Lecture 1: January 6
Overview of Course

Math 690-40, Spring 2022

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Three Faces of Probability

1. Voter model and coalescing random walk: on $\mathbb{Z}^d$ and on random graphs
2. Impact of two choices and explosive percolation
3. Voter model perturbations, connections with PDEs
Voter model and coalescing random walks

Let $G$ be a graph with constant degree.

**Voter model:** Each individual $x \in G$ has an opinion. e.g., 1 (in favor) or 0 (against). At times of a rate one Poisson process $T_n^X$, $n \geq 1$, the voter at $x$ adopts the opinion of random chosen neighbor. If $A$ is the initial set of individuals with opinion 1, $\xi_t^A$ is the set with opinion 1 at time $t$.

**Coalescing random walk.** Let $B$ be the set of sites occupied by particles at time 0. Particles perform independent random walks in which at rate 1 they jump to a randomly chosen neighbor. If two particles are on the same site they coalesce to 1. $\eta_t^B$ set of sites with particles at time $t$.

**Duality.** $P(\xi_t^A \cap B \neq \emptyset) = P(A \cap \eta_t^B \neq \emptyset)$
Holley and Liggett (1975)

\[ G = \mathbb{Z}^d, \] nearest neighbors, e.g., in \( d = 2, \ x \pm (1, 0), \ x \pm (0, 1) \)

Coordinate description: \( \xi_t(x) \in \{0, 1\} \) is opinion at \( x \) at time \( t \).

In \( d = 1, 2 \) the system **clusters**. \( P(\xi_t(x) \neq \xi_t(y)) \to 0 \).

\( d \geq 3 \). **coexistence** Suppose \( \xi_0^\theta(x) \) are independent and \( = 1 \) with probability \( \theta \). As \( t \to \infty \), \( \xi_t^\theta \) converges in distribution to \( \nu_\theta \), a stationary distribution.

Follows from duality and the fact that random walks are recurrent in \( d = 1, 2 \) and transient in \( d \geq 3 \)

Note that in contrast to irreducible Markov chains on finite state spaces, there is a one parameter family of stationary distributions.
Coalescing random walk on $\mathbb{Z}^d$

Suppose $\zeta_0^1 = \mathbb{Z}^d$. Superscript 1, sites are occupied with probability 1. Let $p_t = P(x \in \zeta_t^1)$ independent of $x$ by translation invariance.

$$p_t \sim \begin{cases} 
  c_1 t^{-1/2} & d = 1 \\
  (c_2 \log t)/t & d = 2 \\
  c_d/t & d \geq 3
\end{cases}$$

$a(t) \sim b(t)$ means $a(t)/b(t) \to 1$.

We will be interested in the rate of decay of the density of coalescing random walks on random graphs.
Coalescing random walk on $(\mathbb{Z} \mod N)^d$

Let $|\zeta^1_t|$ be the number of particles at time $t$. Let $\tau_N = \min\{t : |\zeta^1_t| = 1\}$. By duality this is the time the voter model reaches consensus starting from all sites having different opinions, e.g., $\xi_0(x) = x$.

$$\tau_N \sim s_N = \begin{cases} C_1 N^2 & d = 1 \\ C_2 N^2 \log N & d = 2 \\ C_d N^d & d \geq 3 \end{cases}$$

(⋆) In $d \geq 2$, $\tau_N / s_N$ converges in distribution to

$$\sum_{k=2}^{\infty} \text{exponential}(k(k-1)/2)$$

Exponential($\lambda$) has distribution function $F(t) = 1 - e^{-\lambda t}$ and mean $1/\lambda$.

(⋆) holds for a large number of random graphs.
Random Graphs, \( G = \{1, 2, \ldots, N\} \).

**Sparse Erdös-Rényi** If \( i \neq j \) there is an edge from \( i \) to \( j \) with probability \( \lambda/N \). Different edges are independent.

**Random regular graph.** Let \( r \geq 3 \). Attach \( r \) half-edges to each \( i \). If \( rN \) is even, pair the half-edges at random (sum of the degrees = 2 times number of edges).

**Configuration model.** Let \( d_1, d_2, \ldots d_n \) be i.i.d. and condition on \( d_1 + \cdots + d_n \) is even. Attach \( d_i \) half-edges to \( i \). Pair the half-edges at random.

**Power law random graphs.** \( P(d_i = k) \sim Ck^{-\alpha}, \alpha > 2 \).
Random walks on random graphs

Mean-field behavior, (⋆), holds when the time for the random walk to converge to equilibrium is much smaller than the time it takes for two random walks starting at randomly chosen vertices to hit.

To check this condition we need to be able to estimate the time it takes for the random walk to converge to equilibrium. To do this we need to review some basic results for estimating the rate of convergence.

The keywords here are: spectral gap, Dirichlet form, conductance, variational formula, Cheeger’s inequality.
Part 2. Impact of two choices 1: Balls in Boxes

Suppose we put $\lambda N$ balls at random into $N$ boxes. Then the number of balls in a typical box is $\approx \text{Poisson}(\lambda)$ and the most populated box has $\sim c\lambda \log N$ balls.

Some algorithms in computer science use constructions that give rise to instances of this problem. In that situation one would like to produce a more even distribution.

If on each turn you pick two boxes at random and put the ball in the least loaded box then the largest occupancy is now $O(\log \log n)$. Picking 3 or more boxes only changes the constant, not the order of magnitude.
Start with $G = \{1, 2, \ldots N\}$ and no edges. At time $m/N$, $m = 1, 2, \ldots$ add one edge chosen at random.

As $N \rightarrow \infty$ the graph at time $t$ converges to an Erdös-Rényi graph with mean degree $\lambda = 2t$ (edges have two ends).

This construction is convenient for visualizing (and studying) the phase transition in Erdös-Rényi graphs, which takes place at time $t_c = 1/2$. 
Impact of two choices 2: Explosive Percolation

Suppose we modify the last construction so that now that on each step we pick two edges \((v_1, v_2)\) and \((v_3, v_4)\). Let \(\kappa_i\) be the size of the cluster containing \(i\). Some possible algorithms are:

(i) **sum rule**: choose the first edge if \(\kappa_1 + \kappa_2 \leq \kappa_3 + \kappa_4\) and choose the second one otherwise.

(ii) **product rule**: choose the first edge if \(\kappa_1 \cdot \kappa_2 \leq \kappa_3 \cdot \kappa_4\) and choose the second one otherwise.

(iii) **Bohman-Frieze rule**: choose first edge if if \(\kappa_1 = \kappa_2 = 1\) and the second one otherwise.
Simulations by Achlioptas, Dsouza and Spencer in a 2009 paper in Science suggested that in the sum rule or the produce rule is used there is an “explosive transition” going from $n^{1/2}$ vertices in the largest cluster to a cluster of size $n/2$ in a time that is $o(n)$.

In 2011, Riordan and Warnke in another paper published in Science proved that for a wide variety of rules of this type the transition is continuous.

Their proof uses two arguments by contradiction so it does not yield much quantitative information about the phase transition. Our goal is to prove results about the phase transition (critical exponents).
In joint work with undergraduate Braden Hoagland, Durrett has been studying the critical exponents of this phase transition for a collection of systems we call **two choice rules**: on each step you pick a set $A$ of $k$ vertices at random and a set $B$ of $m$ vertices at random. We apply some rule choose a vertex from each set to connect by an edge.

**min rule.** choose the vertex with the smallest degree in each set

**Bohman-Frieze.** pick an isolated vertex if there is one, otherwise pick at random

We have results for a variety of rules but not for

**max rule.** choose the vertex with the largest degree in each set
Part 3. Voter model perturbations

The three particle systems considered in Liggett’s 1999 book: the voter model, the contact process, and the exclusion process, all have special properties that facilitate their analysis.

In 2013, Cox, Durrett, and Perkins developed methods that can be used to study systems that are small perturbations of the voter model. These processes have a dual process that is a coalescing random walk plus births of sets of new particle that occur at a small rate.

Scaling limits of these particle systems converge to partial differential equations of the reaction-diffusion type, which allow these systems to be studied.
Examples

**Nonlinear voter models in** $\mathbb{Z}^2$. Sites change at rate 1, and become a 1 with probability $p_i$ if the number of 1’s at $x$ and its four neighbors is $i$. $p_0 = 0$, $p_5 = 1$, $p_4 = 1 - p_1$, $p_3 = 1 - p_2$. We can prove results when $(p_1, p_2)$ is close to $(1/5, 2/5)$. Even though the region is small there are anumber of interesting examples, potentially one with of two translation invariant stationary distributions. Work with Zoe Huang.

**Latent voter model.** Think of 1 = iPad, 0 = Microsoft surface. Active individuals imitate their neighbor as in the voter model but when they change opinion they enter inert states $1^*$ and $0^*$ in which they will not change their opinion (i.e., buy a new device). Inert individuals become active at rate $\lambda$. In joint work with Ran Huo we proved results when $\lambda$ is large on random graphs generated by the configuration model with $3 \leq d(x) \leq M$. 

The *q*-voter model. The rate at which a site flips is the *q*-the power of the fraction of neighbors with the opposite opinion. Physicists tell us that when *q* > 1 the system clusters, while for *q* < 1 there is coexistence. We are able to prove this in *d* ≥ 3 when *q* is close to 1. This is joint work with Pooja Agarwal and MacKenzie Simper.

In *q*-voter model and in the latent voter model when there is coexistence there is only one nontrivial stationary distribution which has density 1/2.