

On the chaotic character of some parabolic SPDEs

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A simple model for intermittency

(Zeldovich–Ruzmaikin–Sokoloff, 1990)

- ▶ Intermittency occurs when we multiply many roughly-independent r.v.'s ; e.g., ξ_1, ξ_2, \dots i.i.d. with $P\{\xi_1 = 2\} = P\{\xi_1 = 0\} = 1/2$
- ▶ Then

$$u_n := \prod_{j=1}^n \xi_j = \begin{cases} 2^n & \text{with probab. } 2^{-n}, \\ 0 & \text{with probab. } 1 - 2^{-n}. \end{cases}$$

- ▶ Conclusions:
 - ▶ $u_n = 0$ for all n large [a.s.]; in particular, $u_n \rightarrow 0$ a.s.
 - ▶ $n^{-1} \log E(u_n^k) \rightarrow \gamma_k := (k - 1) \log 2$ for all $k > 1$
- ▶ Now replicate this experiment
- ▶ Is this degeneracy because of the many zeros? No

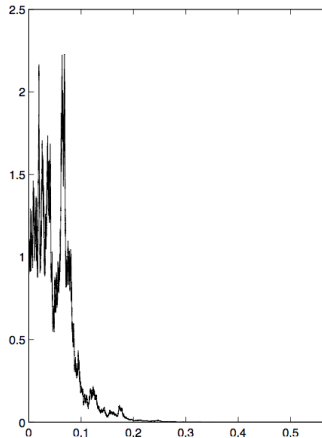
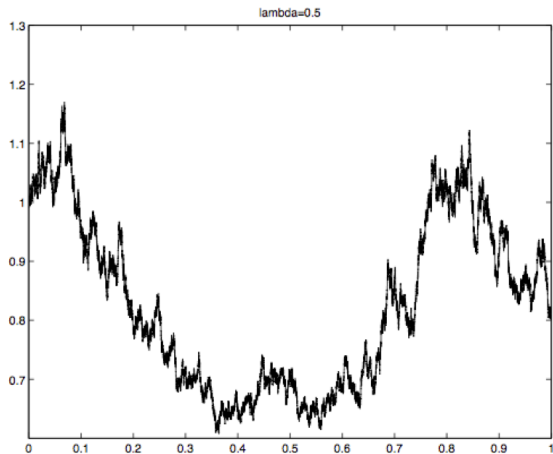
A second simple model for intermittency

(Zeldovich–Ruzmaikin–Sokoloff, 1990)

- ▶ Let b denote 1-D Brownian motion and consider the exponential martingale $u_t := e^{\lambda b_t - (\lambda^2 t/2)}$
- ▶ $u_t \rightarrow 0$ as $t \rightarrow \infty$ [strong law]
- ▶ $t^{-1} \log E(u_t^k) = \frac{1}{2} \lambda^2 k(k-1) \rightarrow \gamma_k := \frac{1}{2} \lambda^2 k(k-1)$ for $k > 1$
- ▶ In the first example, $\gamma_k \approx k \log 2$; in the second, $\gamma_k \approx \frac{1}{2} \lambda^2 k^2$
- ▶ The examples are “similar,”

$$e^{b_t - (t/2)} \approx \prod_j \left(1 - (\Delta b)_j - \frac{1}{2} (\Delta t)_j \right)$$

A simulation $[\dot{u}_t(x) = (\kappa/2)u_t''(x) + \lambda u_t(x)\eta(t), u_0 \equiv 1]$
 $u_t = \exp\{\lambda b_t - (\lambda t/2)\}$ with $\lambda = 0.5$ (left) and $\lambda = 5$ (right)



The model

$$\frac{\partial}{\partial t} u_t(x) = \frac{\kappa}{2} \frac{\partial^2}{\partial x^2} u_t(x) + \sigma(u_t(x)) \eta_t(x),$$

where:

1. $\kappa > 0$;
2. $\sigma : \mathbf{R} \rightarrow \mathbf{R}$ is Lipschitz continuous;
3. η is space-time white noise; i.e., a centered GGWF with

$$\text{Cov}(\eta_t(x), \eta_s(y)) = \delta_0(t - s) \delta_0(x - y)$$

4. $u_0 : \mathbf{R} \rightarrow \mathbf{R}_+$ nonrandom, bounded, and measurable;
5. u exists, is unique and continuous (Walsh, 1986)

The model

$$\partial_t u = (\kappa/2)\partial_{xx}u + \sigma(u)\eta$$

- ▶ Many physically-interesting choices of $\sigma \not\equiv 0$:
 - ▶ σ periodic/quasi-periodic/stationary process [random media];
 - ▶ $\sigma(u) \propto u$ [the parabolic Anderson model/KPZ/Br. Br. motion in random environment];
 - ▶ $\sigma(u) \propto \sqrt{u}$ [super processes];
 - ▶ $\sigma(u) \propto \sqrt{u(1-u)}$ [stoch. KPP];
- ▶ Today, we will say a few things about the first two examples [where σ is Lipschitz]

Weak intermittency

$$\partial_t u = (\kappa/2)\partial_{xx}u + \sigma(u)\eta$$

- ▶ (weak) intermittency [Bertini–Cancrini, 1994; Carmona–Molchanov, 1994; Molchanov, 1991; Foondun–K., 2010; Zel’dovitch et al, 1985, 1988, 1990; ...]:

$$0 < \limsup_{t \rightarrow \infty} \frac{1}{t} \log E \left(|u_t(x)|^k \right) < \infty \quad (k \geq 2, x \in \mathbf{R})$$

- ▶ Weak intermittency implies “localization” on large time scales.
- ▶ Physical intermittency is expected to hold because the SPDE is typically “chaotic,” and for many choices of σ :
 - ▶ For all $t > 0$; and
 - ▶ both in time, *and* space
- ▶ Today: What happens before the onset of localization?

Optimal regularity

- ▶ Can frequently understand parabolic equations via optimal regularity [Lunardi, 1995, and older works by Pazy, Kato, ...]
- ▶ If $\sigma(0) = 0$, then the fact that $u_0(x) \geq 0$ implies that $u_t(x) \geq 0$ [Mueller's comparison principle]
- ▶ If $\sigma(0) = 0$ and $u_0 \in L^2(\mathbf{R})$ then $u_t \in L^2(\mathbf{R})$ a.s. (Dalang–Mueller, 2003, but likely known earlier also)
- ▶ If $u_0 \in C^\alpha(\mathbf{R})$ for some $\alpha > \frac{1}{2}$ and has compact support, and if $\sigma(0) = 0$, then $\sup_{x \in \mathbf{R}} u_t(x) < \infty$ a.s. for all $t > 0$ (Foondun–Kh, 2010+)
- ▶ Today's goal: The solution can be sensitive to the choice of u_0 (we study cases where u_t is unbounded for all $t > 0$)

A reduction

- ▶ $\dot{u} = (\kappa/2)u'' + \sigma(u)\eta$
- ▶ Suppose $\sigma(x_0) = 0$ for some $x_0 > 0$
- ▶ If $u_0(x) \leq x_0$ then $u_t(x) \leq x_0$ [Mueller's comparison theorem]
- ▶ Therefore, today we are interested only in the case that $\sigma(x) > 0$ for all $x > 0$
- ▶ We consider the case that $\inf_{x \in \mathbf{R}} u_0(x) > 0$ only

Theorem (Conus–Joseph–K)

A case of minimum noise

- ▶ $\dot{u} = (\kappa/2)u'' + \sigma(u)\eta$
- ▶ If $\inf_{x \in \mathbf{R}} \sigma(x) > 0$, then

$$\limsup_{|x| \rightarrow \infty} \frac{u_t(x)}{(\log |x|)^{1/6}} \geq \text{const} \cdot \kappa^{-1/12} \quad \text{a.s. for all } t > 0$$

- ▶ \exists weaker versions that allow mild decay for σ ; e.g., suppose $\sigma(x) > 0$ for all $x \geq 0$ and $\exists \gamma \in (0, 1/6)$ such that $\sigma(x) \gg (\log |x|)^{-(1/6)+\gamma}$. Then a.s. for all $t > 0$,

$$\limsup_{|x| \rightarrow \infty} \frac{u_t(x)}{(\log |x|)^\gamma} \geq \text{const} \cdot \kappa^{-1/12}.$$

Theorem (Conus–Joseph–K)

The moderately noisy case

- ▶ $\dot{u} = (\kappa/2)u'' + \sigma(u)\eta$
- ▶ If $0 < \inf_{x \geq 0} \sigma(x) \leq \sup_{x \geq 0} \sigma(x) < \infty$, then

$$\limsup_{|x| \rightarrow \infty} \frac{u_t(x)}{(\log |x|)^{1/2}} \asymp \kappa^{-1/4} \quad \text{a.s. for all } t > 0$$

- ▶ Power of κ suggests the universality class of random walks in weak interactions with their random environment

Theorem (Conus–Joseph–K)

The parabolic Anderson case

- ▶ $\dot{u} = (\kappa/2)u'' + cu\eta$ $[\sigma(x) = cx]$
- ▶ If $c > 0$, then

$$\limsup_{|x| \rightarrow \infty} \frac{\log u_t(x)}{(\log |x|)^{2/3}} \asymp \frac{1}{\kappa^{1/3}} \quad \text{a.s. for all } t > 0$$

- ▶ $u_t(x) \approx \exp\{\text{const} \cdot (\log |x| / \sqrt{\kappa})^{2/3}\}$
- ▶ Power of κ suggests the universality class of random matrix models
- ▶ “fluctuation exponent” $(1/3, 2/3)$

A connection to KPZ

- ▶ The KPZ equation (1986): If $\lambda \in \mathbf{R}$ is fixed then

$$\dot{h} = \frac{\kappa}{2} h'' + \frac{\kappa\lambda}{2} (h')^2 + \eta$$

- ▶ Rigorous meaning (?): A formal Hopf–Cole transformation $[u_t(x) = \exp\{h_t(x)\}]$ yields

$$\dot{u} = \frac{\kappa}{2} u'' + u\eta$$

- ▶ “Therefore,”

$$\limsup_{|x| \rightarrow \infty} \frac{h_t(x)}{(\log |x|)^{2/3}} \asymp \frac{1}{\kappa^{1/3}} \quad \text{a.s. for all } t > 0$$

- ▶ Related result by Balász–Quastel–Seppäläinen & Amir–Corwin–Quastel

Ideas used in proofs

- ▶ **Coupling.** If x_1, \dots, x_N are sufficiently far apart, then $u_t(x_1), \dots, u_t(x_N)$ are “approximately independent”
- ▶ Obtain good tail estimates for $P\{u_t(x) \geq \lambda\}$, when λ is large:
 - ▶ $\log P\{u_t(x) \geq \lambda\} \geq -\text{const} \cdot \kappa^{1/2} \lambda^6$ if σ bounded below
 - ▶ $\log P\{u_t(x) \geq \lambda\} \asymp -\kappa^{1/2} \lambda^2$ if σ bounded above and below
 - ▶ $\log P\{u_t(x) \geq \lambda\} \asymp -\kappa^{1/2} (\log \lambda)^{3/2}$ for parabolic Anderson

Colored noise

$$\dot{u}_t(x) = (\kappa/2)(\Delta u_t)(x) + \sigma(u_t(x))\eta_t(x), \quad t > 0, x \in \mathbf{R}^d$$

- ▶ Now

$$\text{Cov}(\eta_t(x), \eta_s(y)) = \delta_0(s - t)f(x - y)$$

(Dalang, 1999; Hu–Nualart, 2009, . . .)

- ▶ Suppose $f = h * \tilde{h}$ for some $h \in L^2(\mathbf{R}^d)$, so \exists a unique solution for all $d \geq 1$
- ▶ \exists KPZ version also (Medina–Hwa–Kardar–Zhang, 1989)

Theorem (Conus–Joseph–K–Shiu)

The parabolic Anderson case

- ▶ $\dot{u} = (\kappa/2)\Delta u + cu\eta$ $[\sigma(x) = cx]$
- ▶ If $c > 0$, then

$$\limsup_{|x| \rightarrow \infty} \frac{\log u_t(x)}{(\log |x|)^{1/2}} \asymp 1 \quad \text{a.s. for all } t > 0 \text{ and } \kappa \text{ small}$$

- ▶ There are other variations as well
- ▶ “fluctuation exponent” $(0, 1/2)$
- ▶ Are there in-between models? Yes.

Theorem (Conus–Joseph–K–Shiu)

The parabolic Anderson case

- ▶ $\dot{u} = (\kappa/2)\Delta u + cu\eta$ [$\sigma(x) = cx$]
- ▶ $\text{Cov}(\eta_t(x), \eta_s(y)) = \delta_0(t - s) \cdot \|x - y\|^{-\alpha}$
- ▶ The solution $\exists!$ when $\alpha < \min(d, 2)$ [Dalang, 1999]
- ▶ If $c > 0$, then

$$\limsup_{|x| \rightarrow \infty} \frac{\log u_t(x)}{(\log \|x\|)^{2/(4-\alpha)}} \asymp \kappa^{-\alpha/(4-\alpha)} \quad \text{a.s. for all } t > 0$$

- ▶ “fluctuation exponent” $(2\psi - 1, \psi) = (\alpha/(4-\alpha), 2/(4-\alpha))$
- ▶ $f = h * \tilde{h} \Leftrightarrow \alpha = 0$, and $f = \delta_0 \Leftrightarrow \alpha = 1 = \min(d, 2)$
[spectral analogies]