Joint Moments of Characteristic Polynomials of Random Unitary Matrices

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This talk is dedicated to Elizabeth's memory

Characteristic Polynomials of Random Unitary Matrices

Let A be an $N \times N$ unitary matrix. Denote the eigenvalues of A by $e^{i\theta_n}$, $1 \leq n \leq N$, and the characteristic polynomial of A on the unit circle in the complex plane by

$$P_N(A, \theta) = \det(I - Ae^{-i\theta}) = \prod_n (1 - e^{i\theta_n - i\theta}).$$

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Moments:

$$\begin{aligned} M_N(\beta) &= \mathbb{E}_{A \in U(N)} |P_N(A, \theta)|^{2\beta} \\ &= \frac{1}{(2\pi)^N N!} \int_0^{2\pi} \cdots \int_0^{2\pi} \prod_{n=1}^{N} |1 - e^{i(\theta_n - \theta)}|^{2\beta} \\ &\qquad \times \prod_{1 \leq j < k \leq N} |e^{i\theta_j} - e^{i\theta_k}|^2 d\theta_1 \ldots d\theta_N \end{aligned}$$

For $\mathrm{Re}\beta > -1/2$

$$M_N(\beta) = \prod_{j=1}^{N} \frac{\Gamma(j)\Gamma(j+2\beta)}{\Gamma(j+\beta)^2} = \frac{G(1+\beta)^2 G(N+1) G(N+1+2\beta)}{G(1+2\beta) G(N+1+\beta)^2}$$

where $G(s)$ is the Barnes G-function, which satisfies $G(s+1) = \Gamma(s)G(s)$ (JPK & NC Snaith 2000).

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and for $k \in \mathbb{N}$

$$M_N(k) \sim \left(\prod_{m=0}^{k-1} \frac{m!}{(m+k)!} \right) N^{k^2}$$

combinatorial interpretation: for $\beta = k \in \mathbb{N}$, as $N \to \infty$

$$M_N(k) \sim \frac{g_k}{k^2!} N^{k^2}$$

where g_k is the number of ways of filling a $k \times k$ array with the integers $1, 2, \ldots, k^2$ in such a way that the numbers increase along each row and down each column (i.e. the number of $k \times k$ *Young tableaux*).

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c.f. Hardy & Littlewood 1918, Ingham 1926, Conrey & Ghosh 1991, Conrey & Gonek 2000, JPK & NC Snaith 2000, ...

Joint moments

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Set

$$V_N(A,\theta) := \exp\left(\mathrm{i}N\frac{(\theta+\pi)}{2} - \mathrm{i}\sum_{n=1}^{N}\frac{\theta_n}{2}\right)P_N(A,\theta),$$

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($V_N(A,\theta)$ is real-valued for $\theta \in [0, 2\pi)$).
The joint moments of the function $V_U(\theta)$ and its derivative are

$$F_N(k,h) := \mathbb{E}_{A\in U(N)}|V_N(A,0)|^{2k-2h}|V_N'(A,0)|^{2h},$$

where it is assumed that

$$h > -\frac{1}{2} \quad \text{and} \quad k > h - \frac{1}{2}.$$

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These joint moments have been studied by many authors, including Hughes (2001), Conrey Rubinstein & Snaith (2006), Dehaye (2008, 2010), Winn (2012), Riedtmann (2018), Basor *et al.* (2018), Bailey *et al.* (2019).

Asymptotics

Conjecture (Hughes 2001)

When $N \to \infty$, for $k > -1/2$ and $0 \leq h < k + 1/2$

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i.e.

$$F(k,h) := \lim_{N\to\infty} \frac{F_N(k,h)}{N^{k^2+2h}}$$

exists and is non-zero for $k > -1/2$ and $0 \leq h < k + 1/2$

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For integer and half-integer values of h and k, $F_N(k, h)$ is equal to a sum over Young Tableaux, but with a complicated summand (Dehaye (2008, 2010), Winn (2012), and Riedtmann (2018)). The analysis of these formulae in general is a major challenge.

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It has so far not been possible to extend these approaches for a given $h \in \mathbb{N}$, to $k > h - 1/2$, or to non-integer values of h.

Connection to Painlevé equations

Let $L_n^{(\alpha)}(t)$ be the generalized Laguerre polynomial

$$L_n^{(\alpha)}(t) := \frac{e^t}{t^\alpha n!}\frac{d^n}{dt^n}\left(t^{\alpha+n}e^{-t}\right) = \sum_{j=0}^{n} \frac{\Gamma(n+\alpha+1)}{\Gamma(j+\alpha+1)(n-j)!}\frac{(-t)^j}{j!}$$

and define

$$K_n(\epsilon, y) := \frac{(-1)^n}{\pi}\frac{\partial^n}{\partial\epsilon^n}\left(\frac{\epsilon}{\epsilon^2+y^2}\right).$$

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Proposition (Winn 2012)

$$F_N(h,k) = \lim_{\epsilon\to 0}(-1)^{\frac{k(k-1)}{2}}2^{-2h}\int_{-\infty}^{\infty}K_{2h}(\epsilon,y)e^{-N|y|}$$
$$\times \det\left[L_{N+k-1-(i+j)}^{(2k-1)}(-2|y|)\right]_{i,j=0,\dots,k-1}dy,$$

with $N > k-1$.

Theorem – Basor, Bleher, Buckingham, Grava, Its, Its & Keating 2018

Setting

$$\det\left[L^{(2k-1)}_{N+k-1-(i+j)}(-2|y|)\right]_{i,j=0,\cdots,k-1} = \frac{e^{-2k|y|}}{(2\pi i)^k} H_k[w_0],$$

we have that

$$\frac{d}{dx}\ln H_k = \frac{\sigma(x)+kx+Nk}{x},$$

where $\sigma(x)$ is a solution of the σ-Painlevé V equation

$$\left(x\frac{d^2\sigma}{dx^2}\right)^2 = \left(\sigma - x\frac{d\sigma}{dx} + 2\left(\frac{d\sigma}{dx}\right)^2 - 2N\frac{d\sigma}{dx}\right)^2 - 4\frac{d\sigma}{dx}\left(-N+\frac{d\sigma}{dx}\right)\left(-k-N+\frac{d\sigma}{dx}\right)\left(k+\frac{d\sigma}{dx}\right)$$

with asymptotics $\sigma(x) = -Nk + \frac{N}{2}x + \mathcal{O}(x^2)$ as $x \to 0$.

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2. a series of rational and gauge transformations reduces this system of o.d.e.s to the Lax pair of P_{V};
3. identify the Hankel determinant with a particular solution of the σ-form of P_{V}.

Large-Matrix Limit

Theorem – Basor, Bleher, Buckingham, Grava, Its, Its & Keating 2018

For $h \in \mathbb{N}$, $k > h - 1/2$, in general

$$F(h,k) = (-1)^h \frac{G(k+1)^2}{G(2k+1)} \frac{d^{2h}}{dx^{2h}} \left[\exp \int_0^x \left(\frac{\xi(s)}{s} ds \right) \right] \Bigg|_{x=0},$$

where G is the Barnes function and $\xi(x)$ is a particular solution of the σ-Painlevé III equation

$$(x\xi'')^2 = -4x(\xi')^3 + \left(4k^2 + 4\xi\right)(\xi')^2 + x\xi' - \xi,$$

with the initial conditions

$$\xi(0) = 0\,, \quad \xi'(0) = 0.$$

c.f. Bailey, Bettin, Blower, Conrey, Prokhorov, Rubinstein & Snaith (2019)

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Let $\mathbb{W}_N$ denote the Weyl chamber:

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For $N \geq 1$ and $s > -\frac{1}{2}$, the Hua-Pickrell probability measure $\mathfrak{M}_N^{(s)}$ on $\mathbb{W}_N$ is

$$\mathfrak{M}_N^{(s)}(d\mathbf{x}) = \frac{1}{\mathfrak{c}_N^{(s)}} \prod_{j=1}^{N} \frac{1}{(1+x_j^2)^{N+s}} \Delta_N(\mathbf{x})^2 dx_1 \cdots dx_N$$

where $\Delta_N(\mathbf{x}) = \prod_{1 \leq i < j \leq N} (x_j - x_i)$ and $\mathfrak{c}_N^{(s)}$ is a normalisation constant.

Let $s > -\frac{1}{2}$. Then,

$$\frac{1}{N}\sum_{i=1}^{N} \mathrm{x}_i^{(N)} \xrightarrow{\mathrm{d}} \mathrm{X}(s), \quad \text{as } N \to \infty,$$

where $(\mathrm{x}_1^{(N)}, \dots, \mathrm{x}_N^{(N)})$ has law $\mathrm{M}_N^{(s)}$ and $\mathrm{X}(s)$ is a random variable that plays an important role in the work of Pickrell (1991), Vershik (1994), Olshanski & Vershik (1996), Borodin & Olshanski (2001), Qiu (2017), ..., classifying the ergodic measures for the action of the infinite dimensional unitary group on the space of infinite Hermitian matrices.

Connection to joint moments [Assiotis, Keating & Warren (2020)]

Theorem Let $s > -\frac{1}{2}$ and $0 \leq h < s + \frac{1}{2}$. Then,

$$\lim_{N\to\infty} \frac{1}{N^{s^2+2h}} F_N(s,h) \stackrel{\text{def}}{=} F(s,h) = F(s,0) 2^{-2h} \mathbb{E}\left[|\mathsf{X}(s)|^{2h} \right]$$

with the limit $F(s,h)$ satisfying $0 < F(s,h) < \infty$. The function $F(s,0)$ is given by

$$F(s,0) = \frac{G(s+1)^2}{G(2s+1)},$$

where G is the Barnes G-function.

Outline of Proof

The first key ingredient is a representation of $F_N(s,h)$ in terms of $F_N(s,0)$ and the moments $\mathbb{E}\left[\left|\sum_{i=1}^{N}\frac{\mathsf{x}_i^{(N)}}{N}\right|^{2h}\right]$, where $(\mathsf{x}_1^{(N)},\dots,\mathsf{x}_N^{(N)})$ have the same distribution as the non-increasing eigenvalues of a random Hermitian matrix with law $\mathfrak{M}_N^{(s)}$.

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To prove convergence of the moments:

$$\mathbb{E}\left[\left|\sum_{i=1}^{N}\frac{\mathsf{x}_i^{(N)}}{N}\right|^{2h}\right] \longrightarrow \mathbb{E}\left[\left|\mathsf{X}(s)\right|^{2h}\right], \quad \text{as } N\to\infty,$$

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one needs to prove uniform integrability or, as we do, show uniform boundedness for some higher moment.

The averages that we want to control uniformly in N do not converge if we bring the absolute values inside, and it is essential that a cancellation due to symmetry around the origin of the points is taken into account.

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Extending this to the range $-\frac{1}{2} < s \leq 0$ takes more work.

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- Probabilistic interpretation and connections?