit to discuss whatever we were working on with each other, and we started writing the majority of our papers together.

Comparison of metrics between log-concave distributions (motivated by the central limit theorem for convex bodies)
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Wrapping up

Our series of papers on convergence rates for spectral measures of random matrices finished up with:

Theorem (E. and M. Meckes, 2017)

Let $U$ be an $n \times n$ random matrix uniformly chosen from the unitary group. Then

$$c \frac{\log n}{n} \leq \mathbb{E} d_K(\mu_U, \nu) \leq C \frac{\log n}{n},$$

where $d_K$ denotes Kolmogorov distance.

again proved using DPP techniques.
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So

$$d_K(\mu_U, \nu) \approx \frac{\log n}{n} \quad \text{but} \quad W_1(\mu_U, \nu) \lesssim \frac{\sqrt{\log n}}{n}.$$
Wrapping around

In the meantime, Elizabeth got interested in a conjecture of Coram and Diaconis:

*Take n consecutive eigenvalues of a random $2n \times 2n$ unitary matrix and stretch them around the whole unit circle. The result is statistically indistinguishable from the eigenvalues of a random $n \times n$ unitary matrix.*
Wrapping around

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Take \( n \) consecutive eigenvalues of a random \( 2n \times 2n \) unitary matrix and stretch them around the whole unit circle. The result is statistically indistinguishable from the eigenvalues of a random \( n \times n \) unitary matrix.

Inspired by this we proved:

**Theorem (E. and M. Meckes, 2016)**

Let \( N_{n,\theta} \) be the number of eigenvalues of an \( n \times n \) random unitary matrix in an arc of length \( \theta \in (0, \pi) \). Then

\[
W_1 \left( N_{n,\theta}, N_{2n,\theta/2} \right) \leq C \sqrt{n} \theta^2.
\]
And coming full circle

There is a natural model of $n \times n$ Hermitian random matrices with prescribed eigenvalues.
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“Theorem” (E. and M. Meckes, 2020)

Let $\Lambda$ be a fixed $n \times n$ Hermitian matrix and let $U$ be an $n \times n$ random matrix uniformly chosen from the unitary or orthogonal group. Then the marginals of the entries of the random matrix $U \Lambda U^*$ are quantifiably almost Gaussian.
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We discussed perspectives and applications related to quantum information theory, free probability, and randomized linear algebra.
And coming full circle

This is a **quadratic** version of results from Elizabeth’s thesis:

**Theorem (E. Meckes, 2008)**

Let $U$ be an $n \times n$ random matrix uniformly chosen from the orthogonal group, and let $A$ be a fixed $n \times n$ matrix with $\|A\|_{HS} = \sqrt{n}$. Then

$$d_{TV}(\text{Tr}(AU), N(0, 1)) \leq \frac{2\sqrt{3}}{n-1}.$$ 

(Multivariate versions appear in Chatterjee–E. Meckes, 2008.)
And coming full circle

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(Multivariate versions appear in Chatterjee–E. Meckes, 2008.)

The newer theorem considers the distributions of random variables $\text{Tr}(AU^*U^*)$.

The proofs of both are via Stein’s method.
Our penultimate (?) paper

Around 2007 we tossed around an idea for a proof using Stein’s method, but it took us twelve years to figure out which theorem we should be using it to prove:

“Theorem” (E. and M. Meckes, 2021)
Let \( A \) be an \( n \times n \) random matrix whose distribution is spherically symmetric. If \( p \) is a polynomial, then the distribution of the linear eigenvalue statistic \( \text{Tr}(p(A)) \) is quantifiably almost Gaussian.
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An early observation from this project: most of the applications of the *Infinitesimal Version of Stein’s Method of Exchangeable Pairs* are subsumed in a single normal approximation theorem which makes no mention of exchangeable pairs:
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An early observation from this project: most of the applications of the *Infinitesimal Version of Stein’s Method of Exchangeable Pairs* are subsumed in a single normal approximation theorem which makes no mention of exchangeable pairs:

**Proposition (Univariate version, for simplicity)**

Suppose $X$ is a uniform random point in a compact Riemannian manifold $\Omega$, $f : \Omega \to \mathbb{R}$ is smooth, and $\lambda > 0$. Then

$$d_{TV}(f(X), N(0, 1)) \leq \frac{1}{\lambda} \mathbb{E} \left( |\Delta f(X) + \lambda f(X)| + \left\| \nabla f(X) \right\|^2 - \lambda \right).$$
This is a slight generalization of main results from two 2009 papers by Elizabeth.
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It is essentially proved (using exchangeable pairs) in those papers as well as in papers of Fulman (2009 and 2012) and Döbler and Stolz (2011).
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The main results of these papers (plus E. Meckes 2008 and E. and M. Meckes 2020) are all consequences of this master normal approximation lemma and its multivariate version.
Thank you

Elizabeth Samantha Elder Meckes
June 23, 1980 – December 16, 2020