Eigenvalue is not a dirty word

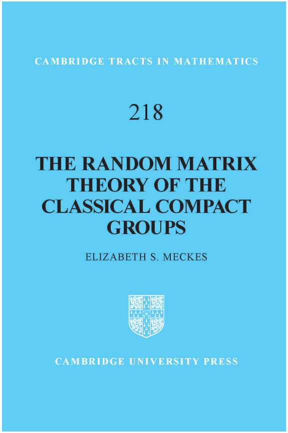
My mathematical collaborations with Elizabeth Meckes

Mark Meckes

Case Western Reserve University
Currently visiting Oxford University

Southeastern Probability Conference
17 May 2021

From the preface of Elizabeth’s book:



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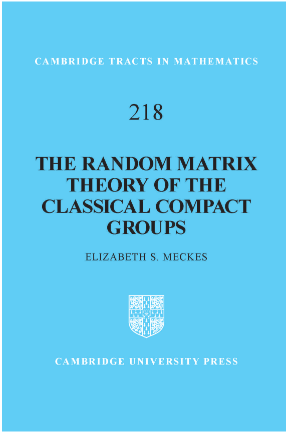

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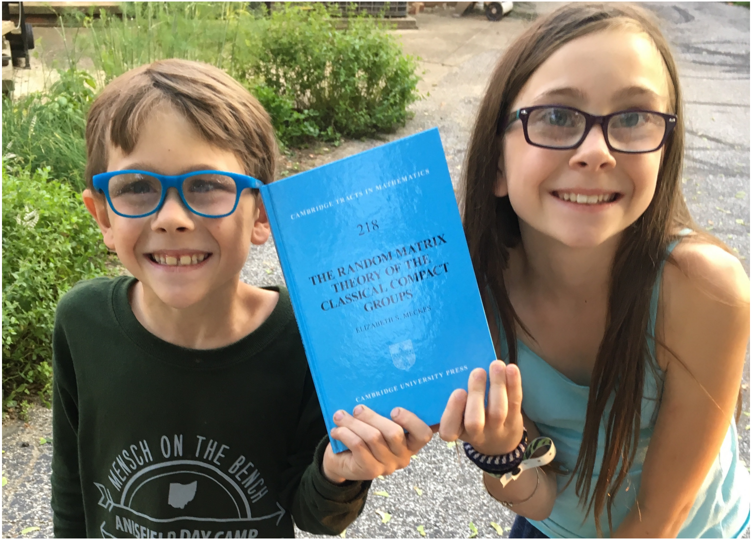
We weren't the only ones at that breakfast table:



Peter and Juliette

You may have met those two before

You may have met those two before

at a conference,

You may have met those two before

at a con-
ference,

or an
AMS
meeting,

You may have met those two before

at a con-
ference,

or an
AMS
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or when
we went
some-
where to
give
seminar
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Meckes family fun fact

Juliette got her first passport as a baby to go with us to a conference in Banff...

Advances in Stochastic Inequalities and Their Applications

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High Dimensional Probability VI

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Advances in Stochastic Inequalities and Their Applications

High Dimensional Probability VI

(This wasn't the first conference for either of them.)

Elizabeth grew up around academic shop talk:

ISSN 0036-8075
27 July 1984
Volume 225, No. 4660

SCIENCE

Intestinal Uptake and Metabolism of Auranofin, A New Oral Gold-Based Antiarthritis Drug

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Katherine Tepperman = "Mom"
R.C. Elder = "Dad"

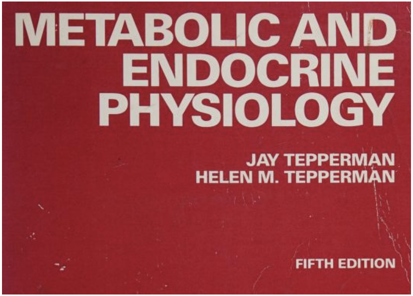
It goes back farther than that:

FOURIER EXPANSIONS OF DOUBLY PERIODIC FUNCTIONS OF THE THIRD KIND.*

BY JOHN D. ELDER.

Annals of Mathematics, vol. 31 no. 4, Oct. 1930, pp. 641–654

John D. Elder = “Grandpa John”



Jay Tepperman =
“Grandpa Jay”

Helen M. Tepperman =
“Grandma”

The dark side to this kind of background...

Young Katherine Tepperman automatically stopped listening to the dinner table conversation whenever she heard "glucose-6-phosphate".

Role of hormones in glucose-6-phosphate dehydrogenase adaptation of rat liver[1]

HELEN M. TEPPERMAN AND JAY TEPPERMAN (With the Technical Assistance of Janet M. DeWitt and Andrew Branch)
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She had to unlearn this habit on her way to becoming a biologist!

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Young Elizabeth Elder (later Meckes) tuned out when she heard "Na, K-ATPase".

Biophysical Journal Volume 79 August 2000 853–862

Combined Allosteric and Competitive Interaction between Extracellular Na^+ and K^+ During Ion Transport by the α_1, α_2, and α_3 Isoforms of the Na, K-ATPase

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She sidestepped having this as a potential problem by going into pure mathematics.

The next generation

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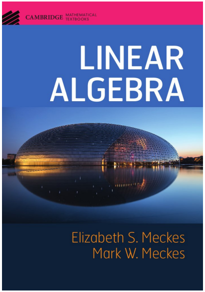

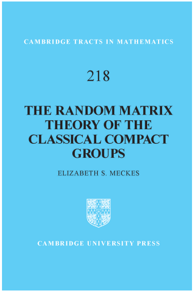

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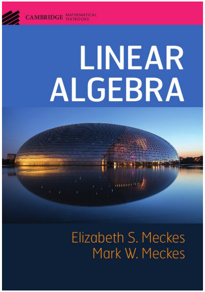

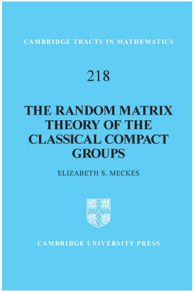

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I asked Elizabeth for help and the result was:

"Theorem" (E. and M. Meckes, 2007)

If n is big and a probability measure μ on $\mathbb{R}^n$ has certain kinds of symmetries, and satisfies some other hypotheses, then certain (one-dimensional) marginals of μ are quantifiably almost Gaussian.

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- Think of this as a Berry–Esseen-type theorem with stochastic independence replaced by a geometric symmetry condition.
- The "other hypotheses" are now known to hold for uniform measures on convex bodies.
- Elizabeth and I each wrote follow-ups to this work but didn't work together on this again.

Some time (and one child) later...

“Theorem” (Chatterjee–Ledoux, 2009)

If $1 \ll k \leq n$ and A is an $n \times n$ Hermitian matrix, then the eigenvalues of almost all $k \times k$ principal submatrices of A are almost the same.

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Elizabeth and I realized we could prove a coordinate-free version of this by combining some basic matrix analysis with tools she was using to study random marginals:

“Theorem” (E. and M. Meckes, 2011)

If $1 \ll k \leq n$ and A is an $n \times n$ Hermitian matrix, then the eigenvalues of almost all k-dimensional compressions of A are almost the same.

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For example:

Theorem (E. and M. Meckes, 2013)

Let U be an $n \times n$ random matrix uniformly chosen from any of the classical compact matrix groups. Then

$$\mathbb{P}\left[W_1(\mu_U, \nu) \geq Cn^{-2/3} + t\right] \leq e^{-cn^2t^2},$$

where $t > 0$, μ_U is the spectral measure of U, ν is the uniform measure on the circle, and W_1 denotes Wasserstein/Kantorovich distance.

Summer 2011

The work for that paper was done during the summer of 2011.

Peter was born on July 20.

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Elizabeth wrote on her web page:

> *I think of this paper as being unofficially dedicated to our children: Peter, who stubbornly refused to be born while most of the work in this paper was done; and Juliette, who told me one morning that it would make her happy if I proved a theorem that day (I'm pretty sure it was what became Theorem 3.5).*

Then the next year...

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Then the next year...

Sandrine Dallaporta used determinantal point process techniques to improve some of our results for eigenvalues of Gaussian random matrices.

We adapted her techniques to the classical compact groups, added in some cool results of Eric Rains, and mixed with log-Sobolev inequalities to prove:

Theorem (E. and M. Meckes, 2013)

Let U be an $n \times n$ random matrix uniformly chosen from any of the classical compact matrix groups and let $1 \leq m \leq n$. Then

$$\mathbb{P}\left[W_1(\mu_{U^m}, \nu) \geq C\frac{\sqrt{m(\log(n/m)+1)}}{n} + t\right] \leq e^{-cn^2t^2/m}.$$

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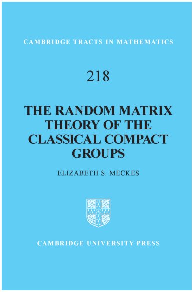
But most of the recent citations go here:



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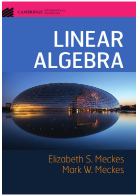
- 

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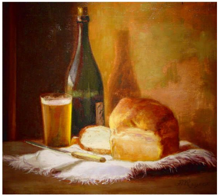
- 

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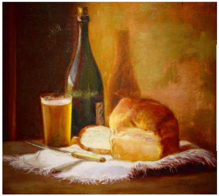
- 

  1.1 Linear Systems of Equations

  Bread, Beer, and Barley

  We begin with a very simple example. Suppose you have 20 pounds of raw barley and you plan to turn some of it into bread and some of it into beer. It takes

Wrapping up

Our series of papers on convergence rates for spectral measures of random matrices finished up with:

Theorem (E. and M. Meckes, 2017)

Let U be an $n \times n$ random matrix uniformly chosen from the unitary group. Then

$$c\frac{\log n}{n} \leq \mathbb{E} d_K(\mu_U, \nu) \leq C\frac{\log n}{n},$$

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So

$$d_K(\mu_U, \nu) \approx \frac{\log n}{n} \qquad \text{but} \qquad W_1(\mu_U, \nu) \lesssim \frac{\sqrt{\log n}}{n}.$$

Wrapping around

In the meantime, Elizabeth got interested in a conjecture of Coram and Diaconis:

> *Take n consecutive eigenvalues of a random $2n \times 2n$ unitary matrix and stretch them around the whole unit circle. The result is statistically indistinguishable from the eigenvalues of a random $n \times n$ unitary matrix.*

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> *Take n consecutive eigenvalues of a random $2n \times 2n$ unitary matrix and stretch them around the whole unit circle. The result is statistically indistinguishable from the eigenvalues of a random $n \times n$ unitary matrix.*

Inspired by this we proved:

Theorem (E. and M. Meckes, 2016)

Let $N_{n,\theta}$ be the number of eigenvalues of an $n \times n$ random unitary matrix in an arc of length $\theta \in (0, \pi)$. Then

$$W_1\left(N_{n,\theta}, N_{2n,\theta/2}\right) \le C\sqrt{n}\theta^2.$$

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Let Λ be a fixed $n \times n$ Hermitian matrix and let U be an $n \times n$ random matrix uniformly chosen from the unitary or orthogonal group. Then the marginals of the entries of the random matrix $U\Lambda U^*$ are quantifiably almost Gaussian.

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We discussed perspectives and applications related to quantum information theory, free probability, and randomized linear algebra.

And coming full circle

This is a quadratic version of results from Elizabeth's thesis:

Theorem (E. Meckes, 2008)

Let U be an $n \times n$ random matrix uniformly chosen from the orthogonal group, and let A be a fixed $n \times n$ matrix with $\|A\|_{HS} = \sqrt{n}$. Then

$$d_{TV}\big(\mathrm{Tr}(AU), N(0,1)\big) \leq \frac{2\sqrt{3}}{n-1}.$$

(Multivariate versions appear in Chatterjee–E. Meckes, 2008.)

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The newer theorem considers the distributions of random variables $\mathrm{Tr}(AU\Lambda U^*)$.

The proofs of both are via Stein's method.

Our penultimate (?) paper

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"Theorem" (E. and M. Meckes, 2021)

Let A be an $n \times n$ random matrix whose distribution is spherically symmetric. If p is a polynomial, then the distribution of the linear eigenvalue statistic $\operatorname{Tr} p(A)$ is quantifiably almost Gaussian.

To be continued...

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An early observation from this project: most of the applications of the *Infinitesimal Version of Stein's Method of Exchangeable Pairs* are subsumed in a single normal approximation theorem which makes no mention of exchangeable pairs:

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An early observation from this project: most of the applications of the *Infinitesimal Version of Stein's Method of Exchangeable Pairs* are subsumed in a single normal approximation theorem which makes no mention of exchangeable pairs:

Proposition (Univariate version, for simplicity)

Suppose X is a uniform random point in a compact Riemannian manifold Ω, $f : \Omega \to \mathbb{R}$ is smooth, and $\lambda > 0$. Then

$$d_{TV}\big(f(X), N(0,1)\big) \leq \frac{1}{\lambda}\mathbb{E}\left(|\Delta f(X) + \lambda f(X)| + \left|\|\nabla f(X)\|^2 - \lambda\right|\right).$$

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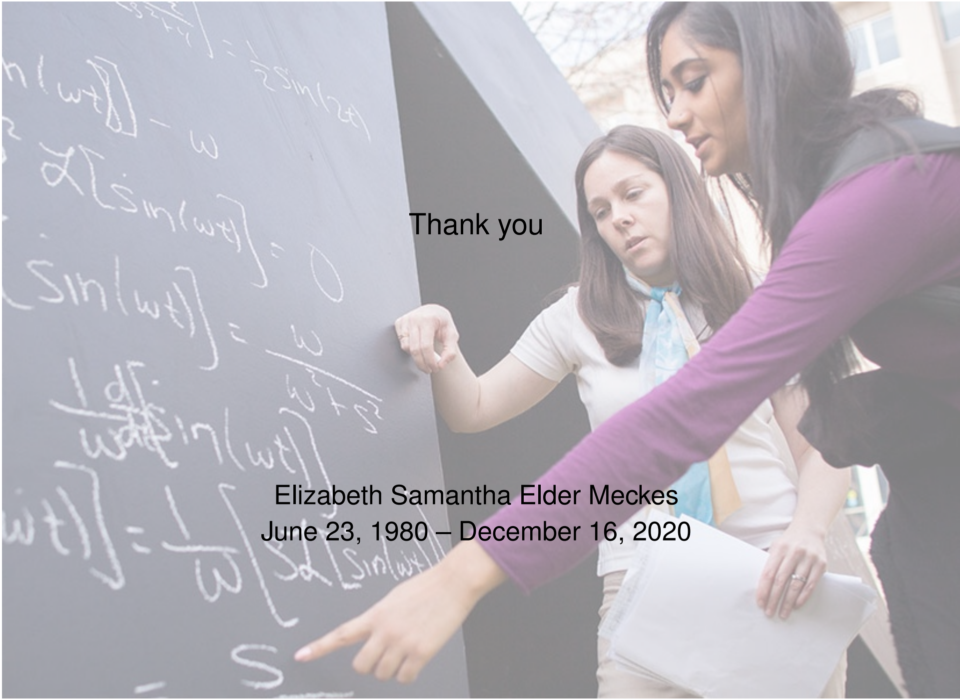
The main results of these papers (plus E. Meckes 2008 and E. and M. Meckes 2020) are all consequences of this master normal approximation lemma and its multivariate version.



Thank you

Elizabeth Samantha Elder Meckes
June 23, 1980 – December 16, 2020