Joint Moments of Characteristic Polynomials of Random Unitary Matrices

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May 15, 2021
This talk is dedicated to Elizabeth’s memory
Let $A$ be an $N \times N$ unitary matrix. Denote the eigenvalues of $A$ by $e^{i\theta_n}$, $1 \leq n \leq N$, and the characteristic polynomial of $A$ on the unit circle in the complex plane by

$$P_N(A, \theta) = \det(I - Ae^{-i\theta}) = \prod_{n}(1 - e^{i\theta_n-i\theta}).$$
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Moments:

$$M_N(\beta) = \mathbb{E}_{A \in U(N)} |P_N(A, \theta)|^{2\beta}$$

$$= \frac{1}{(2\pi)^N N!} \int_{0}^{2\pi} \cdots \int_{0}^{2\pi} \prod_{n=1}^{N} |1 - e^{i(\theta_n - \theta)}|^{2\beta}$$

$$\times \prod_{1 \leq j < k \leq N} |e^{i\theta_j} - e^{i\theta_k}|^2 d\theta_1 \cdots d\theta_N$$
For $\text{Re}\beta > -1/2$

$$M_N(\beta) = \prod_{j=1}^{N} \frac{\Gamma(j) \Gamma(j + 2\beta)}{\Gamma(j + \beta)^2} = \frac{G(1 + \beta)^2 G(N + 1) G(N + 1 + 2\beta)}{G(1 + 2\beta) G(N + 1 + \beta)^2}$$

where $G(s)$ is the Barnes $G$-function, which satisfies $G(s + 1) = \Gamma(s) G(s)$ (JPK & NC Snaith 2000).
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$$M_N(\beta) \sim \frac{G(1 + \beta)^2}{G(1 + 2\beta)} N^{\beta^2}$$

and for $k \in \mathbb{N}$

$$M_N(k) \sim \left( \prod_{m=0}^{k-1} \frac{m!}{(m + k)!} \right) N^{k^2}$$
combinatorial interpretation: for $\beta = k \in \mathbb{N}$, as $N \to \infty$

$$M_N(k) \sim \frac{g_k}{k^2!} N^{k^2}$$

where $g_k$ is the number of ways of filling a $k \times k$ array with the integers $1, 2, \ldots, k^2$ in such a way that the numbers increase along each row and down each column (i.e. the number of $k \times k$ Young tableaux).
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number theoretic application: moments of the Riemann zeta-function

$$\frac{1}{T} \int_0^T |\zeta(1/2 + it)|^{2\beta} dt$$

$c.f.$ Hardy & Littlewood 1918, Ingham 1926, Conrey & Ghosh 1991, Conrey & Gonek 2000, JPK & NC Snaith 2000, ...
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Joint moments

\[ V_N(A, \theta) := \exp\left( i N (\theta + \pi) - i N \sum_{n=1}^{N} \theta_n^2 \right) \]

\( V_N(A, \theta) \) is real-valued for \( \theta \in [0, 2\pi) \).

The joint moments of the function \( V_U(\theta) \) and its derivative are

\[ F_N(k, h) := \mathbb{E}_{A \in U(N)} |V_N(A, 0)|^{2k - 2h} |V'_N(A, 0)|^{2h} , \]

where it is assumed that \( h > -\frac{1}{2} \) and \( k > h - \frac{1}{2} \).

These joint moments have been studied by many authors, including Hughes (2001), Conrey Rubinstein & Snaith (2006), Dehaye (2008, 2010), Winn (2012), Riedtmann (2018), Basor et al. (2018), Bailey et al. (2019).
Joint moments

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Conjecture (Hughes 2001)

When $N \to \infty$, for $k > -1/2$ and $0 \leq h < k + 1/2$

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i.e.

$$F(k, h) := \lim_{N \to \infty} \frac{F_N(k, h)}{N^{k^2+2h}}$$

exists and is non-zero for $k > -\frac{1}{2}$ and $0 \leq h < k + \frac{1}{2}$
Hughes (2001) proved the conjectured scaling with $N$ for integer values of $h$ and $k$, but was not able to establish a tractable general formula for $F(k, h)$. 
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For integer and half-integer values of $h$ and $k$, $F_N(k, h)$ is equal to a sum over Young Tableaux, but with a complicated summand (Dehaye (2008, 2010), Winn (2012), and Riedtmann (2018)). The analysis of these formulae in general is a major challenge.
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It has so far not been possible to extend these approaches for a given $h \in \mathbb{N}$, to $k > h - 1/2$, or to non-integer values of $h$. 
Let $L_n^{(\alpha)}(t)$ be the generalized Laguerre polynomial

$$L_n^{(\alpha)}(t) := \frac{e^t}{t^{\alpha}n!} \frac{d^n}{dt^n} \left( t^{\alpha+n}e^{-t} \right) = \sum_{j=0}^{n} \frac{\Gamma(n + \alpha + 1)}{\Gamma(j + \alpha + 1)(n-j)!} \frac{(-t)^j}{j!}$$

and define

$$K_n(\epsilon, y) := \frac{(-1)^n}{\pi} \frac{\partial^n}{\partial \epsilon^n} \left( \frac{\epsilon}{\epsilon^2 + y^2} \right).$$
Connection to Painlevé equations

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\]

Proposition (Winn 2012)

\[
F_N(h, k) = \lim_{\epsilon \to 0} (-1)^{k(k-1)/2} 2^{-2h} \int_{-\infty}^{\infty} K_{2h}(\epsilon, y) e^{-N|y|} \times \det \left[ L_{N+k-1-(i+j)}^{(2k-1)}(-2|y|) \right]_{i,j=0,\ldots,k-1} dy,
\]

with \( N > k - 1 \).
Setting

\[
\det \left[ L_{N+k-1-(i+j)}^{(2k-1)}(-2|y|) \right]_{i,j=0,\ldots,k-1} = \frac{e^{-2k|y|}}{(2\pi i)^k} H_k[w_0],
\]

we have that

\[
\frac{d}{dx} \ln H_k = \frac{\sigma(x) + kx + Nk}{x},
\]

where \(\sigma(x)\) is a solution of the \(\sigma\)-Painlevé V equation

\[
\left( x \frac{d^2 \sigma}{dx^2} \right)^2 = \left( \sigma - x \frac{d\sigma}{dx} + 2 \left( \frac{d\sigma}{dx} \right)^2 - 2N \frac{d\sigma}{dx} \right)^2
\]

\[
- 4 \frac{d\sigma}{dx} \left( -N + \frac{d\sigma}{dx} \right) \left( -k - N + \frac{d\sigma}{dx} \right) \left( k + \frac{d\sigma}{dx} \right)
\]

with asymptotics \(\sigma(x) = -Nk + \frac{N}{2} x + O(x^2)\) as \(x \to 0\).
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Outline of Proof

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2. a series of rational and gauge transformations reduces this system of o.d.e.s to the Lax pair of \( P_V \);

3. identify the Hankel determinant with a particular solution of the \( \sigma \)-form of \( P_V \).
For \( h \in \mathbb{N}, k > h - 1/2 \), in general

\[
F(h, k) = (-1)^h \frac{G(k + 1)^2}{G(2k + 1)} \frac{d^{2h}}{dx^{2h}} \left[ \exp \int_0^x \left( \frac{\xi(s)}{s} \right) ds \right] \bigg|_{x=0},
\]

where \( G \) is the Barnes function and \( \xi(x) \) is a particular solution of the \( \sigma \)-Painlevé III equation

\[
(x\xi'')^2 = -4x(\xi')^3 + (4k^2 + 4\xi)(\xi')^2 + x\xi' - \xi,
\]

with the initial conditions

\[
\xi(0) = 0, \quad \xi'(0) = 0.
\]
Let $W_N$ denote the Weyl chamber:

$$W_N = \{ x = (x_1, x_2, \ldots, x_N) \in \mathbb{R}^N : x_1 \geq x_2 \geq \cdots \geq x_N \}.$$

For $N \geq 1$ and $s > -\frac{1}{2}$, the Hua-Pickrell probability measure $M_N(s)$ on $W_N$ is

$$M_N(s)(dx_1 \cdots dx_N) = \frac{1}{c_N(s)} N! \prod_{j=1}^N (1 + x_j^2)^{N+s} \Delta_N(x) \, dx_1 \cdots dx_N,$$

where $\Delta_N(x) = \prod_{1 \leq i < j \leq N} (x_j - x_i)$ and $c_N(s)$ is a normalisation constant.
Let $\mathbb{W}_N$ denote the Weyl chamber:

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For $N \geq 1$ and $s > -\frac{1}{2}$, the Hua-Pickrell probability measure $\mathcal{M}_N^{(s)}$ on $\mathbb{W}_N$ is

$$\mathcal{M}_N^{(s)}(d\mathbf{x}) = \frac{1}{c_N^{(s)}} \prod_{j=1}^{N} \frac{1}{(1 + x_j^2)^{N+s}} \Delta_N(\mathbf{x})^2 d\mathbf{x} = \cdots d\mathbf{x}_N$$

where $\Delta_N(\mathbf{x}) = \prod_{1 \leq i < j \leq N} (x_j - x_i)$ and $c_N^{(s)}$ is a normalisation constant.
Let $s > -\frac{1}{2}$. Then,

$$\frac{1}{N} \sum_{i=1}^{N} x_i^{(N)} \xrightarrow{d} X(s), \quad \text{as } N \to \infty,$$

where $(x_1^{(N)}, \ldots, x_N^{(N)})$ has law $M_N^{(s)}$ and $X(s)$ is a random variable that plays an important role in the work of Pickrell (1991), Vershik (1994), Olshanski & Vershik (1996), Borodin & Olshanski (2001), Qiu (2017), ..., classifying the ergodic measures for the action of the infinite dimensional unitary group on the space of infinite Hermitian matrices.
Theorem Let $s > -\frac{1}{2}$ and $0 \leq h < s + \frac{1}{2}$. Then,

$$\lim_{N \to \infty} \frac{1}{Ns^2 + 2h} F_N(s, h) \overset{\text{def}}{=} F(s, h) = F(s, 0)2^{-2h}\mathbb{E} \left[ |X(s)|^{2h} \right]$$

with the limit $F(s, h)$ satisfying $0 < F(s, h) < \infty$. The function $F(s, 0)$ is given by

$$F(s, 0) = \frac{G(s + 1)^2}{G(2s + 1)},$$

where $G$ is the Barnes G-function.
The first key ingredient is a representation of $F_N(s, h)$ in terms of $F_N(s, 0)$ and the moments $\mathbb{E} \left[ \left| \sum_{i=1}^{N} \frac{x_{i}^{(N)}}{N} \right|^{2h} \right]$, where $(x_{1}^{(N)}, \ldots, x_{N}^{(N)})$ have the same distribution as the non-increasing eigenvalues of a random Hermitian matrix with law $M^{(s)}_{N}$.
The first key ingredient is a representation of $F_N(s, h)$ in terms of $F_N(s, 0)$ and the moments $\mathbb{E} \left[ \left| \sum_{i=1}^{N} x_i^{(N)}/N \right|^{2h} \right]$, where $(x_1^{(N)}, \ldots, x_N^{(N)})$ have the same distribution as the non-increasing eigenvalues of a random Hermitian matrix with law $\mathcal{M}_N^{(s)}$.

To prove convergence of the moments:

$$\mathbb{E} \left[ \left| \sum_{i=1}^{N} x_i^{(N)}/N \right|^{2h} \right] \rightarrow \mathbb{E} \left[ |X(s)|^{2h} \right], \quad \text{as } N \rightarrow \infty,$$

one needs to prove uniform integrability or, as we do, show uniform boundedness for some higher moment.
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The averages that we want to control uniformly in $N$ do not converge if we bring the absolute values inside, and it is essential that a cancellation due to symmetry around the origin of the points is taken into account.
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Due to the remarkable property of consistency of the Hua-Pickrell measures \( M_N^{(s)} \), for all \( N \geq 1 \) the diagonal elements of the random matrices in question turn out to be exchangeable, identically distributed random variables with the Pearson IV distribution.
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In particular, they do not grow with $N$ as the eigenvalues $(x_1^{(N)}, \ldots, x_N^{(N)})$ do.
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Extending this to the range $-\frac{1}{2} < s \leq 0$ takes more work.
Questions

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- Probabilistic interpretation and connections?