Tales of
Random projections of
high-dimensional measures

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Brown University

based on joint works with Nina Gantert, Steven Soojin Kim and Yin-Ting Liao

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2021 Southeastern Probability Conference

In honor of Elizabeth Meckes

May 17-18, 2021

An Early Encounter with E. Meckes

Women in Probability Conference
Oct 5–7, 2008, Cornell, Ithaca
organized by R. Durrett

Elizabeth's talk title at that conference

When is Normal Normal?

Elizabeth's talk title at that conference

When is Normal Normal?

A more recent talk at the Simons Institute, October 2020

Projections of Probability Distributions:

A Measure-theoretic Version of Dvoretzky's theorem

The Main Objects of Interest in this Talk

What do we mean by projections?

One-dimensional projections

High-dimensional vector

$X^{(n)}$ taking values in $\mathbb{R}^n$

e.g. uniformly distributed in a convex body in $\mathbb{R}^n$

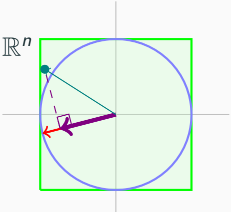
(compact convex set with a non-empty interior)

One-dimensional projections

Let $\theta^{(n)}$ be a vector on S^{n-1}

The projection is then

$$W_\theta^{(n)} = \langle X^{(n)}, \theta^{(n)} \rangle$$



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The Stiefel manifold of orthonormal k-frames in $\mathbb{R}^n$

$$\mathbb{V}_{n,k} := \{A \in \mathbb{R}^{n \times k} : A^T A = I_k\},$$

where I_k is the $k \times k$ identity matrix.

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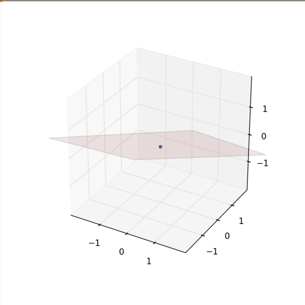
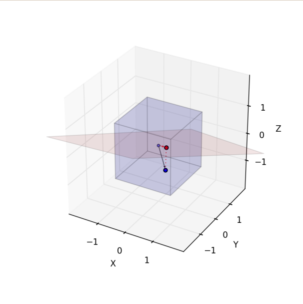
$$\mathbb{V}_{n,k} := \{A \in \mathbb{R}^{n \times k} : A^T A = I_k\},$$

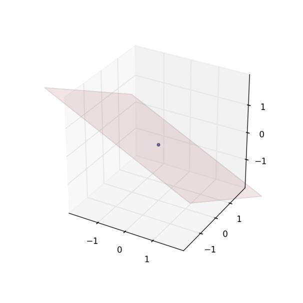
where I_k is the $k \times k$ identity matrix.
For $k < n$, choose $a_{n,k} \in \mathbb{V}_{n,k}$. Then

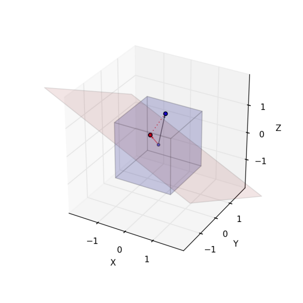
$W_{\mathbf{a}}^{(n)} = \mathbf{a}_{\mathbf{n},\mathbf{k}} X^{(n)}$ defines a k-dimensional projection

Projections onto lower-dimensional bases/subspaces









Motivation and Context

Theme of this talk

Understand high-dimensional objects by looking at their (random) lower-dimensional projections

Motivation, Context and Elizabeth Meckes' Work

Motivation and Context

First Motivation

High-dimensional Probability and Statistics

Understanding high-dimensional data
by studying its
lower-dimensional projections
is of relevance, e.g., in
sparse recovery, information retrieval, statistics, projection-pursuit

Understanding high-dimensional data

Projection-Pursuit Algorithm

Kruskal (1969)

Friedman and Tukey (1974)

Diaconis and Friedman (1984, 1987)

Projection Pursuit: Find the “interesting” directions

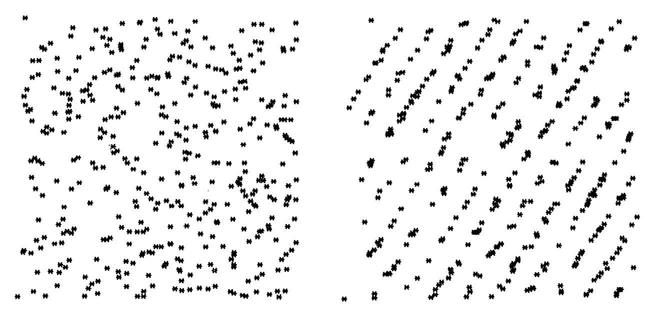


Figure from Projection Pursuit by P. Huber

The focus of this talk:

Second Motivation

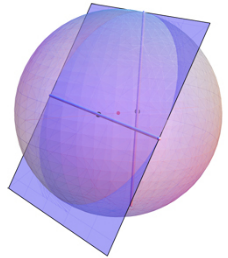
Asymptotic Convex Geometry

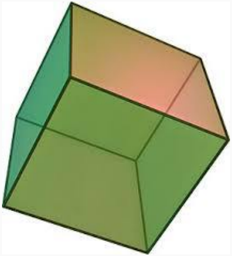
or

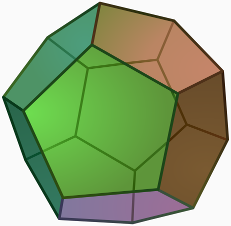
Asymptotic Geometric Analysis

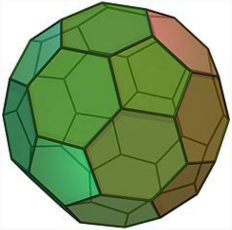
concerned with
the geometry of Banach spaces
and convex bodies
in high dimensions

Motivation – Study of Convex Bodies in High Dimensions









Dvoretsky's Theorem

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(Aryeh Dvortesky '61; Vitali Milman '71)

Every sufficiently high-dimensional normed vector space has subspaces that are approximately Euclidean

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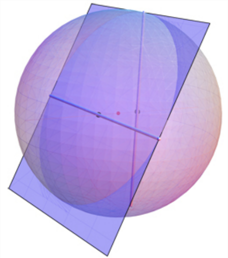
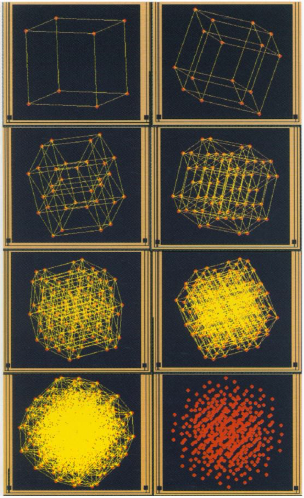
Every sufficiently high-dimensional normed vector space has subspaces that are approximately Euclidean

Every convex body (compact convex set with non-empty interior) of dimension N has a section $d(N)$ with $d(N) \to \infty$ as $N \to \infty$ that is arbitrarily close to being isometric to an ellipsoid.



Two-dimensional Projections of the Cube



Most projections look Gaussian

A Rigorous Universality Result

The CLT for Convex Sets [Klartag '07]

There exist $\varepsilon_n, \delta_n \to 0$ such that for every isotropic logconcave random vector $X^{(n)}$, there exists a measurable subset $A \subset \mathbb{S}^{n-1}$ with measure $\sigma_{n-1}(A) \geq 1 - \delta_n$, such that for all $\theta^n \in A$,

$$d_{TV}\left(\langle X^n, \theta^n \rangle, Z^n\right) \leq \varepsilon_n,$$

where $Z^n \sim \mathcal{N}(0, \mathbb{I}_n)$ is the standard Gaussian in $\mathbb{R}^n$.

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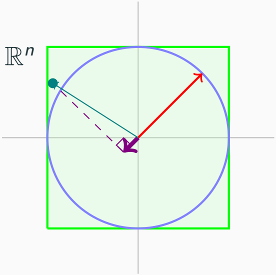
where $Z^n \sim \mathcal{N}(0, \mathbb{I}_n)$ is the standard Gaussian in $\mathbb{R}^n$.

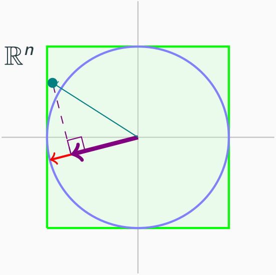
Can be viewed as a
"measure-theoretic Dvoretzky theorrem", to quote E. Meckes (2012)
where the Gaussian distribution now plays the role that the Euclidean
norm did in Dvoretzky

Some Intuition

$X^{(n)} = (X_1, \ldots, X_n) \sim \mathrm{Unif}([-1,1]^n)$, $\theta^{(n)} \in S^{n-1}$, $\iota^{(n)} = (1, \ldots, 1)/\sqrt{n}$



$\langle X^{(n)}, \iota^{(n)} \rangle$ recovers the usual CLT



$\langle X^{(n)}, \theta^{(n)} \rangle$

History of the CLT for convex sets

- Early results on approximate Gaussian marginals by **Borel; Sudakov; Weiszacker; Diaconis & Freedman; Klartag;**

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- Early results on approximate Gaussian marginals by **Borel; Sudakov; Weiszacker; Diaconis & Freedman; Klartag;**
- **Anttila, Ball and Perissinaki** (2003) and **Brehm and Voigt** (2000) showed that if $X^{(n)}$ is symmetric and satisfies a "thin-shell condition" then most projections are almost Gaussian.

Thin-shell condition

$X^{(n)}$ satisfies an ε-thin-shell estimate if there exists $m > 0$ such that

$$\mathbb{P}\left(\left|\frac{\|X^{(n)}\|}{\sqrt{n}} - m\right| > \varepsilon m\right) < \varepsilon.$$

- **Anttila, Ball and Perissinaki** verified the thin-shell condition for symmetric convex bodies such as ℓ_p^n balls, $1 < p < \infty$, and other uniformly convex bodies with some restrictions on their modulus of convexity.

History of the CLT for convex sets

1. **Klartag (2007)** proved the CLT for convex sets by showing that isotropic log-concave measures satisfy the thin-shell condition His work also allowed for multi-dimensional projections
2. E. Meckes wrote two single-author papers on this topic:
   - "Projections of probability distributions: A measure-theoretic Dvoretzky theorem," E. Meckes, *Geometric aspects of functional analysis*, 317-326, 2012.
   - "Approximation of projections of random vectors," E. Meckes, *Journal of Theoretical Probability*, 25 (2), 333-352, 2013.

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3. in addition to an earlier joint paper with Mark Meckes:
   "The central limit problem for random vectors with symmetries", E.S. Meckes and M.W. Meckes, *Journal of Theoretical Probability*, 20 (4), 697-720, 2007.

Meckes' Result on Multidimensional Projections

$X^{(n)}$ an n-dimensional random vector, $a_{k,n} \in \mathbb{V}_{k,n}$ Stiefel manifold

$$W^{(n)}{}_a = a_{k,n} X^{(n)}$$

- She used Stein's method to get quantitative bounds on the distance between $W_a^{(n)}$ and the k-dimensional Gaussian distribution.
- This allowed her to study the case when $k = k_n$ grows with the dimension
- She unearthed the beautiful **phase transition** result that Gaussian projections are guaranteed if and only if for some $\delta < 2$,

$$k_n \leq \delta \frac{\log n}{\log \log n}.$$

Moreover, this condition is **tight** !!

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  Moreover, this condition is **tight** !!
- Klartag (2007) had showed that when specialized to logconcave measures, Gaussian behavior for high-dimensional random projections is possible even when $k_n = n^\alpha$

Beyond Universality

Beyond the CLT ...

1. The CLT for convex sets is a beautiful universality result that shows "most" marginals of a convex body are Gaussian.
2. But it is in a way bad news, as it says that looking at (fluctuations of) projections does not allow one to distinguish between different convex bodies ...

Beyond the CLT ...

1. The CLT for convex sets is a beautiful universality result that shows "most" marginals of a convex body are Gaussian.
2. But it is in a way bad news, as it says that looking at (fluctuations of) projections does not allow one to distinguish between different convex bodies ...
3. Alternative: Try to establish large deviation principles and see if they contain interesting geometric information

Large deviation principles

Recall the Definition

Large deviation principle (LDP)

A sequence of random variables $(W^{(n)})_{n\in\mathbb{N}}$ is said to satisfy a large deviation principle with speed s_n and a good rate function (GRF) $\mathbb{I}:\mathbb{R}\mapsto[0,\infty)$ if for any measurable set A

$$\begin{aligned}-\inf_{x\in A^o}\mathbb{I}(x) &\le \liminf_{n\to\infty}\frac{1}{s_n}\log\mathbb{P}(W^{(n)}\in A)\\ &\le \limsup_{n\to\infty}\frac{1}{s_n}\log\mathbb{P}(W^{(n)}\in A)\le -\inf_{x\in\bar{A}}\mathbb{I}(x),\end{aligned}$$

where $\mathbb{I}$ is lower semi-continuous and with compact level sets.

For a nice set A,

$$P(W^{(n)}\in A)\approx e^{-s_n I(A)}.$$

A Classical LDP: Cramér's Theorem

Consider an i.i.d. sequence $\{X_i\}$

$$\mathbb{P}\left(\frac{1}{n}\sum_{i=1}^{n} X_i \geq x\right) \approx e^{-nI(x)}$$

- The probability of $O(1)$ fluctuations of the empirical mean shows an **exponential decay**, whose rate depends on the distribution of X_i

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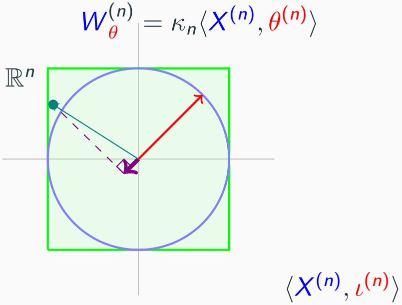
Theorem (Cramér ('38))
If $\Lambda(t) = \log \mathbb{E}[e^{tX_1}]$ is finite in a neighborhood of 0, $\{\frac{1}{n}\sum_{i=1}^{n} X_i\}_{n\in\mathbb{N}}$ satisfies an LDP with speed n and rate function

$$\mathbf{I}_\iota(\mathbf{x}) = \Lambda^*(x) \doteq \sup_{t\in\mathbb{R}}[xt - \Lambda(t)].$$

Large Deviation Principles for Random Projections

Consider first 1-dimensional projections: $\theta^{(n)} \in S^{n-1}$



When $X^{(n)}$ is a product measure $\mu^{\otimes n}$, **Cramér's theorem** (1938)

- implies an LDP with $\kappa_n = n^{-1/2}$ if $\theta^{(n)} = \iota^{(n)} = (1, 1, \ldots, 1)/\sqrt{n}$

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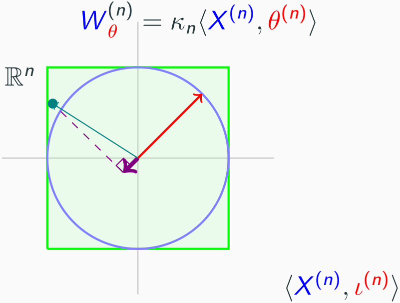

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- implies an LDP with $\kappa_n = n^{-1/2}$ if $\theta^{(n)} = \iota^{(n)} = (1, 1, \ldots, 1)/\sqrt{n}$
- and (nearly) implies an LDP if $\Theta^{(n)}$ is a random vector on S^{n-1} distributed according to σ_{n-1} the unique rotation invariant measure on S^{n-1}

Beyond Product Measures: Random Projections

Question:

What sequences of random variables $\{X^{(n)}\}_{n\in\mathbb{N}}$ are such that their multidimensional projections satisfy a large deviation principle (LDP)?

The Stiefel manifold of orthonormal k-frames in $\mathbb{R}^n$

$$\mathbb{V}_{n,k} := \{A \in \mathbb{R}^{n\times k} : A^T A = I_k\},$$

where I_k is the $k \times k$ identity matrix.

Random orthonormal frames/bases are chosen with respect to the invariant measure $\sigma_{n,k}$ on the (compact) Stiefel manifold.

Asymptotic thin shell condition

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Recall Question:

What sequences of random variables $\{X^{(n)}\}_{n\in\mathbb{N}}$ are such that their multidimensional projections satisfy a large deviation principle (LDP)?

Assumption A

The sequence of scaled norms $\{\|X^{(n)}\|_2/\sqrt{n}\}$ satisfies an LDP at speed s_n with rate function $J_X : \mathbb{R} \to [0,\infty]$.

We say Assumption A* holds if Assumption A holds with $s_n = n$.

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We say Assumption A* holds if Assumption A holds with $s_n = n$.
Suppose J_X has a unique minimum at $m > 0$.
Fix $\varepsilon > 0$. Then for n large enough, $X^{(n)}$ satisfies the ε-thin-shell estimate, that is,

$$\mathbb{P}\left(\left|\frac{\|X^{(n)}\|_2}{\sqrt{n}} - m\right| \geq \varepsilon\right) \leq \varepsilon, \qquad \text{for } n \text{ large.}$$

(for $X^{(n)}$ uniform on an isotropic convex body, $m = 1$)

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In some cases, we need a rescaled version of Assumption A.

Assumption B

For certain sequence $\{b_n\}$, the sequence of scaled norms $\{b_n\|X^{(n)}\|_2/\sqrt{n}\}$ satisfies an LDP at speed s_n with rate function $J_X : \mathbb{R} \to [0, \infty]$.

LDPs for random projections onto growing subspaces

Let $\mathbb{V}_{n,k} = \{A \in \mathbb{R}^{n \times k} : A^T A = I_k\}$ denote the Stiefel manifold of k-frames in $\mathbb{R}^n$.

LDPs for random projections onto growing subspaces

The random matrix $\mathbf{A}_{n,k_n}^T \in \mathbb{V}_{n,k_n}$ linearly projects a vector from n to k_n dimensions.

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Three regimes:

1. $\{k_n\}$ is *constant* at k;
2. $\{k_n\}$ *grows sublinearly*, $1 \ll k_n \ll n$;
3. $\{k_n\}$ *grows linearly* with rate λ, for some $\lambda \in (0, 1]$, $k_n/n \to \lambda$.

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Goal: To prove LDP for

(i) $\{n^{-1/2}\mathbf{A}_{n,k_n}^T X^{(n)}\}$ when k_n is constant at k.

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(i) $\{n^{-1/2}\mathbf{A}_{n,k_n}^T X^{(n)}\}$ when k_n is constant at k.
(ii) $\{L^n := \frac{1}{k_n}\sum_{j=1}^{k_n} \delta_{(\mathbf{A}_{n,k_n}^T X^{(n)})_j}\}$ when k_n is growing.

LDPs for random projections onto growing subspaces

The random matrix $\mathbf{A}^T_{n,k_n} \in \mathbb{V}_{n,k_n}$ linearly projects a vector from n to k_n dimensions.

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Goal: To prove LDP for

(i) $\{n^{-1/2}\mathbf{A}^T_{n,k_n}X^{(n)}\}$ when k_n is constant at k.
(ii) $\{L^n := \frac{1}{k_n}\sum_{j=1}^{k_n}\delta_{(\mathbf{A}^T_{n,k_n}X^{(n)})_j}\}$ when k_n is growing.
(iii) $\{n^{-1/2}\|\mathbf{A}^T_{n,k_n}X^{(n)}\|_q\}$ in all regimes.

Constant regime

Theorem [constant, $k_n \equiv k$] (Kim, Liao, R '20)

Suppose Assumption **A*/B** holds, with sequence $\{s_n\}$ and GRF J_X.
Then $\{n^{-1/2}\mathbf{A}_{n,k}^T X^{(n)}\}$ satisfies an LDP in $\mathbb{R}^k$ at speed s_n, with GRF

$$I_{\mathbf{A}X,k}(x) := \begin{cases} \inf_{0<c<1} \left\{ J_X\left(\frac{\|x\|_2}{c}\right) - \frac{1}{2}\log\left(1-c^2\right)\right\}, & \text{if A* holds,} \\ \inf_{c>0} \left\{ J_X\left(\frac{\|x\|_2}{c}\right) + \frac{c^2}{2}\right\}, & \text{if B holds.} \end{cases}$$

Constant regime

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Define $Y_{q,k}^n := n^{-1/2}\|\mathbf{A}_{n,k}^T X^{(n)}\|_q$

Corollary [LDP for q-norms of the projection]

$\{Y_{q,k}^n\}_{n\in\mathbb{N}}$ satisfies an LDP at speed s_n with GRF

$$\mathbb{J}_{Y_{q,k}}(x) := \inf_{z\in\mathbb{R}^k}\left\{I_{\mathbf{A}X,k}(z) : \|z\|_q = x\right\}, \quad x \in \mathbb{R}_+.$$

Examples satisfying the asymptotic thin shell condition

Examples: 1. Product measures & 2. ℓ_p^n balls

Proposition [i.i.d. case] (corollary of Cramér '38)

Let $X_1, X_2, \dots$ be a sequence of i.i.d. real-valued random variables, and let $X^{(n)} := (X_1, \dots, X_n)$. Suppose $\Lambda(t) := \log \mathbb{E}[e^{tX_1^2}] < \infty$. Then, $\{X^{(n)}\}$ satisfies Assumption A*, i.e., $\{\|X^{(n)}\|_2/\sqrt{n}\} \sim$ LDP at speed n.

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Proposition [ℓ_p^n balls, $p \in [1, \infty)$] (Kim, Liao, R '20)

Let $X^{(n,p)} \sim$ uniformly on scaled ℓ_p^n ball, $\mathbb{B}_p^n := \{x \in \mathbb{R}^n : \sum |x_i|^p \leq n\}$. Then

1. for $p \in [\mathbf{2}, \infty)$, $\{\|X^{(n,p)}\|_2/\sqrt{n})\}$ satisfies Assumption **A***.
2. for $p \in [\mathbf{1}, \mathbf{2})$, $\{\|X^{(n,p)}\|_2/\sqrt{n})\}$ satisfies Assumption **B**

Examples: 1. Product measures & 2. ℓ_p^n balls

Proposition [i.i.d. case] (corollary of Cramér '38)

Let $X_1, X_2, \ldots$ be a sequence of i.i.d. real-valued random variables, and let $X^{(n)} := (X_1, \ldots, X_n)$. Suppose $\Lambda(t) := \log \mathbb{E}[e^{tX_1^2}] < \infty$. Then, $\{X^{(n)}\}$ satisfies Assumption A*, i.e., $\{\|X^{(n)}\|_2/\sqrt{n}\} \sim$ LDP at speed n.

Proposition [ℓ_p^n balls, $p \in [1,\infty)$] (Kim, Liao, R '20)

Let $X^{(n,p)} \sim$ uniformly on scaled ℓ_p^n ball, $\mathbb{B}_p^n := \{x \in \mathbb{R}^n : \sum |x_i|^p \leq n\}$. Then

1. for $p \in [\mathbf{2}, \infty)$, $\{\|X^{(n,p)}\|_2/\sqrt{n})\}$ satisfies Assumption **A***.
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Proof relies on a probabilistic representation for ℓ_p^n balls

$$X^{(n,p)} \stackrel{(d)}{=} n^{1/p} U^{1/n} \frac{\xi^{(n,p)}}{\left\|\xi^{(n,p)}\right\|_p},$$

where $U \sim \text{Uniform}[0,1]$ and $\xi^{(n,p)} = (\xi_1^{(p)}, \ldots, \xi_n^{(p)})$ where $\{\xi_i^{(p)}\}$ are i.i.d. and has density $f_p(x) := \frac{1}{2p^{1/p}\Gamma(1+1/p)} \exp(-|x|^p/p)$.

LDPs for Euclidean norms $Y^{(n,p)}_{2,k_n} = n^{-1/2}\|\mathbf{A}^T_{n,k}X^{(n,p)}\|_2$

a double phase transition in the LDP speed: define $\kappa_p = 2p/(2+p)$

Theorem (Kim, Liao, R '20) related to Example 2

For $p \in [1,2)$, $\{Y^{(n,p)}_{2,k_n}\} \sim$ LDP with

Projection subspace k_n	LDP speed s_n	LDP rate function
$k_n \equiv k$	n^{κ_p}	$\kappa_p^{-1} x^{\kappa_p}$
$1 \ll k_n \ll n^{\kappa_p}$	$\mathbf{n^{\kappa_p}}$	$\kappa_p^{-1} x^{\kappa_p}$
$\mathbf{k_n = n^{\kappa_p}}$	n^{κ_p}	$\kappa_p^{-1} \frac{x^p}{\bar{c}(x)^p} - \log(\bar{c}(x))^{\dagger}$
$n^{\kappa_p} \ll k_n \ll n$	$n^p k_n^{-p/2}$	$\frac{x^p}{p}$
$k_n \sim \lambda n$	$n^{p/2}$	$\frac{x^p}{p\lambda^{p/2}}$

$\dagger$: $\bar{c}(x) \in [1 + x^{p/(p+2)}, \infty)$ is the unique positive solution to $c^{p+2} - c^p - x^p = 0$.

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$\dagger$: $\bar{c}(x) \in [1 + x^{p/(p+2)}, \infty)$ is the unique positive solution to $c^{p+2} - c^p - x^p = 0$.

Observation : $x^{2p/(p+2)} \leq \bar{c}(x)^2 - 1 = x^p/\bar{c}(x)^p \leq x^p$

LDPs carry geometric information

One-dimensional (k=1) projections of ℓ_p^n balls
Studied earlier by **Gantert-Kim-R '17**

1. When $p > 2$, one-dimensional projections of ℓ_p^n balls satisfy an LDP at speed n;
2. When $p \in (1, 2)$, one-dimensional projections of ℓ_p^n balls satisfy an LDP at speed n^{κ_p};

Norms of high-dimensional (k=1) projections of ℓ_p^n balls
Studied by **Alonso-Gutierrez-Prochno-Thale '18**) and **Kim-Liao-R '19**

1. When the subspace is growing the speed of the LDP of the norms of projections also depends on the relative growth of the subspace dimension
2. The above double phase transition result provides the full picture
   Open Question: What feature of the geometry of the ℓ_p^n ball is captured by κ_p?

Example 3.: Superquadratic Orlicz balls

Definition

We say V is an *Orlicz function* if $V : \mathbb{R} \to \mathbb{R}_+$ is convex and satisfies $V(0) = 0$ and $V(x) = V(-x)$ for $x \in \mathbb{R}$. Further, we say $V : \mathbb{R} \to \mathbb{R}_+$ is *superquadratic* if V is differentiable, $V(x)/x^2$ is strictly increasing

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Unlike ℓ_p^n balls, does not admit a probabilistic representation in terms of iid random variables!

Asymptotic Thin Shell Cond. holds for superquad. Orlicz balls

$$\begin{aligned}\mathcal{J}(u,v) &:= \sup_{s\in\mathbb{R}, t\in\mathbb{R}} \left\{ su + tv - \log\left(\int_{\mathbb{R}} e^{sV(x)+tx^2} dx\right)\right\} \\ &= \sup_{s<0, t\in\mathbb{R}} \left\{ su + tv - \log\left(\int_{\mathbb{R}} e^{sV(x)+tx^2} dx\right)\right\} \quad \text{for} \quad u, v \in \mathbb{R}_+\end{aligned}$$

Proposition [Assumption A* holds] (Kim, Liao, R '20)

Suppose $X^{(n)} \sim \text{Uniform}(\mathbb{B}_V^n)$. Then $\{\|X^{(n)}\|_2/\sqrt{n}\} \sim$ LDP at speed n with GRF $J_X = J_{X,V}$, where

$$J_{X,V}(z) := \mathcal{J}(1, z^2) - \inf_{x\in\mathbb{R}_+} \mathcal{J}(1, x), \quad z \in \mathbb{R}_+.$$

Example 4.: Gibbs measures

Gibbs measures arising as equilibria of interacting diffusions

Define a Hamiltonian $\mathbf{H}_n : \mathbb{R}^n \to (-\infty, \infty]$ given by

$$\mathbf{H}_n(x) := \frac{1}{n} \sum_{i=1}^{n} F(x_i) + \frac{1}{n^2} \sum_{i=1}^{n} \sum_{j=1, j \neq i}^{n} G(x_i, x_j), \quad x \in \mathbb{R}^n.$$

Further, for $n \in \mathbb{N}$, let $P_n \in \mathcal{P}(\mathbb{R}^n)$ be the probability measure given by

$$P_n(dx) := \frac{1}{Z_n} e^{-n\mathbf{H}_n(x)} \ell(dx), \quad x \in \mathbb{R}^n,$$

where $\ell \in \mathcal{P}(\mathbb{R})$ is a non-atomic, sigma-finite probability measure on $\mathbb{R}$

Let $Q_n \in \mathcal{P}(\mathbb{R})$ be the pushforward measure induced by P_n under the mapping $\mathbb{R}^n \ni (x_1, \ldots, x_n) \mapsto \frac{1}{n} \sum_{i=1}^{n} \delta_{x_i} \in \mathcal{P}(\mathbb{R})$

Example: Gibbs measures

Theorem (Dupuis, Laschos, R '20)

Under certain assumptions for the potentials F and G, $\{Q_n\}$ satisfies an LDP in the Wasserstein space $\mathcal{P}_2(\mathbb{R})$ at speed n with GRF $\mathcal{I}_*$ defined by

$$\mathcal{I}_*(\mu) := \mathcal{I}(\mu) - \inf_{\mu \in \mathcal{P}_2(\mathbb{R})} \mathcal{I}(\mu),$$

$$\mathcal{I}(\mu) := H(\mu|\ell) + \frac{1}{2}\int_{\mathbb{R}\times\mathbb{R}} G(x,y)\mu(dx)\mu(dy) + \int_{\mathbb{R}} F(x)\mu(dx),$$

with H the relative entropy.

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Corollary (Assumption A* holds for Gibbs measure)

Suppose $X^{(n)}$ is drawn from P_n. Then, $\{\|X^{(n)}\|_2/\sqrt{n}\} \sim$ LDP at speed n with GRF

$$J_X(x) := \inf\left\{\mathcal{J}_*(\mu) : \mu \in \mathcal{P}_2(\mathbb{R}), x = \sqrt{M_2(\mu)}\right\}, \quad x \geq 0,$$

where $M_2(\mu)$ is the second moment of μ.

IV. Refined Large Deviations

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So far ...

- Uniform measures on (suitably scaled sequences of) convex bodies seem to satisfy many limit theorems that hold for product measures ...
- e.g. CLTs, LDPs ... with the latter containing more geometric information about the high-dimensional body

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- But LDPs do not capture all geometric information ...
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Can one get more information from lower-dimensional projections?

Perhaps look at refined estimates: sharp large deviations?

Consider an i.i.d. sequence of non-lattice random variables $\{X_i\}$.

Theorem (Bahadur, Ranga-Rao '60)

Let Λ be the log moment generating function of X_1. Let $a > 0$ be such that $a = \Lambda'(\tau_a)$ for some positive τ_a, and $\sigma_a^2 = \Lambda''(\tau_a)$. Then we have the following refinement of LDP

$$\mathbb{P}\left(\frac{1}{n}\sum_{i=1}^{n} X_i \geq a\right) = \frac{e^{-n\mathbb{I}(a)}}{\sigma_a \tau_a \sqrt{2\pi n}}[1 + o(1)].$$

Sharp large deviations accompanying Cramér's theorem

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Question: Can we obtain a similar sharp estimate for (annealed) random projections?

A general result: Sharp density condition (SDC)

$$\mathbb{P}\left(n^{-1/2}\mathbf{A}_{n,1}^T X^{(n)} > a\right) \stackrel{?}{=} \frac{e^{-n\mathbb{I}_X^{an}(a)}}{\sigma_a \tau_a \sqrt{2\pi n}}[1 + o(1)]$$

Assumption SDC

1. $\{\|X^{(n)}\|_2^2/n\}$ satisfies an LDP with rate function $J(x)$. Define $D_J := \{x \in \mathbb{R} : J(x) < \infty\}$

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Assumption SDC

1. $\{\|X^{(n)}\|_2^2/n\}$ satisfies an LDP with rate function $J(x)$. Define $D_J := \{x \in \mathbb{R} : J(x) < \infty\}$
2. The random variable $\|X^{(n)}\|_2^2/n$ has a density f^n and let there exist a differentiable function h and a constant $\alpha \in \mathbb{R}$ such that the following asymptotic estimate holds:

$$f^n(x) = n^{\alpha}h(x)e^{-nJ(x)}(1+o(1))$$

uniformly in any compact neighborhood of x in D_J.

Remark: Assumption SDC-1. implies the asymptotic thin-shell condition

Let $\Theta^{(n)}$ be uniformly distributed on S^{n-1}.

Recall Asymptotic Thin Shell Theorem (Kim, Liao, R '20)

Suppose **Assumption A*** holds, with GRF J_X. Then the projection $\{n^{-1/2}\sum_{i=1}^n X_i^{(n)}\Theta_i^n\}$ satisfies an LDP at speed n, with GRF

$$I_X^{\text{an}}(x) := \inf_{0<c<1}\left\{J_X\left(\frac{\|x\|_2}{c}\right) - \frac{1}{2}\log\left(1-c^2\right)\right\},$$

that is, this implies

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A Refinement: Theorem (Liao and R '21)

Suppose $\{X^{(n)}\}$ satisfies Assumption SDC. For $a>0$ such that $\mathbb{I}_X^{\mathrm{an}}(a)<\infty$. Then, there exists $\gamma_a^{\mathrm{an}}\in\mathbb{R}_+$ such that

$$\mathbb{P}\left(n^{-1/2}\sum_{i=1}^n X_i^{(n)}\Theta_i^n > a\right) = \frac{1}{\gamma_a^{\mathrm{an}} n^{1-\alpha}} e^{-n\mathbb{I}_X^{\mathrm{an}}(a)}(1+o(1)).$$

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Compare with Product measure (Bahadur, Ranga-Rao):

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A. Yes. ℓ_p^n balls, $p > 2$, and superquadratic Orlicz balls

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$$V^2(x)/x \to \infty, \qquad \text{as } x \to \infty.$$

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Remark: When $V(x) = |x|^p$, B_V^n is indeed the ℓ_p^n ball of radius $n^{1/p}$

Theorem (Liao and R' 21)

Superquadratic Orlicz balls satisfy the SDC with $\alpha = 1/2$.

A Geometric Consequence - Intersection of ℓ_p^n balls

A phase transition result for intersections of ℓ_p^n balls

- Intersection of ℓ_p^n balls (**Schechtman-Zinn** '90; **Schechtman-Schmuckenschläger** '91)
  For $p \in (0, \infty]$, $q \in (0, \infty)$, there exists $c_{pq} > 0$ such that

$$\left|\hat{B}_p^n \cap t\hat{B}_q^n\right| \to \begin{cases} 0, & \text{if} \quad t < c_{pq}, \\ 1, & \text{if} \quad t > c_{pq}, \end{cases}$$

  where $\hat{B}_p^n$ is the normalized ℓ_p^n ball with volume 1.

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  where $\hat{B}_p^n$ is the normalized ℓ_p^n ball with volume 1.
- Critical case: when $t = c_{pq}$ (Schmuckenschläger '01)
  $$\left|\hat{B}_p^n \cap c_{pq}\hat{B}_q^n\right| \to \frac{1}{2}.$$

Remark: The proof makes use of the special probabilistic representation for ℓ_p^n balls, the WLLN and the CLT

Resolving an Open Problem: Intersections of Orlicz Balls

Theorem (Liao and R '21)

Given Orlicz functions V_1 and V_2 such that $V_1(x)/V_2(x) \to \infty$ as $x \to \infty$ and $R_1 > 0$, for every $R_2 > 0$, there exists an explicit constant $c_{R_1} > 0$ such that as $n \to \infty$

$$\frac{\left|B^n_{V_1}(R_1) \cap B^n_{V_2}(R_2)\right|}{\left|B^n_{V_1}(R_1)\right|} \to \begin{cases} 0, & \text{if} \quad c_{R_1} > R_2 \\ \frac{1}{2} & \text{if} \quad c_{R_1} = R_2 \\ 1, & \text{if} \quad c_{R_1} < R_2. \end{cases}$$

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Remark: No explicit probabilistic representation in the case of Orlicz balls, and so the proof is quite different.
the 0 and 1 limits use large deviations estimates (see also **Kabluchko-Prochno '20**), but the critical case requires sharp large deviation estimates.

Summary & Future work

- CLT for convex sets is a universal and beautiful result, but nonuniversal large deviation results enables one to classify or distinguish between different measures.
- A general sufficient condition was developed for (annealed) large deviation principles to hold for random projections – asymptotic thin shell condition.
- Various examples are shown to satisfy this condition, including those not admitting a convenient representation.
- Sharp large deviation estimates obtained for (norms) of random projections under SDC, which was verified for Orlicz balls
- Open: Verification of the asymptotic thin shell condition and SDC for broader classes of convex bodies.
- Considered applications to asymptotic convex geometry (volumetric properties for intersections of convex bodies)
- Future directions: further applications to high-dimensional statistics and data science ...

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