UNC Lecture 4

Voter models, coalescing random walk

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We construct the voter model from a **graphical representation**. At the times $T_n^x$, $n \geq 1$ of a rate 1 Poisson process $x$ decides to change its opinion, picks a neighbor $y_n^x$ at random, and at time $t = T_n^x$ we set

$$\xi_t(x) = \xi_t(y_n^x).$$

This construction allows us to define a **dual process** $\zeta^{x,t}_s$ that works backwards in time to determine the source of the opinion at $x$ at time $t$.

$$\xi_t(x) = \xi_{t-s}(\zeta^{x,t}_s)$$

For fixed $x$ and $t$, $\zeta^{x,t}_s$ is a random walk that jumps at rate 1 and to a neighbor chosen at random. If $\zeta^{x,t}_s = \zeta^{y,t}_s$ for some $s$ then the two random walks will stay together at later times. For these reason the $\zeta^{x,t}_s$ are called **coalescing random walks**.
Dual process = coalescing random walk

\[ \begin{array}{cccccccc}
0 & 0 & 0 & 1 & 0 & 1 & 0 & 1
\end{array} \]

\[ \begin{array}{cccccccc}
1 & 1 & 1 & 1 & 1 & 1 & 0 & 0
\end{array} \]

\[ t \]

\[ x \] imitates \[ y \]

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Set-valued duality

Let $\xi^A_t$ be the points with opinion 1 at time $t$ when initially sites in $A$ have opinion 1. Let $\zeta^B,t = \bigcup_{x \in B} \zeta^x,t$

$$\{\xi^A_t \cap B \neq \emptyset\} = \{A \cap \zeta^B,t\}$$

There is a voter in $A$ at time $t$ with opinion 1 if and only some $x$ at time $t$ traces its opinion back to a site in $A$ at time 0.

$$P(\xi^A_t \cap B \neq \emptyset) = P(A \cap \zeta^B_t \neq \emptyset)$$

Construction from graphical representation implies “additivity”

$$\xi^A_t \cup \xi^B_t = \xi^{A \cup B}_t$$
Clustering versus Coexistence

If we consider the voter model on $\mathbb{Z}^d$ with the usual nearest neighbors then as Holley and Liggett (1975) have shown the recurrence of random walks in $d \leq 2$ and transience in $d > 3$ implies

**Theorem 7.1.1.** In $d \leq 2$ the voter model approaches complete consensus, i.e., $P(\xi_t(x) = \xi_t(y)) \to 1$. In $d \geq 3$ if we start from product measure with density $p$ (i.e., we assign opinions 1 and 0 independently to sites with probabilities $p$ and $1 - p$) then as $t \to \infty$, $\xi^p_t$ converges in distribution to $\xi^p_\infty$, a stationary distribution with a fraction $p$ of the sites have opinion 1.

There is a one parameter family of stationary distributions since density is conserved. (Notice that the voter model is “irreducible.”)
Kingman’s Coalescent

If we start the coalescing random walk on the complete graph with $k$ walkers each jumping at rate one, then the time until the first coalescence is approximately exponential with rate $k(k - 1)/n$. Let $T_m$ be the first time the coalescing random walk has only $m$ particles.

$$E(T_{m-1} - T_m) = n/(m(m - 1))$$

If we let $\tau_m$ be the time $T_m$ measured in units of $n$ generations then

$$E(\tau_{m-1} - \tau_m) = 1/(m(m - 1)) = \frac{1}{m - 1} - \frac{1}{m}$$

so telescoping the series $E\tau_1 = \sum_{m=2}^{n} \frac{1}{m-1} - \frac{1}{m} = 1 - 1/n$

Comes down from $\infty$ in finite time. (Entrance boundary.)
Cox (1989) studied the nearest neighbor voter model on a finite torus \( (\mathbb{Z} \mod M)^d \). Since the state space is finite the chain will eventually be absorbed in a state where all individuals have the same opinion. Suppose that all sites initially have different opinions, e.g., let \( \bar{\xi}_0(x) = x \). Using duality we see that the time to reach consensus, \( \tau_M \) is the same as the time it takes coalescing random walk to be reduced to one particle if we start with one particle at each site.

**Theorem 7.2.1.** The time to reach consensus \( \tau_N \) satisfies

\[
\tau_M = \Theta(s_M) \quad \text{where} \quad s_M = \begin{cases} 
M^2 & d = 1 \\
M^2 \log M & d = 2 \\
M^d & d \geq 3
\end{cases}
\]

The number of points \( n = M^d \) so the answers are \( n^2, n \log n, n \).
**Metatheorem.** If the time for a random walk to converge to equilibrium (the mixing time) is much larger than the time for random walks from two randomly chosen sites to hit, then coalescing random walk behaves like Kingman’s coalescent.

CRW on $\mathbb{Z}^d$. True in $d \geq 2$. Mixing time $= \Theta(N^2)$. $d \geq 3$ hitting time $N^d$. $d = 2$ hitting time $N^2 \log N$.

$d = 2$ is delicate: random walk is recurrent which produces the log factor and there is no stationary distribution for the voter model. Most random graphs are highly transient so we ignore this case.
Hitting time for two walks: Torus

Translation invariance means you can compute things.

\[ H_M^2 \text{ time for two randomly located particles to hit } \approx_d T_0 \]

\[ s_M = M^d \text{ in } d \geq 3, \ M^2 \log M \text{ in } d = 2 \]

\[ G = \sum_{k=0}^{\infty} p^k(0, 0) \text{ in } d \geq 3, \ \frac{2}{\pi} \text{ in } d = 2 \]

If \( a_M \to \infty \) and in \( d = 2 \) if \( a_M \gg \frac{M}{\sqrt{\log M}} \) then

\[ P_x(H_M^2/s_M > t) \to e^{-t/G} \]

Proof: Compute Laplace transform.

Ted was a student of Frank Spitzer.
Hitting time for two walks: general

Let $G$ a random graph with $n$ vertices generated by the configuration model. Let $d(x)$ be the degree of $x$, $D = \sum_{x=1}^{n} d(x)$, $\pi(x) = d(x)/D$. Let $X_t^1$ and $X_t^2$ be independent random walks that jump at rate 1, $X_0^i = d \pi$. $A = \{(x, x) : x \in G\}$ and $T_A = \inf\{t : (X_t^1, X_t^2) \in A\}$.

Aldous’ clumping heuristic. If $t_n$ is the mixing time of random walk

$$E_\pi T_A \approx \frac{1}{\pi(A)} \cdot \frac{1}{P_A(T_A \gg t_n)}$$

Kac’s formula $E_A T_A = 1/\pi(A)$. Once you hit the origin once it is easy to hit it again. The number of times is geometric($P_A(T_A \gg t_n)$)

Need to increase waiting time by multiplying by the clump size.
Two simple examples

\[ E_\pi T_A \approx \frac{1}{\pi(A)} \cdot \frac{1}{P_A(T_A \gg t_n)} \]

Random walk torus \( d \geq 3 \). \( G = 1/P_0(T_0 = \infty), \pi(A) = 1/n \)

Random \( r \)-regular graphs., \( r \geq 3 \). \( \pi(A) = 1/n. \) ans = \( n/G \)

\[ E_\pi T_A \sim \frac{r - 1}{r - 2} n \]

Random graph is locally an \( r \)-tree. Difference of locations moves away from 0 on tree with probability \((r - 1)/r \) and back toward it with probability \(1/r \).

\[ P_A(T_A \gg t_n) = (r - 2)/(r - 1) \]
Sood and Redner (2005) \( p_k \sim Ck^{-\gamma}, 2 \leq \gamma \leq 3 \)

\[
E_{\pi} T_A \approx \begin{cases} 
  n / \log n & \gamma = 3 \\
  n^{(2\gamma-4)/(\gamma-1)} & 2 < \gamma < 3 \\
  (\log n)^2 & \gamma = 2
\end{cases}
\]

Proof for \( 2 < \gamma < 3 \): \( \bar{d} = Ed < \infty, D \approx n\bar{d} \)

\[
\pi(A) \approx \sum_{x=1}^{n} \frac{d(x)^2}{n^2} \approx \frac{n^{2/(\gamma-1)}}{n^2} = n^{-(2\gamma-4)/(\gamma-1)}
\]

\[
P(d(x)^2 > y) = P(d(x) > \sqrt{y}) \sim Cy^{-(\gamma-1)/2} \text{ so if } 2 < \gamma < 3 \text{ sum is in domain of attraction of one-sided stable } \alpha = (\gamma - 1)/2.
\]

\( \min d(x) = 3, t_n = \log n, P(T_A \gg t_n) \to 1, \text{ Ans } = 1/\pi(A). \)
Exponential limits

$T_A/E_\pi T_A \Rightarrow \text{exponential}(1)$.

Proposition 23 of Aldous and Fill (2002) implies

$$\sup_t \left| P_\pi(T_A > t) - \exp\left(-t/E_\pi T_A\right) \right| \leq \tau_2/E_\pi T_A$$

where $\tau_2$ is the relaxation time, i.e., $1$ over the spectral gap.

Low tech solution: $T^n_A/E_\pi T^n_A$ is tight. Want to show that every subsequential limit has the lack of memory property (and mean of limit is the limit of the means).

Let $G_n(a, b) = \text{no visit to } A \text{ in } an, bn)$. Mixing faster than hitting implies $G_n(a, b)$ and $G_n(b + \delta, c)$ are asymptotically independent.
Cooper, Elsässer, Ono, and Radzik (2012)

**Theorem 7.4.1.** Let $G$ be a connected graph with $n$ vertices, average vertex degree $\bar{d}$ and maximum degree $\Delta = O(n^{1-\epsilon})$. Let $\nu = (\sum_{v \in V} d^2(v))/(\bar{d}^2 n)$. Let $C(n)$ be the coalescence time starting with one particle at each site, independent continuous time random walk at rate 1. Then

$$EC(n) = O \left( \frac{n}{\nu (1 - \lambda_1)} \right)$$

where $\lambda_1$ is the second largest eigenvalue.

If $Ed^2 < \infty$, $\nu = \Theta(1)$. In power law case $\nu = n\pi(A)$.

Coalescence time and hitting time for two particles are of the same order.
A nice trick for multiple random walks

To study the coalescence of $k \geq 2$ walks on a graph, we replace the $k$ walks by one walk on a new graph $Q_k$ with vertex set $V^k$. Vertices $v, w \in Q_k$ adjacent if there is a $j$ so that $v_i = w_i$ for $i \neq j$ and $v_j \sim w_j$.

Although we are interested in coalescence, our $k$ random walks will be independent. For any starting positions $u = (u_1, \ldots u_k)$ for the walks, let

$$S_k = \{(v_1, v_2, \ldots v_k) : v_i = v_j \text{ for some } 1 \leq i < j \leq n\}.$$

On contraction all edges including those that have become loops or parallel edges are retained. To use results from the previous section it is convenient to contract the set $S_k$ to be a single vertex $\gamma$ making a new graph $\Gamma_k$. 
**Estimation of hitting probabilities**

**Lemma 7.4.2.** Let $F$ be a graph with spectral graph $1 - \lambda_1$

$$E_{\pi} H_v \leq \frac{1}{1 - \lambda_1} \cdot \frac{1}{\pi_v}$$

**Lemma 7.4.4.** There is a constant $c_k > 0$ so that if $k \leq \log^4 n$

$$\hat{\pi}_\gamma \geq \frac{c_k k^2 \nu}{2n}$$

Let $M_k$ be the time of the first meeting in $G$ and let $T_\Gamma$ be the time to reach equilibrium on $\Gamma$,

$$EM_k \leq T_\Gamma + (1 + o(1)) E_{\pi} H_\gamma \leq \frac{C}{1 - \lambda_1(G)} \cdot \left( k \log n + \frac{n}{\nu k^2} \right)$$
Lemma 7.4.7. Let $k_* = \log^4 n$. The probability there is a set of $k_*$ particles that have not had any collisions by time

$$T_* = k_*^{3/2} (T_G + 3E_\pi H_\gamma)$$

is \leq \exp(-\lfloor t^*/(T_G + 3E_\pi H_\gamma) \rfloor) \leq e^{-\log^6 n}.$$
Let $X_t$ be a continuous time Markov chain with generator $Q$ and stationary distribution $\pi$, let $t_m^{Q}$ be the mixing time, let $m(Q)$ be the expected meeting time of two independent copies of the chain $Q$, each starting from the stationary distribution, Define the Wasserstein distance between the distributions of two random variables with finite means by

$$d_W(X, Y) = \int |P(X > x) - P(Y > x)| \, dx$$

$$d_W(X, Y) = \sup\{ |Ef(X) - Ef(Y) | : fis 1-Lipshitz \}$$
let $Z_i$, $i \geq 2$ be independent exponential($i$)

(Mean-field limit for transitive reversible Markov chains). Start with one copy of the Markov chain at each site and let $C$ be the time needed for all the particles to coalesce to 1. If $\rho(Q) = t_{mix}^Q / m(Q)$ then

$$d_W \left( \frac{C}{m(Q)}, \sum_{i \geq 2} Z_i \right) = O \left( [\rho(Q) \ln(1/\rho(Q))]^{1/6} \right)$$