# Large deviations and fluctuation exponents for some polymer models

Timo Seppäläinen

Department of Mathematics
University of Wisconsin-Madison

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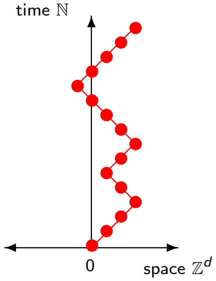
## Directed polymer in a random environment




simple random walk path $(x(t), t)$, $t \in \mathbb{Z}_+$

space-time environment $\{\omega(x,t) : x \in \mathbb{Z}^d,\ t \in \mathbb{N}\}$

inverse temperature $\beta > 0$

quenched probability measure on paths

$$Q_n\{x(\cdot)\} = \frac{1}{Z_n} \exp\Big\{\beta \sum_{t=1}^{n} \omega(x(t),t)\Big\}$$

partition function $\quad Z_n = \sum_{x(\cdot)} \exp\Big\{\beta \sum_{t=1}^{n} \omega(x(t),t)\Big\}$

(summed over all $n$-paths)

$\mathbb{P}$ probability distribution on $\omega$, often $\{\omega(x,t)\}$ i.i.d.





**Key quantities again:**

- Quenched measure $Q_n\{x(\cdot)\} = Z_n^{-1} \exp\Big\{\beta \sum_{t=1}^{n} \omega(x(t),t)\Big\}$
- Partition function $Z_n = \sum_{x(\cdot)} \exp\Big\{\beta \sum_{t=1}^{n} \omega(x(t),t)\Big\}$

**Questions:**

- Behavior of walk $x(\cdot)$ under $Q_n$ on large scales: fluctuation exponents, central limit theorems, large deviations
- Behavior of $\log Z_n$ (now also random as a function of $\omega$)
- Dependence on $\beta$ and $d$

## Large deviations

**Question:** describe quenched limit $\lim_{n\to\infty} n^{-1}\log Z_n$ ($\mathbb{P}$-a.s.)

Large deviation perspective.

Generalize: $E_0$ = expectation under background RW $X_n$ on $\mathbb{Z}^\nu$.

$$n^{-1}\log Z_n = n^{-1}\log E_0\big[e^{\beta\sum_{k=0}^{n-1}\omega_{X_k}}\big]$$
$$= n^{-1}\log E_0\big[e^{\sum_{k=0}^{n-1} g(\omega_{X_k})}\big]$$
$$= n^{-1}\log E_0\big[e^{\sum_{k=0}^{n-1} g(T_{X_k}\omega,\, Z_{k+1,k+\ell})}\big]$$

Introduced shift $(T_x\omega)_y = \omega_{x+y}$, steps $Z_k = X_k - X_{k-1} \in \mathcal{R}$, $Z_{1,\ell} = (Z_1, Z_2, \dots, Z_\ell)$.

$g(\omega, z_{1,\ell})$ is a function on $\boldsymbol{\Omega}_\ell = \Omega\times\mathcal{R}^\ell$.





Define empirical measure $R_n = n^{-1}\sum_{k=0}^{n-1}\delta_{T_{X_k}\omega,\, Z_{k+1,k+\ell}}$.

It is a probability measure on $\boldsymbol{\Omega}_\ell$.

Then $n^{-1}\log Z_n = n^{-1}\log E_0\big[e^{nR_n(g)}\big]$

**Task:** understand large deviations of $P_0\{R_n\in\cdot\}$ under $\mathbb{P}$-a.e. fixed $\omega$ (quenched).

**Process:** Markov chain $(T_{X_n}\omega, Z_{n+1,n+\ell})$ on $\boldsymbol{\Omega}_\ell$ under a fixed $\omega$.

**Evolution:** pick random step $z$ from $\mathcal{R}$, then execute move $S_z : (\omega, z_{1,\ell}) \mapsto (T_{z_1}\omega, z_{2,\ell}z)$.

Defines kernel $p$ on $\boldsymbol{\Omega}_\ell$: $p(\eta, S_z\eta) = |\mathcal{R}|^{-1}$.





## Entropy

For $\mu\in\mathcal{M}_1(\boldsymbol{\Omega}_\ell)$, $q$ Markov kernel on $\boldsymbol{\Omega}_\ell$, usual relative entropy on $\boldsymbol{\Omega}_\ell^2$:

$$H(\mu\times q\,|\,\mu\times p) = \int_{\boldsymbol{\Omega}_\ell}\sum_{z\in\mathcal{R}} q(\eta, S_z\eta)\log\frac{q(\eta,S_z\eta)}{p(\eta,S_z\eta)}\,\mu(d\eta).$$

The effect of $\mathbb{P}$ in the background?

Let $\mu_0 = \Omega$-marginal of $\mu\in\mathcal{M}_1(\boldsymbol{\Omega}_\ell)$. Define

$$H_{\mathbb{P}}(\mu) = \begin{cases}\inf\{H(\mu\times q\,|\,\mu\times p) : \mu q = \mu\} & \text{if } \mu_0 \ll \mathbb{P}\\ \infty & \text{otherwise.}\end{cases}$$

Infimum taken over Markov kernels $q$ that fix $\mu$.

$H_{\mathbb{P}}$ is convex but not lower semicontinuous.





**Assumptions.**

- Environment $\{\omega_x\}$ IID under $\mathbb{P}$.
- $g$ local function on $\boldsymbol{\Omega}_\ell$, $\mathbb{E}|g|^p < \infty$ for some $p > \nu$.

**Theorem.** (Rassoul-Agha, S, Yilmaz) Deterministic limit

$$\Lambda(g) = \lim_{n\to\infty} n^{-1}\log E_0\big[e^{nR_n(g)}\big] \qquad \text{exists } \mathbb{P}\text{-a.s.}$$

and $$\Lambda(g) = H_{\mathbb{P}}^{\#}(g) \equiv \sup_{\mu}\sup_{c>0}\{E^\mu[g\wedge c] - H_{\mathbb{P}}(\mu)\}.$$

**Remarks.**

- With higher moments of $g$ admit mixing $\mathbb{P}$.
- $\Lambda(g) > -\infty$.
- IID directed + above moment $\Rightarrow$ $\Lambda(g)$ finite.

Quenched weak LDP (large deviation principle) under $Q_n$.

$$Q_n(A) = \frac{1}{E_0\left[e^{nR_n(g)}\right]} E_0\left[e^{nR_n(g)} \mathbf{1}_A(\omega, Z_{1,\infty})\right]$$

Rate function $\quad I(\mu) = \inf_{c>0} \{ H_{\mathbb{P}}(\mu) - E^{\mu}(g \wedge c) + \Lambda(g) \}.$

**Theorem.** (RSY) Assumptions as above and $\Lambda(g)$ finite. Then $\mathbb{P}$-a.s. for compact $F \subseteq \mathcal{M}_1(\mathbf{\Omega}_\ell)$ and open $G \subseteq \mathcal{M}_1(\mathbf{\Omega}_\ell)$:

$$\overline{\lim_{n\to\infty}}\, n^{-1} \log Q_n\{R_n \in F\} \le -\inf_{\mu\in F} I(\mu)$$

$$\underline{\lim_{n\to\infty}}\, n^{-1} \log Q_n\{R_n \in G\} \ge -\inf_{\mu\in G} I(\mu)$$

IID environment, directed walk: full LDP holds.





**Return to $d+1$ dim directed polymer in i.i.d. environment.**

**Question:** Is the path $x(\cdot)$ diffusive or not, that is, does it scale like standard RW?

**Early results:** diffusive behavior for $d \ge 3$ and small $\beta > 0$:

1988 Imbrie and Spencer: $n^{-1}E^Q(|x(n)|^2) \to c \quad \mathbb{P}$-a.s.

1989 Bolthausen: quenched CLT for $n^{-1/2}x(n)$.

**In the opposite direction:** if $d = 1, 2$, or $d \ge 3$ and $\beta$ large enough, then $\exists\, c > 0$ s.t.

$$\overline{\lim_{n\to\infty}}\, \max_z Q_n\{x(n) = z\} \ge c \quad \mathbb{P}\text{-a.s.}$$

(Carmona and Hu 2002, Comets, Shiga, and Yoshida 2003)





## Definition of fluctuation exponents $\zeta$ and $\chi$

- Fluctuations of the path $\{x(t) : 0 \le t \le n\}$ are of order $n^{\zeta}$.
- Fluctuations of $\log Z_n$ are of order $n^{\chi}$.
- Conjecture for $d = 1$: $\quad \zeta = 2/3 \quad$ and $\quad \chi = 1/3$.

**Results:** these exact exponents for three particular 1+1 dimensional models.





## Earlier results for $d = 1$ exponents

Past rigorous bounds give $3/5 \le \zeta \le 3/4$ and $\chi \ge 1/8$:

- Brownian motion in Poissonian potential: Wüthrich 1998, Comets and Yoshida 2005.
- Gaussian RW in Gaussian potential: Petermann 2000 $\zeta \ge 3/5$, Mejane 2004 $\zeta \le 3/4$
- Licea, Newman, Piza 1995-96: corresponding results for first passage percolation

## Rigorous $\zeta = 2/3$ and $\chi = 1/3$ results

exist for three "exactly solvable" models:

(1) Log-gamma polymer: $\beta = 1$ and $e^{-\omega(x,t)} \sim$ Gamma, plus appropriate boundary conditions.

(2) Polymer in a Brownian environment (joint with B. Valkó). Model introduced by O'Connell and Yor 2001.

(3) Continuum directed polymer, or Hopf-Cole solution of the Kardar-Parisi-Zhang (KPZ) equation:

  (i) Initial height function given by two-sided Brownian motion (joint with M. Balázs and J. Quastel).

  (ii) Narrow wedge initial condition (Amir, Corwin, Quastel).

Next details on (3.i), then details on (1).





## Hopf-Cole solution to KPZ equation

KPZ eqn for height function $h(t,x)$ of a 1+1 dim interface:

$$h_t = \tfrac{1}{2} h_{xx} - \tfrac{1}{2} (h_x)^2 + \dot{W}$$

where $\dot{W}$ = Gaussian space-time white noise.

Initial height $h(0,x)$ = two-sided Brownian motion for $x \in \mathbb{R}$.

$Z = \exp(-h)$ satisfies $Z_t = \tfrac{1}{2} Z_{xx} - Z\,\dot{W}$ that can be solved.

**Define** $h = -\log Z$, the **Hopf-Cole solution** of KPZ.

Bertini-Giacomin (1997): $h$ can be obtained as a weak limit via a smoothing and renormalization of KPZ.

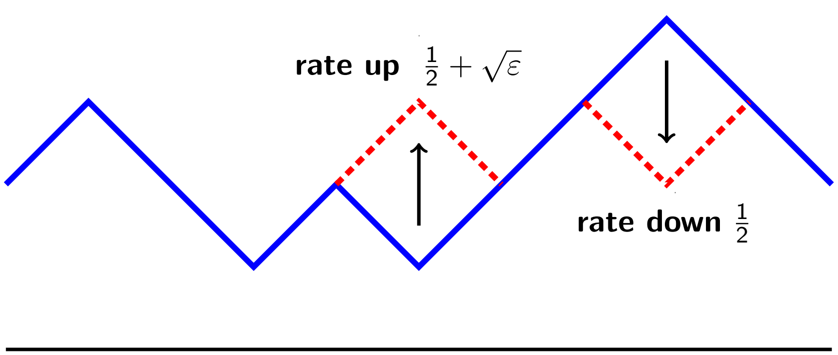




## WASEP connection

$\zeta_\varepsilon(t,x)$ height process of weakly asymmetric simple exclusion s.t.

$$\zeta_\varepsilon(x+1) - \zeta_\varepsilon(x) = \pm 1$$








## WASEP connection

Jumps:

$$\zeta_\varepsilon(x) \longrightarrow \begin{cases} \zeta_\varepsilon(x) + 2 & \text{with rate } \frac{1}{2} + \sqrt{\varepsilon} \quad \text{if } \zeta_\varepsilon(x) \text{ is a local min} \\ \zeta_\varepsilon(x) - 2 & \text{with rate } \frac{1}{2} \quad \text{if } \zeta_\varepsilon(x) \text{ is a local max} \end{cases}$$

Initially: $\zeta_\varepsilon(0,x+1) - \zeta_\varepsilon(0,x) = \pm 1$ with probab $\frac{1}{2}$.

$$h_\varepsilon(t,x) = \varepsilon^{1/2} \left( \zeta_\varepsilon(\varepsilon^{-2} t, [\varepsilon^{-1} x]) - v_\varepsilon t \right)$$

**Theorem** (Bertini-Giacomin 1997) As $\varepsilon \searrow 0$, $h_\varepsilon \Rightarrow h$

# Fluctuation bounds

From coupling arguments for WASEP

$$C_1 t^{2/3} \le \mathrm{Var}(h_\varepsilon(t,0)) \le C_2 t^{2/3}$$

**Theorem** (Balázs-Quastel-S) For the Hopf-Cole solution of KPZ,

$$C_1 t^{2/3} \le \mathrm{Var}(h(t,0)) \le C_2 t^{2/3}$$

Lower bound comes from control of rescaled correlations

$$S_\varepsilon(t,x) = 4\varepsilon^{-1}\,\mathrm{Cov}\big[\eta(\varepsilon^{-2}t,\,\varepsilon^{-1}x)\,,\,\eta(0,0)\big]$$

where $\eta(t,x) \in \{0,1\}$ is the occupation variable of WASEP





Rescaled correlations again:

$$S_\varepsilon(t,x) = 4\varepsilon^{-1}\,\mathrm{Cov}\big[\eta(\varepsilon^{-2}t,\,\varepsilon^{-1}x)\,,\,\eta(0,0)\big]$$

$$E\big[\langle \varphi', h_\varepsilon(t)\rangle\langle \psi', h_\varepsilon(0)\rangle\big]$$

$$= \tfrac{1}{2}\int\Big[\int \varphi\Big(\frac{y+x}{2}\Big)\psi\Big(\frac{y-x}{2}\Big)\,dy\Big]\,S_\varepsilon(t,x)\,dx$$

Let $\varepsilon \searrow 0$. On the left increments of $h_\varepsilon$ so total control !

On the right $S_\varepsilon(t,x)dx \Rightarrow S(t,dx)$ with control of moments:

$$\int |x|^m\,S_\varepsilon(t,x)\,dx \sim O(t^{2m/3}), \qquad 1 \le m < 3.$$

(Second class particle estimate.)





After $\varepsilon \searrow 0$ limit

$$E\big[\langle \varphi', h(t)\rangle\langle \psi', h(0)\rangle\big] = \tfrac{1}{2}\iint \varphi\Big(\frac{y+x}{2}\Big)\psi\Big(\frac{y-x}{2}\Big)\,dy\,S(t,dx)$$

From mean zero, stationary $h$ increments

$$\tfrac{1}{2}\partial_{xx}\mathrm{Var}(h(t,x)) = S(t,dx) \quad \text{as distributions.}$$

With some control over tails we arrive at the result:

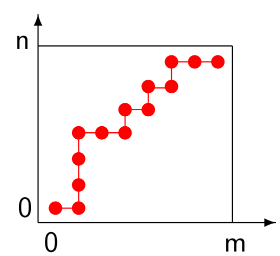
$$\mathrm{Var}(h(t,0)) = \int |x|\,S(t,dx) \sim O(t^{2/3}).$$





# 1+1 dimensional lattice polymer with log-gamma weights

**Fix both endpoints.**




$\Pi_{m,n}$ = set of admissible paths

independent weights $Y_{i,j} = e^{\omega(i,j)}$

environment $(Y_{i,j} : (i,j) \in \mathbb{Z}_+^2)$

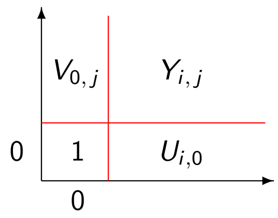
$$Z_{m,n} = \sum_{x_\cdot}\prod_{k=1}^{m+n} Y_{x_k}$$

quenched measure $Q_{m,n}(x_\cdot) = Z_{m,n}^{-1}\prod_{k=1}^{m+n} Y_{x_k}$

averaged measure $P_{m,n}(x_\cdot) = \mathbb{E}Q_{m,n}(x_\cdot)$

## Weight distributions

- Parameters $0 < \theta < \mu$.
- **Bulk weights** $Y_{i,j}$ for $i, j \in \mathbb{N}$
- **Boundary weights** $U_{i,0} = Y_{i,0}$ and $V_{0,j} = Y_{0,j}$.




$$U_{i,0}^{-1} \sim \text{Gamma}(\theta)$$
$$V_{0,j}^{-1} \sim \text{Gamma}(\mu - \theta)$$
$$Y_{i,j}^{-1} \sim \text{Gamma}(\mu)$$

- Gamma($\theta$) density: $\Gamma(\theta)^{-1} x^{\theta-1} e^{-x}$ on $\mathbb{R}_+$
- $\Psi_n(s) = (d^{n+1}/ds^{n+1}) \log \Gamma(s)$
- $\mathbb{E}(\log U) = -\Psi_0(\theta)$ and $\mathbb{V}\text{ar}(\log U) = \Psi_1(\theta)$





## Variance bounds for log $Z$

With $0 < \theta < \mu$ fixed and $N \nearrow \infty$ assume

$$|\, m - N\Psi_1(\mu - \theta)\,| \le CN^{2/3} \quad \text{and} \quad |\, n - N\Psi_1(\theta)\,| \le CN^{2/3} \qquad (1)$$

**Theorem**

For $(m, n)$ as in (1), $\quad C_1 N^{2/3} \le \mathbb{V}\text{ar}(\log Z_{m,n}) \le C_2 N^{2/3}$ .

**Theorem**

Suppose $n = \Psi_1(\theta)N$ and $m = \Psi_1(\mu - \theta)N + \gamma N^{\alpha}$ with $\gamma > 0$, $\alpha > 2/3$. Then

$$N^{-\alpha/2}\Big\{\log Z_{m,n} - \mathbb{E}\big(\log Z_{m,n}\big)\Big\} \Rightarrow \mathcal{N}\big(0, \gamma\Psi_1(\theta)\big)$$





## Fluctuation bounds for path

$v_0(j)$ = leftmost, $v_1(j)$ = rightmost point of $x_\centerdot$ on horizontal line:

$$v_0(j) = \min\{i \in \{0, \dots, m\} : \exists k : x_k = (i, j)\}$$

$$v_1(j) = \max\{i \in \{0, \dots, m\} : \exists k : x_k = (i, j)\}$$

**Theorem**

Assume $(m, n)$ as previously and $0 < \tau < 1$. Then

$$\text{(a)}\ P\Big\{v_0(\lfloor \tau n \rfloor) < \tau m - bN^{2/3} \ \text{ or } \ v_1(\lfloor \tau n \rfloor) > \tau m + bN^{2/3}\Big\} \le \frac{C}{b^3}$$

(b) $\forall \varepsilon > 0\ \exists \delta > 0$ such that

$$\varlimsup_{N \to \infty} P\{\ \exists k \text{ such that } |x_k - (\tau m, \tau n)| \le \delta N^{2/3}\ \} \le \varepsilon.$$





## Results for log-gamma polymer summarized

With reciprocals of gammas for weights, both endpoints of the polymer fixed and the right boundary conditions on the axes, we have identified the one-dimensional exponents

$$\zeta = 2/3 \qquad \text{and} \qquad \chi = 1/3.$$

Next step is to

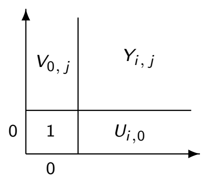
- eliminate the boundary conditions and
- consider polymers with fixed length and free endpoint

In both scenarios we have the upper bounds for both log $Z$ and the path.

But currently do not have the lower bounds.

Next some key points of the proof.

# Burke property for log-gamma polymer with boundary




Given initial weights ($i, j \in \mathbb{N}$):

$U_{i,0}^{-1} \sim \text{Gamma}(\theta)$, $V_{0,j}^{-1} \sim \text{Gamma}(\mu - \theta)$

$Y_{i,j}^{-1} \sim \text{Gamma}(\mu)$

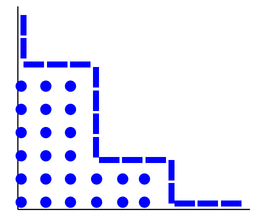
Compute $Z_{m,n}$ for all $(m,n) \in \mathbb{Z}_+^2$ and then define

$$U_{m,n} = \frac{Z_{m,n}}{Z_{m-1,n}} \qquad V_{m,n} = \frac{Z_{m,n}}{Z_{m,n-1}} \qquad X_{m,n} = \left( \frac{Z_{m,n}}{Z_{m+1,n}} + \frac{Z_{m,n}}{Z_{m,n+1}} \right)^{-1}$$

For an undirected edge $f$: $T_f = \begin{cases} U_x & f = \{x - e_1, x\} \\ V_x & f = \{x - e_2, x\} \end{cases}$







down-right path $(z_k)$ with edges $f_k = \{z_{k-1}, z_k\}$, $k \in \mathbb{Z}$

interior points $\mathcal{I}$ of path $(z_k)$

**Theorem**

Variables $\{T_{f_k}, X_z : k \in \mathbb{Z},\ z \in \mathcal{I}\}$ are independent with marginals

$U^{-1} \sim \text{Gamma}(\theta)$, $V^{-1} \sim \text{Gamma}(\mu - \theta)$, and $X^{-1} \sim \text{Gamma}(\mu)$.

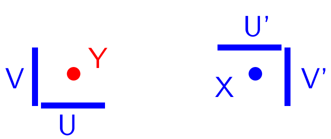
"Burke property" because the analogous property for last-passage is a generalization of Burke's Theorem for M/M/1 queues, via the last-passage representation of M/M/1 queues in series.





# Proof of Burke property

Induction on $\mathcal{I}$ by flipping a growth corner:




$U' = Y(1 + U/V)$ $\quad V' = Y(1 + V/U)$

$X = (U^{-1} + V^{-1})^{-1}$

**Lemma.** Given that $(U, V, Y)$ are independent positive r.v.'s, $(U', V', X) \overset{d}{=} (U, V, Y)$ iff $(U, V, Y)$ have the gamma distr's.

**Proof.** "if" part by computation, "only if" part from a characterization of gamma due to Lukacs (1955). ☐

This gives all $(z_k)$ with finite $\mathcal{I}$. General case follows.





# Proof of off-characteristic CLT

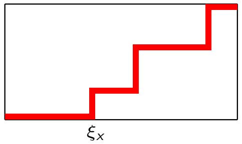
Recall that $\begin{cases} n = \Psi_1(\theta) N \\ m = \Psi_1(\mu - \theta) N + \gamma N^{\alpha} \end{cases}$ $\qquad \gamma > 0,\ \alpha > 2/3.$

Set $m_1 = \lfloor \Psi_1(\mu - \theta) N \rfloor$. Since $Z_{m,n} = Z_{m_1,n} \cdot \prod_{i=m_1+1}^{m} U_{i,n}$

$$N^{-\alpha/2}\, \overline{\log Z_{m,n}} = N^{-\alpha/2}\, \overline{\log Z_{m_1,n}} + N^{-\alpha/2} \sum_{i=m_1+1}^{m} \overline{\log U_{i,n}}$$

First term on the right is $O(N^{1/3 - \alpha/2}) \to 0$. Second term is a sum of order $N^{\alpha}$ i.i.d. terms. ☐

## Variance identity




Exit point of path from $x$-axis

$\xi_x = \max\{k \geq 0 : x_i = (i,0) \text{ for } 0 \leq i \leq k\}$

For $\theta, x > 0$ define positive function

$$L(\theta, x) = \int_0^x \big(\Psi_0(\theta) - \log y\big) x^{-\theta} y^{\theta-1} e^{x-y}\, dy$$

**Theorem.** For the model with boundary,

$$\mathbb{V}\mathrm{ar}\big[\log Z_{m,n}\big] = n\Psi_1(\mu-\theta) - m\Psi_1(\theta) + 2\, E_{m,n}\Big[\sum_{i=1}^{\xi_x} L(\theta, Y_{i,0}^{-1})\Big]$$





## Variance identity, sketch of proof

$N = \log Z_{m,n} - \log Z_{0,n}$

$W = \log Z_{0,n}$ $\qquad$ $E = \log Z_{m,n} - \log Z_{m,0}$

$S = \log Z_{m,0}$

$$\begin{aligned}
\mathbb{V}\mathrm{ar}\big[\log Z_{m,n}\big] &= \mathbb{V}\mathrm{ar}(W+N)\\
&= \mathbb{V}\mathrm{ar}(W) + \mathbb{V}\mathrm{ar}(N) + 2\,\mathbb{C}\mathrm{ov}(W,N)\\
&= \mathbb{V}\mathrm{ar}(W) + \mathbb{V}\mathrm{ar}(N) + 2\,\mathbb{C}\mathrm{ov}(S+E-N,N)\\
&= \mathbb{V}\mathrm{ar}(W) - \mathbb{V}\mathrm{ar}(N) + 2\,\mathbb{C}\mathrm{ov}(S,N) \qquad (E, N \text{ ind.})\\
&= n\Psi_1(\mu-\theta) - m\Psi_1(\theta) + 2\,\mathbb{C}\mathrm{ov}(S,N).
\end{aligned}$$





To differentiate w.r.t. parameter $\theta$ of $S$ while keeping $W$ fixed, introduce a separate parameter $\rho \ (= \mu - \theta)$ for $W$.

$$-\mathbb{C}\mathrm{ov}(S,N) \;=\; \frac{\partial}{\partial\theta}\mathbb{E}(N) \;=\; \widetilde{\mathbb{E}}\Big[\frac{\partial}{\partial\theta}\log Z_{m,n}(\theta)\Big]$$

when $\quad Z_{m,n}(\theta) = \displaystyle\sum_{x_\bullet \in \Pi_{m,n}} \prod_{i=1}^{\xi_x} H_\theta(\eta_i)^{-1} \cdot \prod_{k=\xi_x+1}^{m+n} Y_{x_k}$ $\quad$ with

$$\eta_i \sim \text{IID Unif}(0,1), \quad H_\theta(\eta) = F_\theta^{-1}(\eta), \quad F_\theta(x) = \int_0^x \frac{y^{\theta-1}e^{-y}}{\Gamma(\theta)}\, dy.$$

Differentiate: $\quad \displaystyle\frac{\partial}{\partial\theta}\log Z_{m,n}(\theta) \;=\; -\,E^{Q_{m,n}}\Big[\sum_{i=1}^{\xi_x} L(\theta, Y_{i,0}^{-1})\Big].$





Together:

$$\begin{aligned}
\mathbb{V}\mathrm{ar}\big[\log Z_{m,n}\big] &= n\Psi_1(\mu-\theta) - m\Psi_1(\theta) + 2\,\mathbb{C}\mathrm{ov}(S,N)\\
&= n\Psi_1(\mu-\theta) - m\Psi_1(\theta) + 2\, E_{m,n}\Big[\sum_{i=1}^{\xi_x} L(\theta, Y_{i,0}^{-1})\Big].
\end{aligned}$$

This was the claimed formula. $\square$

## Sketch of upper bound proof

The argument develops an inequality that controls both $\log Z$ and $\xi_x$ simultaneously. Introduce an auxiliary parameter $\lambda = \theta - bu/N$. The weight of a path $x_\cdot$ such that $\xi_x > 0$ satisfies

$$W(\theta) = \prod_{i=1}^{\xi_x} H_\theta(\eta_i)^{-1} \cdot \prod_{k=\xi_x+1}^{m+n} Y_{x_k} \;=\; W(\lambda) \cdot \prod_{i=1}^{\xi_x} \frac{H_\lambda(\eta_i)}{H_\theta(\eta_i)}.$$

Since $H_\lambda(\eta) \le H_\theta(\eta)$,

$$Q^{\theta,\omega}\{\xi_x \ge u\} = \frac{1}{Z(\theta)} \sum_{x_\cdot} \mathbf{1}\{\xi_x \ge u\} W(\theta) \;\le\; \frac{Z(\lambda)}{Z(\theta)} \cdot \prod_{i=1}^{\lfloor u \rfloor} \frac{H_\lambda(\eta_i)}{H_\theta(\eta_i)}.$$





For $1 \le u \le \delta N$ and $0 < s < \delta$,

$$\mathbb{P}\big[Q^\omega\{\xi_x \ge u\} \ge e^{-su^2/N}\big] \le \mathbb{P}\Big\{ \prod_{i=1}^{\lfloor u \rfloor} \frac{H_\lambda(\eta_i)}{H_\theta(\eta_i)} \ge \alpha \Big\} + \mathbb{P}\Big( \frac{Z(\lambda)}{Z(\theta)} \ge \alpha^{-1} e^{-su^2/N} \Big).$$

Choose $\alpha$ right. Bound these probabilities with Chebychev which brings $\mathbb{V}\mathrm{ar}(\log Z)$ into play. In the characteristic rectangle $\mathbb{V}\mathrm{ar}(\log Z)$ can be bounded by $E(\xi_x)$. The end result is this inequality:

$$\mathbb{P}\big[Q^\omega\{\xi_x \ge u\} \ge e^{-su^2/N}\big] \le \frac{CN^2}{u^4} E(\xi_x) + \frac{CN^2}{u^3}$$

Handle $u \ge \delta N$ with large deviation estimates. In the end, integration gives the moment bounds. **END.**





## Polymer in a Brownian environment

**Environment:** independent Brownian motions $B_1, B_2, \dots, B_n$

**Partition function (without boundary conditions):**

$$Z_{n,t}(\beta) = \int_{0<s_1<\cdots<s_{n-1}<t} \exp\big[\beta\big( B_1(s_1) \;+\; B_2(s_2) - B_2(s_1) \;+\; B_3(s_3) - B_3(s_2) \;+\; \cdots +\; B_n(t) - B_n(s_{n-1})\big)\big]\, ds_{1,n-1}$$