



# **Southeastern Probability Conference**

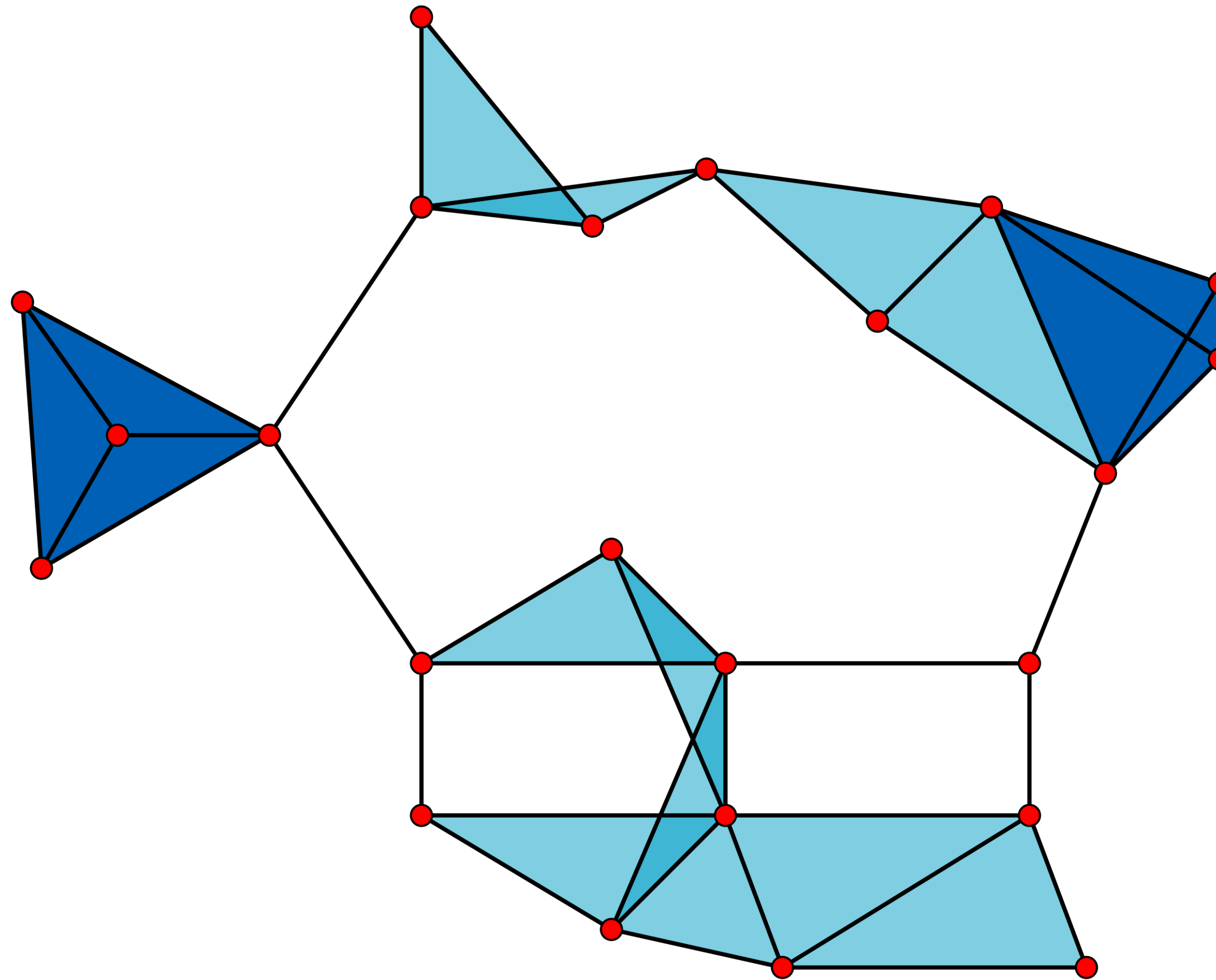
**Limit theorems for Betti numbers of random  
simplicial complexes**

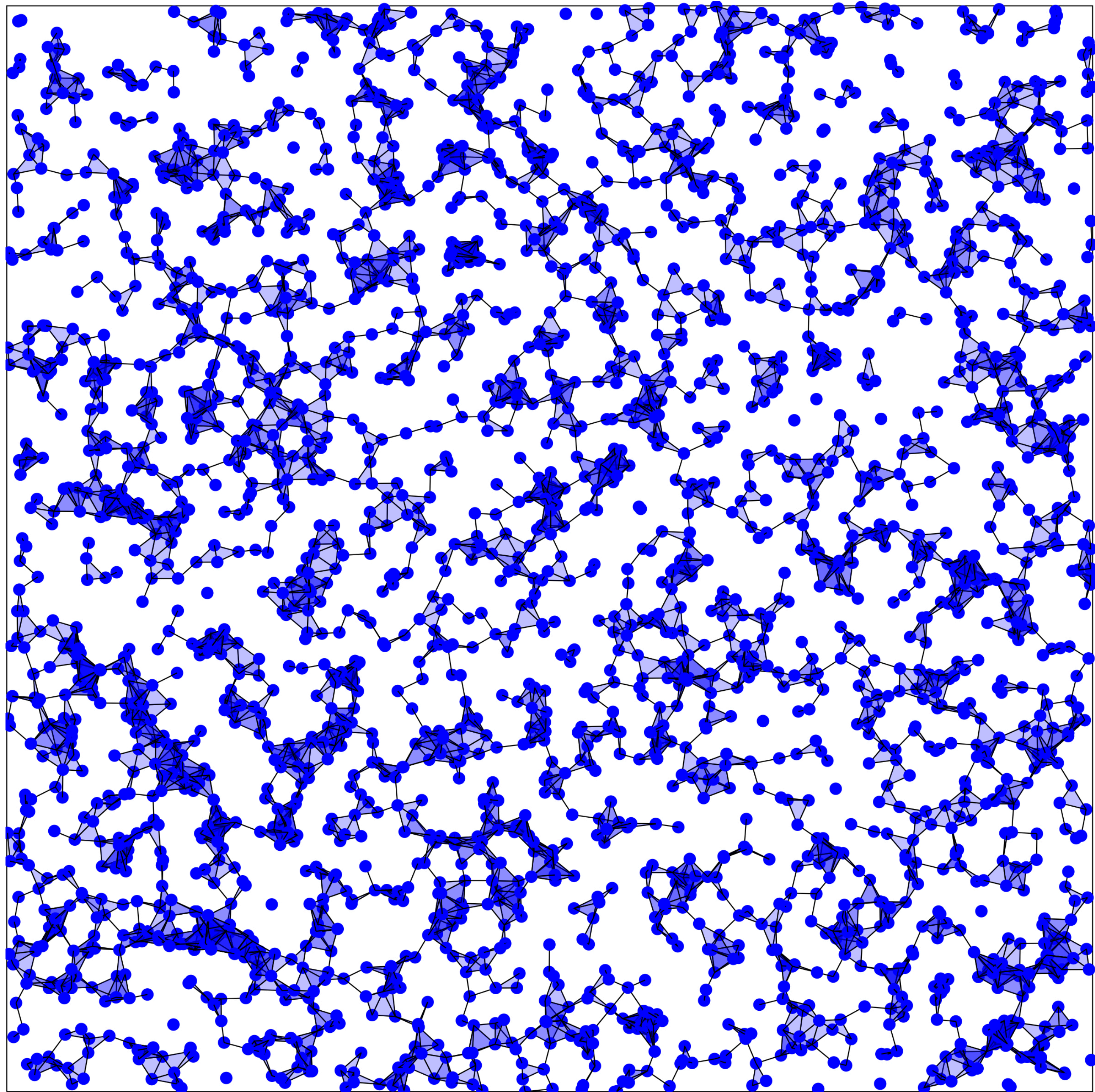
**(based on joint work with Elizabeth Meckes)**

Over the past 15 years or so “stochastic topology”, the study of random shapes, has been an active field.

This has some proposed applications, including to topological data analysis.

We're often interested in combinatorial stochastic topology, e.g. random simplicial complexes.



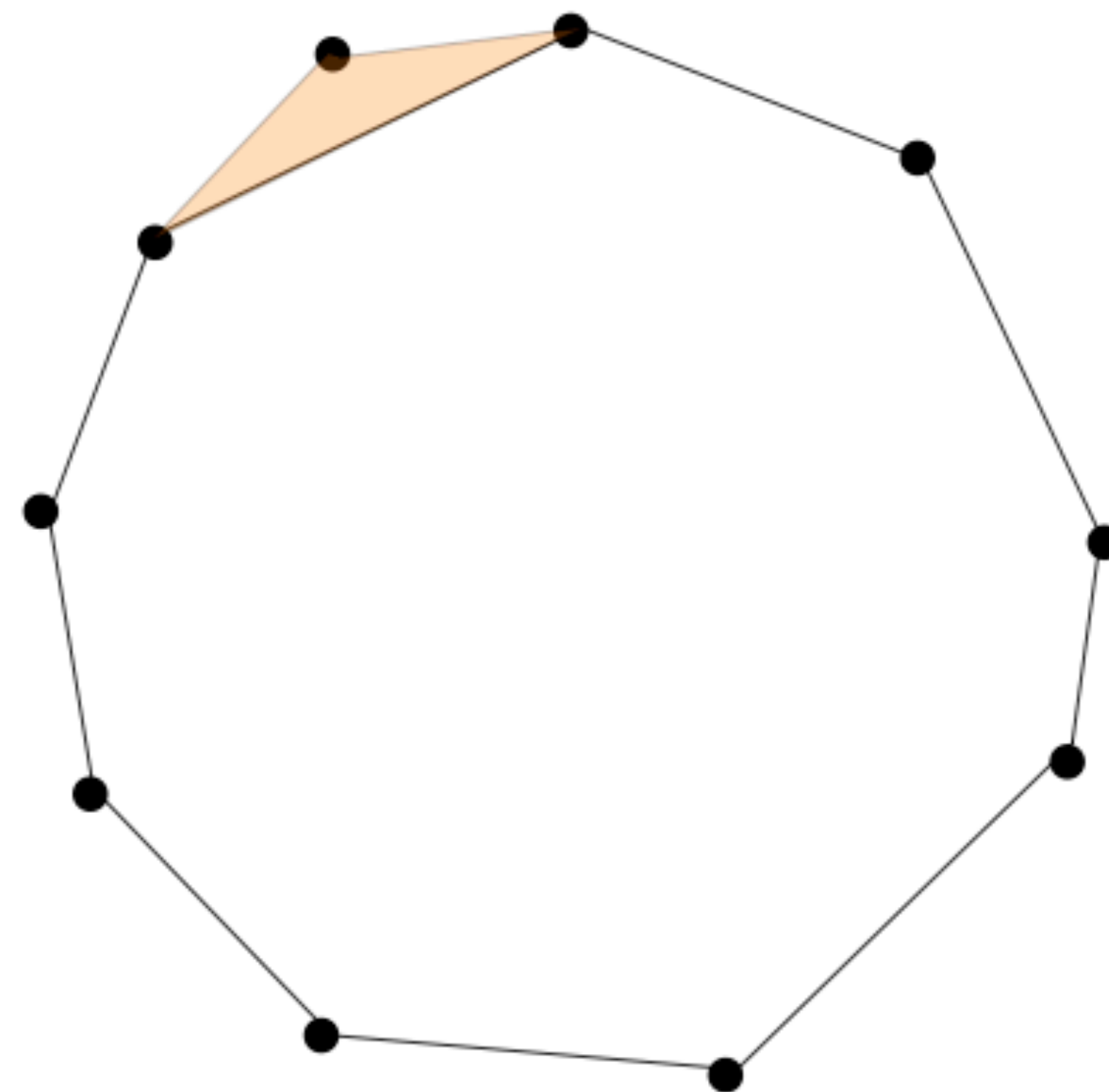
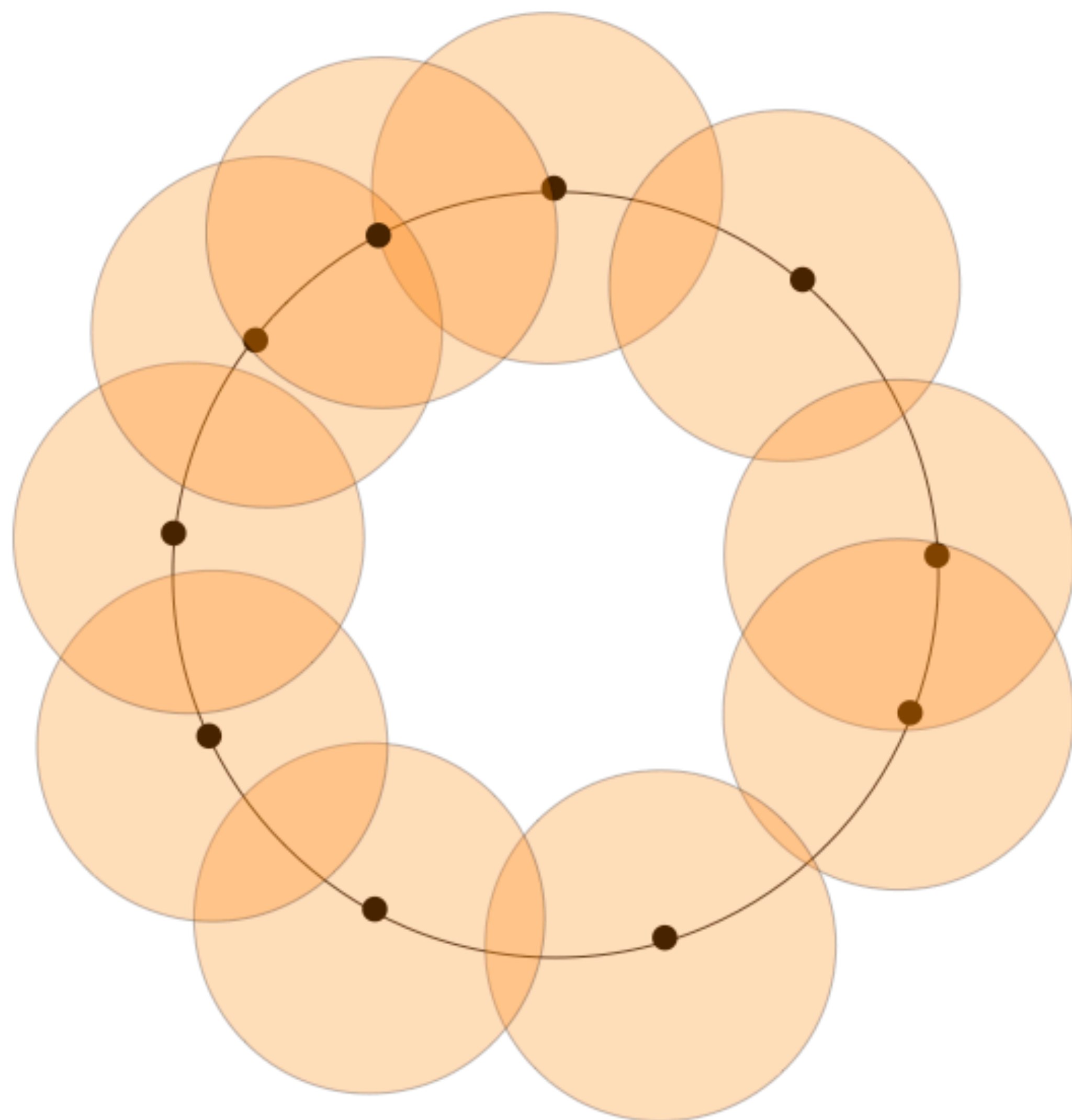




AIM Workshop on *Topological Complexity of Random Sets*  
August 2009

Let  $X_n$  denote a set of  $n$  i.i.d. random points in  $\mathbb{R}^d$ . E.g. let  $X_n$  be a set of  $n$  i.i.d. uniform random points in the unit  $d$ -dimensional cube  $[0,1]^d$ .

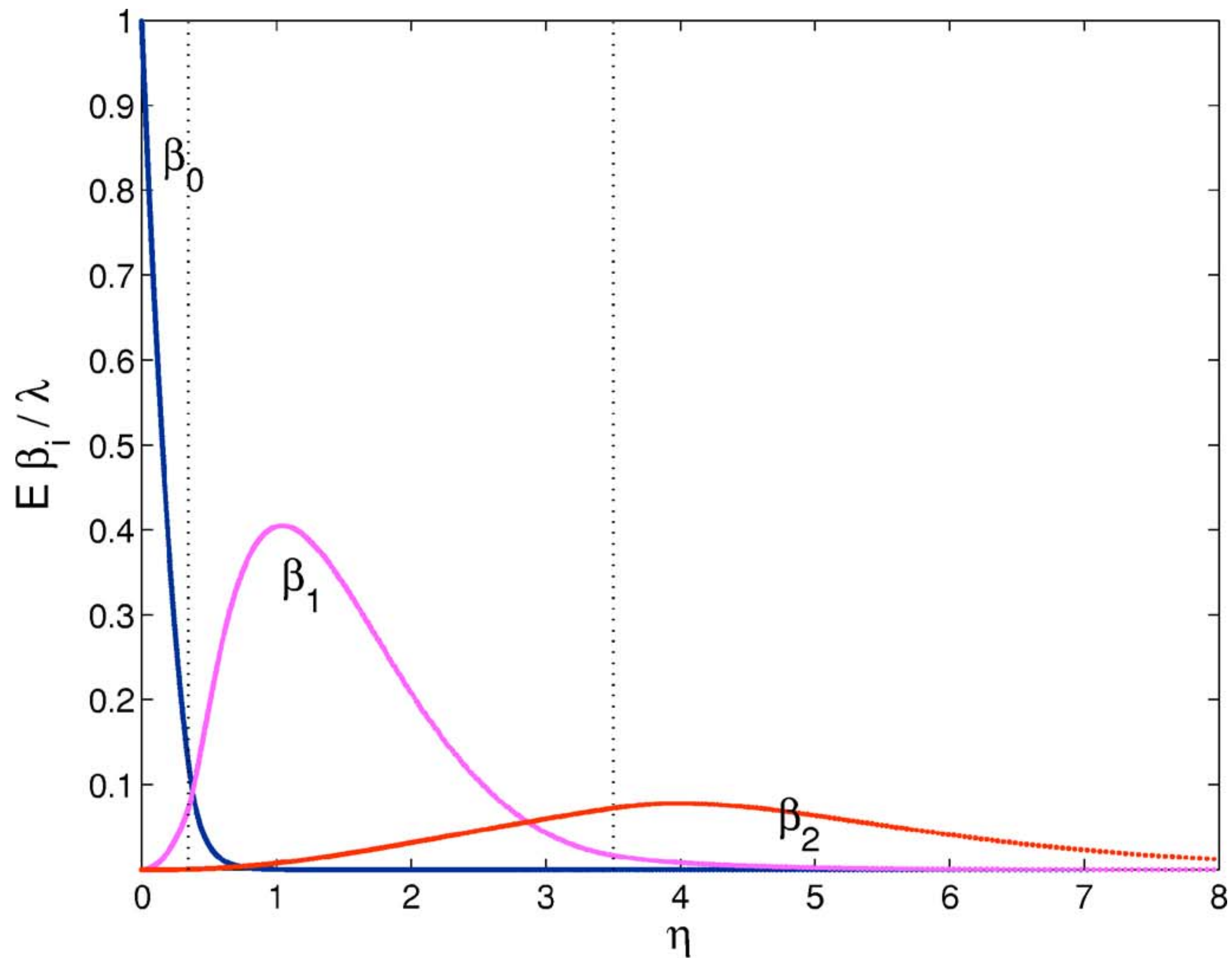
Let  $C(n, r)$  denote the *random Čech complex*. I.e. take the union of balls of radius  $r$  around the points in  $X_n$ .



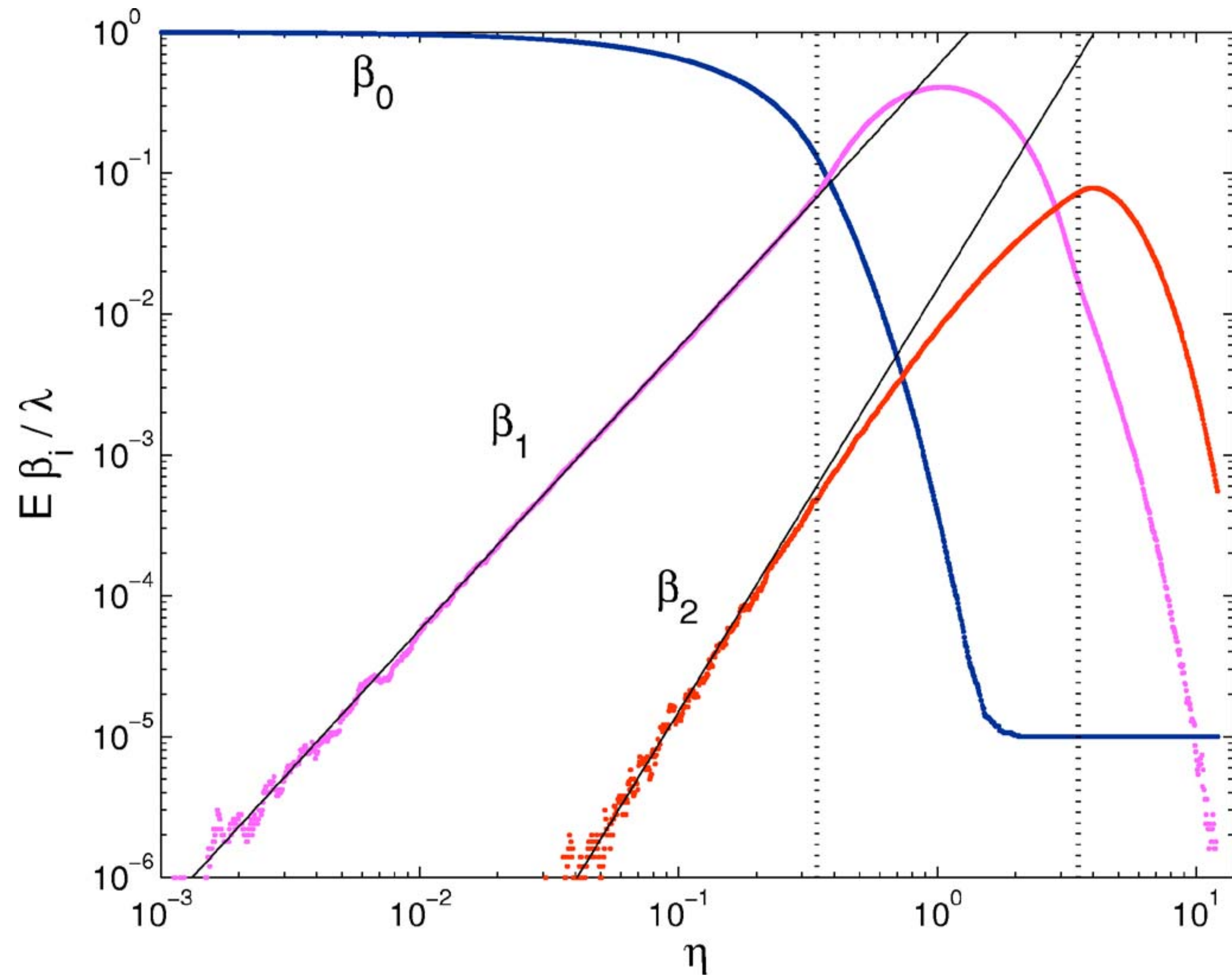
I had been studying topological properties of  $C(n, r)$ , namely: what is the *homology* like?

Are there any  $k$ -dimensional holes, and  
if so how many?

I didn't realize at the time that random Čech complexes had been previously studied..



Robins, Vanessa. Betti number signatures of homogeneous Poisson point processes. *Phys. Rev. E.* 74 (2006), no. 1, 87–127.



Robins, Vanessa. Betti number signatures of homogeneous Poisson point processes. *Phys. Rev. E.* **74** (2006), no. 1, 87–127.

**Theorem.** (K.) Let  $r = n^{-\alpha}$ ,  $d \geq 2$ , and  $1 \leq k \leq d - 1$ .

If  $\alpha < 1/d$  or  $\alpha > \frac{k+2}{d(k+1)}$  then w.h.p.  $H_k(C(n, r)) = 0$ .

If  $\frac{1}{d} < \alpha < \frac{k+2}{d(k+1)}$  then w.h.p.  $H_k(C(n, r)) \neq 0$ .

Kahle, M. Random geometric complexes. Disc.  
Comp. Geom. 45 (2011), 553–573.

**Theorem.** (K.) If  $r \ll n^{-1/d}$  then  $\mathbb{E}[\beta_k] \asymp n^{k+2} r^{d(k+1)}$ .

Here  $\beta_k = \beta_k [C(n, r)]$  denotes the  $k$ th Betti number,  
i.e. the number of  $k$ -dimensional holes.

Kahle, M. Random geometric complexes. Disc.  
Comp. Geom. 45 (2011), 553–573.

After the AIM meeting, I asked Meckes if we could try to prove a CLT for Betti numbers.

We exchanged several emails and made some headway, and she invited me to visit her at CWRU in March 2010. During this time, we worked out most of the main ideas..

**Theorem.** If  $r \ll n^{-1/d}$  then  $\frac{\beta_k - \mathbb{E}[\beta_k]}{\sqrt{\text{Var}[\beta_k]}} \rightarrow \mathcal{N}(0,1)$ .

Kahle, M. and Meckes, E. Limit theorems for Betti numbers of random simplicial complexes. *Homology Homotopy Appl.* **15** (2013), 343–374.

Kahle, M. and Meckes, E. Erratum to “Limit theorems for Betti numbers of random simplicial complexes”. *Homology Homotopy Appl.* **18** (2016), 129–142.

Some interesting headway has been made on random geometric complexes since then..

Yogeshwaran, D.; Subag, Eliran; Adler, Robert J. Random geometric complexes in the thermodynamic regime. Probab. Theory Related Fields 167 (2017), no. 1-2, 107–142.

This paper proves a CLT for the Betti numbers in the regime where  $r \asymp n^{-1/d}$ .

“The proofs combine probabilistic arguments from the theory of stabilizing functionals of point processes and topological arguments exploiting the properties of Mayer–Vietoris exact sequences.”

Some interesting headway has been made on random geometric complexes since then..

Bobrowski, Omer. Homological connectivity of random Čech complexes. [arXiv:1906.04861](https://arxiv.org/abs/1906.04861)

This paper studies random Čech complexes in a flat torus  $\mathbb{T}^d$ , and the threshold for homology  $H_k(C(n, r))$  to become isomorphic with the homology of the ambient torus  $H_k(\mathbb{T}^d)$ .

**Theorem** (Bobrowski).

Consider the random Cech complex  $C(n, r)$  in a flat torus  $\mathbb{T}^d$  made by identifying the opposite sides of  $[0, 1]^d$ . Let  $\mathcal{H}_{k,r}$  be the event that  $H_k(C(n, r)) \cong H_k(\mathbb{T}^d)$ .

Set  $\Lambda = \omega_d n r^d$ . Let  $1 \leq k \leq d - 2$ .

- If  $\Lambda \geq \log n + (k - 1) \log \log n + \omega(1)$  then  $\mathbb{P}(\mathcal{H}_{k,r}) \rightarrow 1$ .
- If  $\Lambda \leq \log n + (k - 1) \log \log n - \omega(1)$  then  $\mathbb{P}(\mathcal{H}_{k,r}) \rightarrow 0$ .
- If  $\Lambda \geq \log n + (d - 1) \log \log n + \omega(1)$  then  $\mathbb{P}(\mathcal{H}_{d-1,r}) \rightarrow 1$ .
- If  $\Lambda \leq \log n + (d - 1) \log \log n - \omega(1)$  then  $\mathbb{P}(\mathcal{H}_{d-1,r}) \rightarrow 0$ .

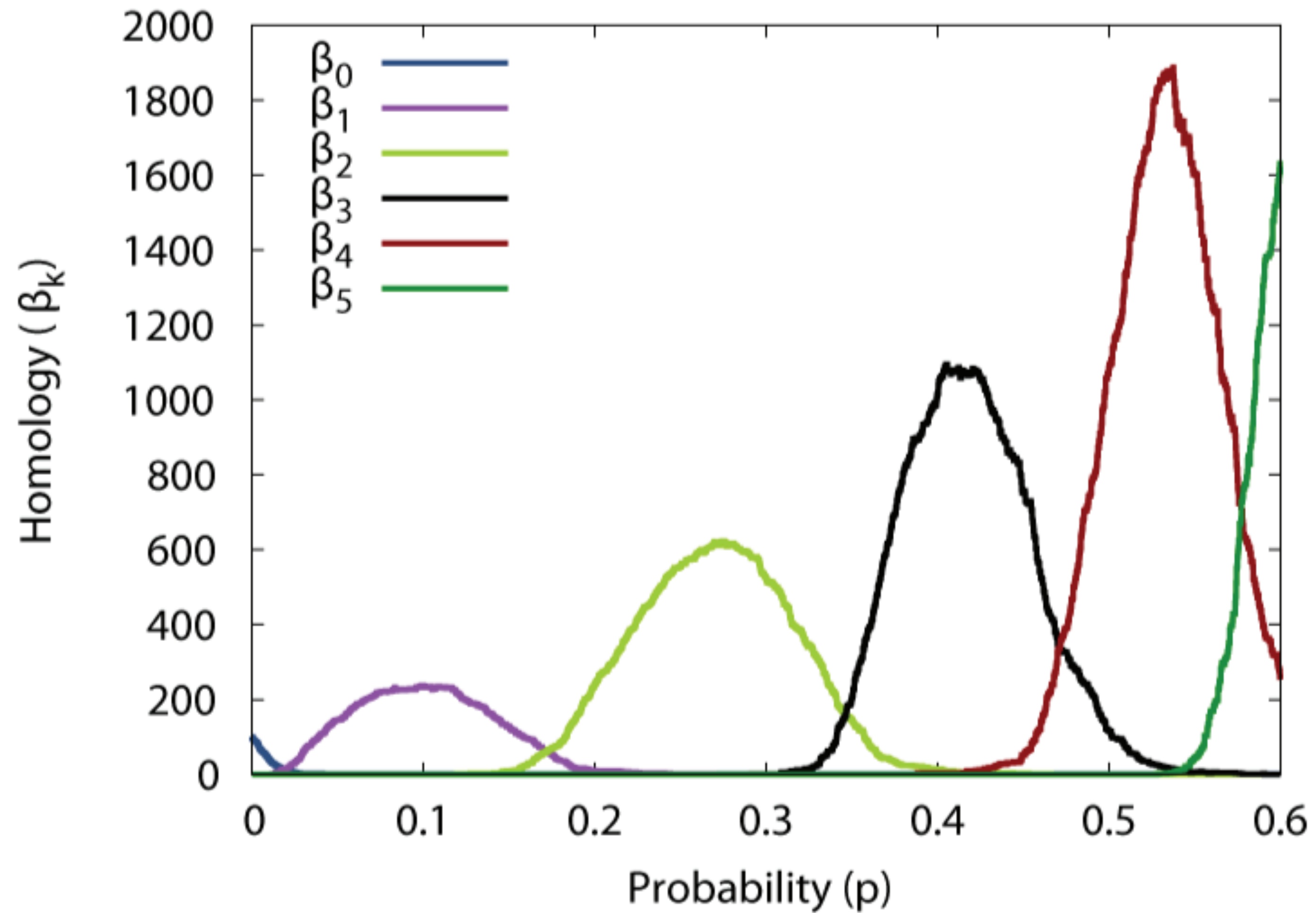
Bobrowski, Omer. Homological Connectivity in Random Cech Complexes. [arXiv:1906.04861](https://arxiv.org/abs/1906.04861)

Let  $G(n, p)$  denote the Erdős–Rényi random graph.

I.e.  $G(n, p)$  has vertex set  $[n] = \{1, 2, \dots, n\}$  and every edge has probability  $p$ , independently.

Let  $X(n, p)$  denote the random clique complex.

I.e.  $X(n, p)$  is the maximal simplicial complex compatible with  $G(n, p)$ .



The Betti numbers of a random clique complex  $\beta_k(X(n, p))$  when  $n = 100$ .

**Theorem.** (K.-Meckes) If  $n^{-1/k} \ll p \ll n^{-1/(k+1)}$  then

$$\frac{\beta_k - \mathbb{E}[\beta_k]}{\sqrt{\text{var}[\beta_k]}} \rightarrow \mathcal{N}(0,1).$$

Kahle, M. and Meckes, E. Limit theorems for Betti numbers of random simplicial complexes. *Homology Homotopy Appl.* **15** (2013), 343–374.

Kahle, M. and Meckes, E. Erratum to “Limit theorems for Betti numbers of random simplicial complexes”. *Homology Homotopy Appl.* **18** (2016), 129–142.

## **The bouquet-of-spheres conjecture.**

If  $n^{-1/k} \ll p \ll n^{-1/(k+1)}$  and  $k \geq 3$  then w.h.p.  $X(n, p)$  is homotopy equivalent to a bouquet of  $k$ -dimensional spheres.

“... However, that does not explain why so many simplicial complexes that arise in combinatorics are homotopy equivalent to a wedge of spheres. I have often wondered if perhaps there is some deeper explanation for this.” — Robin Forman

“Wonderful family and friends, some theorems.”  
— Elizabeth Meckes’s six word memoir

