Chapter 5

Contact Process

The contact process is often used as a simple spatial model for the spread of species. In this case the state at time t, ξ_t, is the set of occupied sites, and sites in ξ_t^c are vacant. In the contact process on a graph G, occupied sites $x \in \xi_t$ become vacant at rate 1, and give birth onto each vacant neighbor at rate λ. The contact process can also be viewed as a spatial SIS epidemic model. In this case ξ_t is the set of infected sites, and sites in ξ_t^c are susceptible. With most diseases, individuals have some immunity to reinfection after they recover, but we are not really concerned with applications here. The contact process in which deaths occur at a constant rate, and the birth rate is linearis a very simple and fundamentally important example of a stochastic spatial model.

5.1 Basic properties

sec:basicp

Harris (1974) introduced the contact process on $G = \mathbb{Z}^d$ in 1974. Let ξ_t^x be the process starting from only x occupied. Harris defined the critical value

$$\lambda_c = \inf\{\lambda : P(\xi_t^x \neq \emptyset \text{ for all } t) > 0\},$$

(this value is independent of x on connected graphs.) He proved that on $\mathbb{Z}^d$ we have $0 < \lambda_c < \infty$. On $\mathbb{Z}^d$, or on any graph with all degrees $\leq M$, the lower bound is trivial $\lambda_c \geq 1/M$. In the early days of the theory, upper bounds on critical values seemed harder than lower bounds, but when it came to random graphs with unbounded degree distributions lower bounds were easier because one only had to find a mechanism that ensured survival. On graphs with power law degree distribution or even subexponential distributions the critical value is 0, so the question of lower bounds is moot. The question for random graphs with exponentially decaying degree distributions, considered in Section 5.8, turned out to be quite difficult and it took many years to show that $\lambda_c > 0$.

A rich theory has been developed for the contact process on $\mathbb{Z}^d$ and on regular trees. See Liggett's 1999 book for a summary of much of what is known in these settings. In this section we will review some of the definitions and results that are the most important for our work. If you want to see more details consult Liggett's book. I am not a big fan of calling things beautiful, but this book is.

The first item on our agenda is a special construction of the process from a **graphical representation** that is built using independent Poisson processes.

- For each site there is an independent rate 1 Poisson process T_n^x, $n \geq 1$. At the arrivals of this process if there is a particle at x, it will die. To facilitate later definitions we write a dot at x at the times T_n^x. In early versions of the construction people wrote δ for death.

- For each oriented edge (x, y) we have a rate λ Poisson process $T_n^{x,y}$. At these times we draw an arrow from x to y to indicate that if x is occupied there will be a birth from x to y. If y is vacant it will become occupied. If y is occupied there is no change.

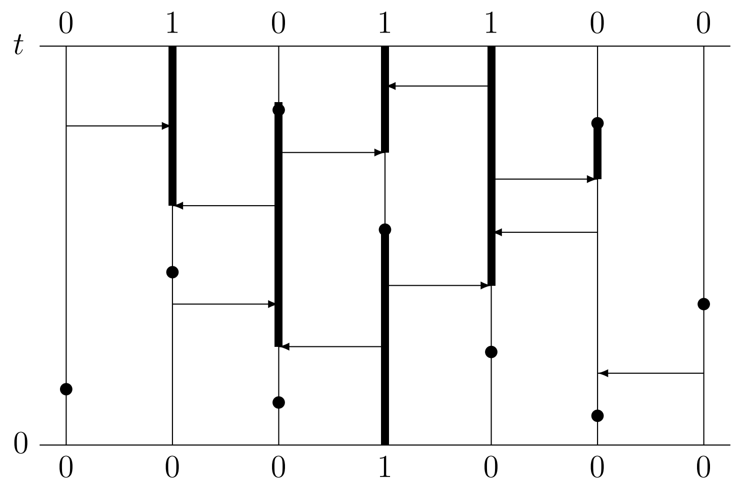

Figure 5.1: Graphical representation for the contact process. We think of fluid flowing up the structure and across arrows in the direction of the orientation, but being stopped by dots. Thus the contact process is a percolation process that is discrete in space and continuous and oriented in time,

Let ξ_t^A denote the contact process starting from A occupied at time 0. A point $y \in \xi_t^A$ if for some $x \in A$ there is path from $(x, 0)$ to (y, t) that goes up the graphical representation without passing through $\bullet$s and crosses edges in the direction of their orientation. A nice feature of the graphical representation is that it allows us to construct all of the ξ_t^A on the same space in such a way that

$$\text{if } A \subset B \text{ then } \xi_t^A \subset \xi_t^B. \tag{5.1.1}$$

cpmono

When a set-valued Markov process has this property, it is called **attractive**. The term originated from the Ising model in which each site was in state 1 (spin up) or -1 (spin down). In this case attractive meant that the spins had a tendency to align. An important consequence of a process being attractive is

upinvm **Theorem 5.1.1.** *If we let ξ_t^1 be the system starting from all sites occupied then ξ_t^1 converges in distribution to a limit ξ_∞^1, which is a stationary distribution.*

The reader can find the proofs of these and other assertions in Liggett's (1999) book. Due to monotonicity property in (5.1.1), ξ_∞^1 is the largest possible stationary distribution. Thus, if $\xi_\infty^1 = \delta_\emptyset$ then there are no nontrivial stationary distributions.

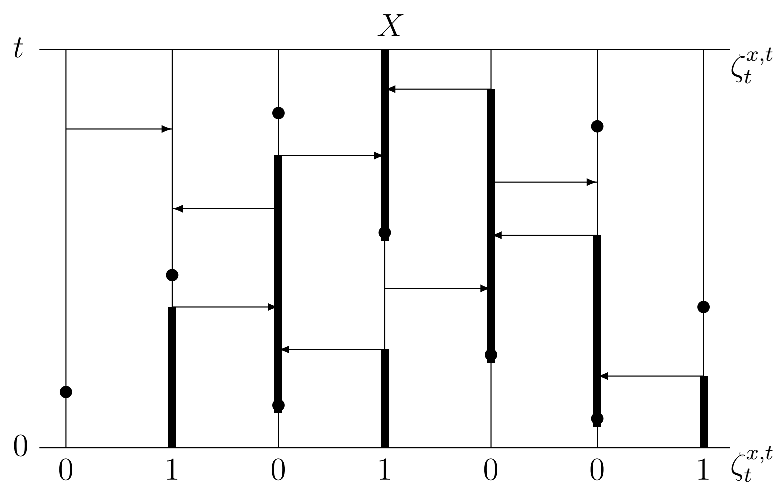

Figure 5.2: Contact process duality. We think of fluid flowing down the structure, across arrows in the direction opposite orientation, but being stopped by dots.

A second important consequence of the construction is that it allows us to define for each x a **dual process** $\zeta_s^{x,t}$, $s \le t$, that works backwards in time to answer the question "Is the site x occupied at time t?" The dual process can be constructed by a variant of the rule used for the contact process: $y \in \zeta_s^{x,t}$ if there is a path from (x,t) to $(y, t-s)$ that goes down the graphical representation without passing through $\bullet$s and crossses edges in the direction opposite their orientation.

We extend the definition of the dual to an initial set B by setting

$$\zeta_s^{B,t} = \cup_{x \in B} \zeta_s^{x,t}$$

A little thought shows that

$$\{\xi_t^A \cap B = \emptyset\} = \{A \cap \zeta_t^{B,t} = \emptyset\} \qquad (5.1.2)$$ cdualeq

The almost sure equality in (5.1.2) is convenient for establishing the equation but it is useful to rewrite the equality without the superscript t. To do this we note that if $t < t'$ then the joint distribution of the $\zeta_s^{x,t}$, $s \le t$ with $x \in \mathbb{Z}^d$ is the same as that of $\zeta_s^{x,t'}$, with $x \in \mathbb{Z}^d$ when $s \le t'$, so using the Kolmogorov extension theorem there is a family of processes ζ_s^x whose joint distributions agree with $\zeta_s^{x,t}$ on $s \le t$.

$$P(\xi_t^A \cap B \neq \emptyset) = P(A \cap \zeta_t^B \neq \emptyset) \qquad (5.1.3)$$ setdual

With future generalizations in mind we will ignore for the moment that contact process on a graph G is self dual. Taking $A = G$ and $B = \{x\}$ the duality equation becomes

$$P(x \in \xi_t^1) = P(\zeta_t^x \neq \emptyset)$$

Since the empty set is an absorbing state for the dual

$$\lim_{t\to\infty} P(x \in \xi_t^1) = P(\zeta_t^x \neq \emptyset \text{ for all } t)$$

In words the density of particles in ξ_∞^1 is the probability that the dual process lives forever starting from one site. Replacing x by a finite set B

$$\lim_{t\to\infty} P(\xi_t^1 \cap B \neq \emptyset) = P(\zeta_t^B \neq \emptyset \text{ for all } t)$$

The fact that this holds for all B implies that ξ_t^1 converges in distribution.

A consequence ot the construction from a graphical representation is that the interaction is **additive**

$$\xi_t^A \cup \xi_t^B = \xi_t^{A\cup B}$$

Harris (1978) identifed the processes with this property and showed that they had dual processes. The study of this class of processes was greatly advance by the work of Griffeath (1979). In Chapter 7 we will encounter another member of this family, the voter model.

5.2 Mean-field theory

The first step in the investigation of an interacting particle system is to examine its properties when we get rid of the spatial correlations that make it difficult to analyze. To motivate the simplifications we will use, we begin with an exact equation.

$$\frac{d}{dt} P(x \in \xi_t) = -P(x \in \xi_t) + \lambda \sum_{y\sim x} P(x \notin \xi_t, y \in \xi_t) \qquad (5.2.1)$$ conexacteq

where $y \sim x$ means y is a neighbor of x. Let $\mathcal{N}_x = \{y : y \sim x\}$ Suppose we are on a vertex transitive graph. This implies (i) that the number of neighbors $|\mathcal{N}_x|$ does not depend on x and (ii) if we start from all sites occupied then at any time t the probabilities $P(y \in \xi_t)$ does not depend on y, so we can call it $u(t)$. Using the fact that $P(x \notin \xi_t, y \in \xi_t) \leq P(y \in \xi_t)$, (5.2.1) becomes

$$\frac{du}{dt} \leq (\beta - 1)u(t)$$

where $\beta = \lambda|\mathcal{N}_x|$. If $\beta < 1$ and $u(0) = 1$ then the last equation implies that $u(t) \to 0$. Using Theorem 5.1.1 now, we conclude that on any vertex transitive graph the critical total birth rate for the existence of a nontrivial stationary distribution $\beta_c \geq 1$.

We get more information from m the differential equation (5.2.1) if we have an equality rather than an inequality. The first and simplest thing to do is to pertend that the states of sites are independent and to compute the behavior of $u(t) = P(x \in \xi_t)$. When we use our independence assumption (5.2.1) becomes

$$\begin{aligned} \frac{du}{dt} &= -u(t) + \beta u(t)(1 - u(t)) \\ &= u(t)(\beta - 1 - \beta u(t)) \equiv g(u(t)) \end{aligned} \tag{5.2.2}$$ conMFE

where $\equiv$ indicates that the second equality defines the function g.

Theorem 5.2.1. *Let $\beta_0 = (\beta - 1)/\beta$. If $u(0) \in (0, 1]$ and $\beta > 1$ then in (5.2.2) $u(t) \to \beta_0$ as $t \to \infty$. In words, the mean-field contact process has a nontrivial equilibirum frequency $\beta_0 > 0$ in addition to the trivial fixed point at 0.*

Proof. When $\beta > 1$ it is easy to see that $g(u) > 0$ for $u < \beta_0$ and $g(u) < 0$ when $u > \beta_0$. From this the result follows easily. The monotonicity of $u(t)$ implies that the limit exists, and a simple argument shows that it can only converge to a point v where $u(v) = 0$. □

The contact process on the complete graph

provides a second approach to mean-field theory. The complete graph $\mathbb{K}_n$ on n vertices has vertex set $\{1, \ldots, n\}$ and all pairs of neighbors are connected by an edge. On the complete graph it is convenient to reformulate the definition of the contact process as: an occupied site produces new particles at rate β and sends them to a site chosen at random from the graph (including itsef), since this makes the number of neighbors n instead of $n - 1$. The number of occupied sties, X_t ,is a birth and death chain with birth and death rates

$$b_m = \beta \cdot m \cdot \left(1 - \frac{m}{n}\right) \quad \text{and} \quad d_m = m, \tag{5.2.3}$$ bdrates

where m is the number of occupied sites.

The first step in analyzing a birth and death chain is to find a function $\phi(x)$, often called the **natural scale**, so that $\phi(X_t)$ is a martingale, i.e., $(d/dt)E\phi(X_t) = 0$. In terms of the rates

$$0 = b_m(\phi(m + 1) - \phi(m)) + d_m(\phi(m - 1) - \phi(m)). \tag{5.2.4}$$ ctharm

Rearranging gives

$$(\phi(m+1)-\phi(m)) = \frac{d_m}{b_m}(\phi(m)-\phi(m-1)) = \frac{1}{\beta(1-m/n)}(\phi(m)-\phi(m-1)).$$

Setting $\phi(0)=0$ and $\phi(1)-\phi(0)=1$

$$\phi(m+1)-\phi(m)) = \prod_{k=1}^{m} \frac{1}{\beta(1-k/n)} \approx (1/\beta)^m, \tag{5.2.5}$$ phidiff

when $m \ll n$. Summing we have (again for $m \ll n$.)

$$\phi(m) \approx \sum_{k=0}^{m-1} \phi(k+1)-\phi(k) = \sum_{k=0}^{m-1} (1/\beta)^k = \frac{1-(1/\beta)^m}{1-1/\beta}. \tag{5.2.6}$$ philim

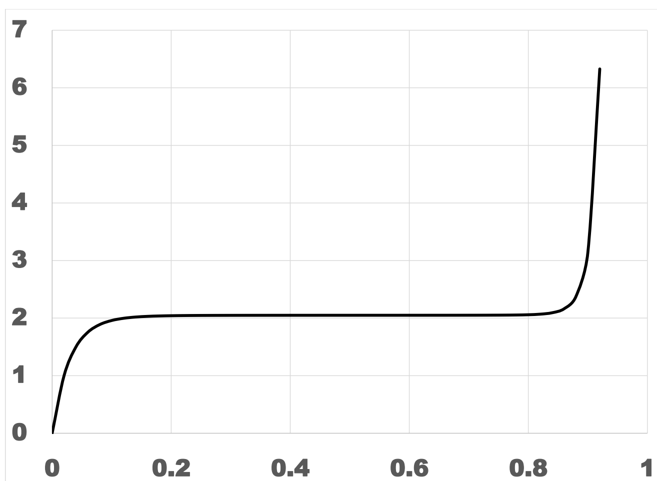

Figure 5.3: Natural scale $\phi(x)$ when $\beta = 2$ and $n = 50$. fig:mftphi

From (5.2.6) we see that

$$\text{if } m \to \infty \text{ and } m/n \to 0 \text{ then } \phi(m) \to \beta/(\beta-1). \tag{5.2.7}$$ phismallm

Eventually we will prove that if $\beta > 1$ the equilibrium frequency of 1's on the complete graph is $(\beta-1)/\beta$ in agreement with the ODE calculation. To do this the following monotonicity result, which follows from the formula for the rates in (5.2.3) is useful.

phimono **Lemma 5.2.2.** *If $\beta > 1$ we have $b_m \geq d_m$ when $\beta(1-m/n) \geq 1$ or $m \leq m_0 = \lfloor n(\beta-1)/\beta \rfloor$. We have $b_m < d_m$ when $m > m_0$.*

Using this result we can extend (5.2.7)

phim0 **Lemma 5.2.3.** *As $n \to \infty$, $\phi(m_0) \to \beta/(\beta-1)$.*

Proof. Lemma 5.2.2 implies that the slope $\phi(m+1)-\phi(m)$ is decreasing when $m \le m_0$. Let $m_1 = C_1 \log(n)$ where $C_1 = -2/\log(1/\beta)$. Since $m_1/n \to 0$, we have $\phi(m_1) \to (\beta-1)/\beta$. Using (5.2.5) $\phi(m_1+1)-\phi(m_1) \to n^{-2}$. Using the fact that $\phi(m+1)-\phi(m)$ is decreasing for $m_1 \le m \le m_0$, and there are only $O(n)$ points in this range, it follows that $\phi(m_0)-\phi(m_1) \to 0$ which completes the proof. □

Let $T_k = \min\{t : X_t = k\}$. Since $\phi(X_t)$ is a martingale, $\phi(0) = 0$, and $\phi(1) = 1$

$$1 = \phi(1) = \phi(m_0) P_1(T_{m_0} < T_0) \approx \frac{\beta}{\beta-1} P_1(T_{m_0} < T_0),$$

by (5.2.3), so as $n \to \infty$

$$P_1(T_{m_0} < T_0) \to \frac{\beta-1}{\beta}.$$

On any finite graph, the contact process dies out but as we will see in Section 5.3 that on $\mathbb{K}_n$ and many other graphs the contact has a **quasi-stationary distribution**or **metastable state** with density $(\beta-1)/\beta$.

Prolonged persistence

Our next goal is to show that when $\beta > 1$ then the contact process on the complete graph with n vertices survives for time $\exp(C(\beta)n)$. This is the best we can hope for since the probability that in one unit of time no births occur and and all particles die is

$$\ge [(1-e^{-\beta})e^{-1}]^n. \tag{5.2.8}$$ ubsurv

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Method 1: Bounding the extinction time using hitting times

The first step is to construct a martingale for the mean-field contact process in continuous time X_t, which has jump rates given in (5.2.3). In order for $\psi(X_t)$ to be a martingale while $X_t \in (0, m_0)$ we must have

$$0 = b_m(\psi(m+1)-\psi(m)) + d_m(\psi(m-1)-\psi(m)) \quad \text{for } 0 < m < m_0.$$

If $\psi(m_0) = 0$ and $\psi(m) = \sum_{k=m}^{m_0-1} s_k$ for $0 < m < m_0$ then $\psi(X_t)$ will be a martingale if

$$0 = -b_m s_m + d_m s_{m-1}.$$

Thus we want

$$s_{m-1} = s_m b_m/d_m = s_m \beta(1-m/n). \tag{5.2.9}$$ srec

To get the recursion started we set $s(m_0) = 1$.

If we let $L = m_0$ and $h_L = P_{L-1}(T_0 < T_L)$ then using the fact that $\psi(X_t)$ is a martingale and $\psi(L) = 0$ we have $\psi(L-1) = h_L \psi(0)$ and hence

$$h_L = \psi(L-1)/\psi(0) \leq s_{L-1}/s_0. \tag{5.2.10}$$ hLbd

Using the recursion (5.2.9)

$$\log(s_0/s_{L-1}) = \sum_{k=1}^{L-1} \log(s_{k-1}/s_k) = \sum_{k=1}^{L-1} \log(\beta(1-k/n)).$$

Note that by the definition of $L = m_0$ the terms in the final sum > 0. Multiplying by n/n and turning the sum into an approximation of an integral the RHS is

$$\approx n \int_0^{\beta_0} \log \beta + \log(1-x)\, dx = \beta_0 \log \beta + \int_0^{\beta_0} \log(1-x)\, dx, \tag{5.2.11}$$ Method1

where $\beta_0 = (\beta-1)/\beta$. The antiderivative of $\log(1-x)$ is $(x-1)\log(1-x) - x$ so the above is

$$\begin{aligned} B &\equiv \beta_0 \log \beta + (\beta_0 - 1)\log(1-\beta_0) - \beta_0 \\ &= \frac{\beta-1}{\beta} \log \beta - \frac{1}{\beta} \log(1/\beta) - \frac{\beta-1}{\beta} = \log(\beta) - \frac{\beta-1}{\beta} \end{aligned}$$

and using (5.2.10) we have $h_L \leq \exp(-nB)$. Since each excursion away from L takes at least $1/L$ unit of time on the average we have

cpcpart1 **Lemma 5.2.4.** *If $\beta > 1$ and $\epsilon > 0$ then the extinction time for the contact on the complete graph with n vertices takes time $\geq \exp(n(B-\epsilon))$ with high probability.*

Method 2: Hitting times for birth and death processes

Our second approach is compute the expected hitting time directly using methods that hold for many borth and death chains. Let $T_{b,a}$ = time to go from a to b. The subscripts are backwards from how I would have chosen them, but it is easier to live with the notation than to rewrite all of the equations in Section 6.7 in Allen (2003). Our first identity is obvious from the definitions:

ETdecomp **Lemma 5.2.5.** *If $a > b$ then $T_{b,a} = T_{a-1,a} + T_{a-2,a-1} + \cdots + T_{b,b+1}$.*

Let μ_i be the death rate at i (rate for jumps $i \to i-1$) and λ_i be the birth rate at i (rate for jumps $i \to i+1$). It is easy to see that

$$ET_{i-1.i} = \frac{\mu_i}{\lambda_i + \mu_i} \cdot \frac{1}{\lambda_i + \mu_i} + \frac{\lambda_i}{\lambda_i + \mu_i} \cdot \left(\frac{1}{\lambda_i + \mu_i} + ET_{i,i+1} + ET_{i-1,i} \right).$$

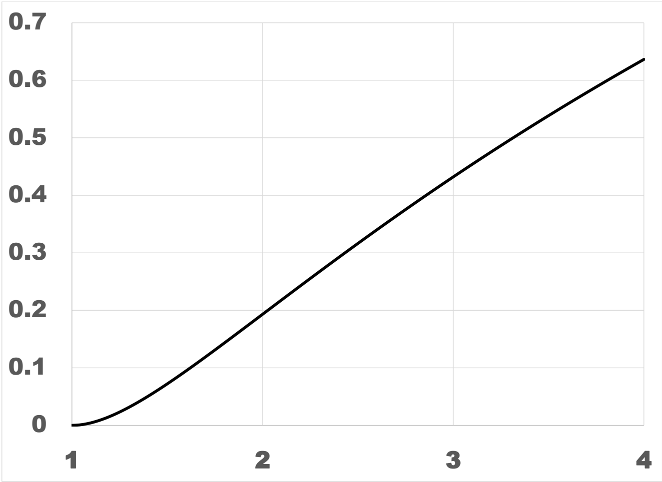

Figure 5.4: Plot of consta B versus β. fig:mftsurv

To explain this equality note that the chain stays at i for a time with mean $1/(\lambda_i + \mu_i)$. If it jumps to $i-1$ (with probability $\mu_i/(\mu_i + \lambda_i)$) then we have gotten to $i-1$. If not, then to get to $i-1$ we have to go from $i+1$ to i and from i to $i-1$. Rearranging gives

$$ET_{i-1.i}\left(1 - \frac{\lambda_i}{\lambda_i + \mu_i}\right) = \frac{1}{\lambda_i + \mu_i} + \frac{\lambda_i}{\lambda_i + \mu_i} \cdot ET_{i,i+1},$$

which gives us our recursion:

$$ET_{i-1.i} = \frac{1}{\mu_i} + \frac{\lambda_i}{\mu_i} \cdot ET_{i-1,i}. \tag{5.2.12}$$ ETrec

Suppose now that we want to compute $ET_{0,n}$ for a birth and death process on $\{0, 1, \dots n\}$. The first jump can only be from n to $n-1$, so using the recursion (5.2.12) for later terms, we have

$$\begin{aligned}
ET_{n-1,n} &= \frac{1}{\mu_n} \\
ET_{n-2,n-1} &= \frac{1}{\mu_{n-1}} + \frac{\lambda_{n-1}}{\mu_{n-1}} \cdot \frac{1}{\mu_n} \\
ET_{n-3,n-2} &= \frac{1}{\mu_{n-2}} + \frac{\lambda_{n-2}}{\mu_{n-2}} \cdot \left(\frac{1}{\mu_{n-1}} + \frac{\lambda_{n-1}}{\mu_{n-1}} \cdot \frac{1}{\mu_n}\right) \\
&= \frac{1}{\mu_{n-2}}\left(1 + \frac{\lambda_{n-2}}{\mu_{n-1}} + \frac{\lambda_{n-2}\lambda_{n-1}}{\mu_{n-1}\mu_n}\right) \\
ET_{n-4,n-3} &= \frac{1}{\mu_{n-3}}\left(1 + \frac{\lambda_{n-3}}{\mu_{n-2}} + \frac{\lambda_{n-3}\lambda_{n-2}}{\mu_{n-2}\mu_{n-1}} + \frac{\lambda_{n-3}\lambda_{n-2}\lambda_{n-1}}{\mu_{n-2}\mu_{n-1}\mu_n}\right)
\end{aligned}$$

Generalizing from these formulas, we see that

$$\begin{aligned} ET_{n-k-1,n-k} &= \frac{1}{\mu_{n-k}} \left(1 + \sum_{j=n-k+1}^{n} \prod_{i=n-k+1}^{j} \frac{\lambda_{i-1}}{\mu_i} \right) \\ &= \frac{1}{\mu_{n-k}} \left(\sum_{j=n-k}^{n} \prod_{i=n-k+1}^{j} \frac{\lambda_{i-1}}{\mu_i} \right) \end{aligned}$$

where in the last step we have absorbed the 1 into the sum by using the convention that the empty product $\prod_{i=n-k+1}^{n-k} = 1$. Using Lemma 5.2.5 we have

$$ET_{0,n} = \sum_{k=0}^{n-1} ET_{n-k-1,n-k}$$

When we combine the last two expressions we get a sum of roughly $n^2/2$ products of λ_{i-1}/μ_i from $i = n-k+1$ to $j \in [n-k+1, n]$. When $\beta > 1$, if $\lambda_k = \beta(1-k/n)$ and $\mu_k = k$, he largest of these products is the one wth $i = 1$ and $j = n\beta_0$.

$$\pi = \prod_{i=1}^{n\beta_0} \beta(1 - i/n). \qquad (5.2.13)$$ oneterm

Using calculation at the end of the section on Method 1 see (5.2.11)

$$B = \log \beta - (\beta - 1)/\beta$$

The sum which gives $T_{0,n}$ is bigger than the term in (5.2.13) and smaller than n^2 times it, so we have

cpcpart2 **Lemma 5.2.6.** *If n is large then he extinction time for the contact on the complete graph takes time $\le Cn^2 \exp(nB)$ with high probability.*

This result and Lemma 5.2.4 provide complementary upper and lower bounds on the survival time.

5.3 Bounded degrees

sec:CPtree

Pemantle (1992) was the first to study the contact process on the tree $\mathbb{T}^d$ in which each vertex has degree $d+1$. Here, and in what follows, we assume $d \ge 2$ since $\mathbb{T}^1 = \mathbb{Z}$. Let 0 be the root of the tree and let P_0 be the probability measure for the process starting from only the root occupied. Pemantle found that the contact process on $\mathbb{T}^d$ has two critical values:

$$\begin{aligned} \lambda_1 &= \inf\{\lambda : P_0(\xi_t \neq \emptyset \text{ for all } t) > 0\}, \\ \lambda_2 &= \inf\{\lambda : \liminf_{t\to\infty} P_0(0 \in \xi_t) > 0\}. \end{aligned}$$

In words, the contact process **survives** when $\lambda > \lambda_1$ but **survives locally** when $\lambda > \lambda_2$, i.e., with positive probability 0 is occupied infinitely many times, though the definition we have chosen is stronger than that.

By deriving bounds on the critical values, he showed that $\lambda_1 < \lambda_2$ when $d \geq 3$. Liggett (1996) settled the case $d = 2$ by showing

$$\lambda_1 < 0.605 < 0.609 < \lambda_2.$$

At about the same time, Stacey (1996) gave a proof that $\lambda_1 < \lambda_2$ that did not rely on numerical bounds on the critical value.

Open problem. Stacey's argument is simple and elegant but relies heavily on the fact that the graph is a tree. It would be nice to generalize it to Galton-Watson trees, but the randomness of the graph seems to pose a substantial problem. A simpler problem is to consider the Big World of Durrett and Jung (2007), which is the free product $\mathbb{Z}^d * \{0, 1\}$. More intuitively, the Big World is the covering space of the Bollobás-Chung (1988) small world which was defined before the more highly cited small world of Watts and Strogatz (1998). In the BC small world one starts with a circle with an even number of vertices, pairs them at random and connects each pair by an edge. The Big World gives the limit as $n \to \infty$ of the view point of a bug walking on the small world. Moving through a long range edge brings it to a new copy of $\mathbb{Z}$. If the verbal description is not enough, eee their paper for a picture.

Contact process on $\{-n, \dots n\}$ with edges connecting nearest neighbors. Suppose it starts from all sites occupied and let $\tau_n = \inf\{t : \xi_t = \emptyset\}$. Combining results of Durrett and Liu (1988) and Durrett and Schonmann (1988) gives the following results

1dsurv **Theorem 5.3.1.** *(i) If $\lambda < \lambda_c$ then there is a constant $\gamma_1(\lambda)$ so that*

$$\tau_n / \log n \to \gamma_1(\lambda) \quad \textit{in probability.}$$

(ii) If $\lambda > \lambda_c$ then there is a constant $\gamma_2(\lambda)$ so that

$$(\log \tau_n)/n \to \gamma_2(\lambda) \quad \textit{in probability.}$$

(iii) When $\lambda > \lambda_c$ there is "metastability":

$$\tau_n / E\tau_n \Rightarrow \textit{exponential}(1)$$

Here $\Rightarrow$ means convergence in distribution. Intuitively, when $\lambda > \lambda_c$ the process on the interval stays exponentially long in a state that looks like the stationary distribution for the process on $\mathbb{Z}$ restricted to the interval $[-n, n]$. The lack of memory property of the survival time suggests that the death of the process comes suddenly and without warning, but we know of no result that makes this precise.

Results on $\mathbb{Z}^d$ with $d > 1$ had to wait for the work of Bezuidenhout and Grimmett (1990), who showed that in $d > 1$ the contact process dies out at the critical value. To do this they introduced a **block construction** that can be used to study the supercritical process. Mountford proved the metastability result in 1993 and that $(\log \tau_n)/n^d \to \gamma(\lambda)$ in 1999.

Stacey (2001) studied the contact process on a tree truncated at height ℓ, $\mathbb{T}^d_\ell$. To be precise, the root has degree d, vertices at distance $0 < k < \ell$ from the root have degree $d+1$, while those at distance ℓ have degree 1. Cranston, Mountford, Mourrat, and Valesin (2014) improved Stacey's result to establish that the time to extinction starting from all sites occupied τ^d_ℓ satisfies

fintree **Theorem 5.3.2.** *(a) For any* $0 < \lambda < \lambda_2(\mathbb{T}^d)$ *there is a* $c \in (0,\infty)$ *so that as* $\ell \to \infty$

$$\tau^d_\ell / \log |\mathbb{T}^d_\ell| \to c \quad \textit{in probability.}$$

(b) For any $\lambda_2(\mathbb{T}^d) < \lambda < \infty$ *there is a* $c \in (0,\infty)$ *so that as* $\ell \to \infty$

$$\log(\tau^d_\ell)/|\mathbb{T}^d_\ell| \to c \quad \textit{in probability.}$$

In case (b), $\tau^d_\ell / E\tau^d_\ell$ *converges to a mean one exponential.*

When a tree is truncated at a finite distance, a positive fraction of the sites are on the boundary. A more natural finite version of a tree is a **random regular graph** in which all vertices have degree $d+1$. In this case there is no boundary and the graph has the same distribution viewed from any point. If there are n vertices, the graph looks like $\mathbb{T}^d$ in neighborhoods of a point that have $\le n^{1/2-\epsilon}$ vertices, see Theorem 1.2.3. Mourrat and Valesin (2016) have shown for a random regular graph, the time to extinction starting from all sites occupied τ_n satisfies:

randreg **Theorem 5.3.3.** *(a) For any* $0 < \lambda < \lambda_1(\mathbb{T}^d)$ *there is a* $C < \infty$ *so that as* $n \to \infty$

$$P(\tau_n < C \log n) \to 1,$$

(b) For any $\lambda_1(\mathbb{T}^d) < \lambda < \infty$ *there is a* $c > 0$ *so that as* $n \to \infty$

$$P(\tau_n > e^{cn}) \to 1.$$

Notice that the threshold in this comes at λ_1, while the one in Stacey's result comes at λ_2. The difference is that when $\lambda \in (\lambda_1, \lambda_2)$ on the infinite tree the origin is in the middle of linearly growing vacant region. On the truncated tree the system dies out when the vacant region is large enough, or more poetically the particles like lemmings run over the edge of cliff. However, on the random regular graph the occupied sites will later return to the origin. Durrett and Jung (2007) investigated the qualitative differences between $\lambda \in (\lambda_1, \lambda_2)$ and $\lambda > \lambda_2$ on the small world graph.

Lalley and Su (2017) also proved Theorem 5.3.3. In addition they established an interesting "cutoff" result about convergence to equilibirum. Their Theorem 1.1 shows that if we start from only 1 occupied then there are constants c_λ and $g_n(\epsilon) \to 0$ so that

$$\begin{array}{ll} P(2 \in \xi_t) & \text{at times} \\ \leq g_n(\epsilon) & \leq (1-\epsilon)c_\lambda \log n \\ \geq \rho^2(1 - g_n(\epsilon) & \geq (1+\epsilon)c_\lambda \log n \end{array}$$

where ρ is the survival probability, which by duality is the fraction of sites that are occupied.

Mourrat and Valesin (2016) extended their result on random reguar graphs given in Theorem 5.3.3to graphs with bounded degree. The next result is their Theorem 1.3.

bddeg **Theorem 5.3.4.** *Let $\mathcal{G}_{n,d}$ be the set of connected graphs with n vertices and degree bounded by $d+1$. (a) For any $0 \in (0, \lambda_c(\mathbb{T}^d))$ there is a constant $C < \infty$ so that*

$$\lim_{n\to\infty} \inf_{G \in \mathcal{G}_{n,d}} P(\tau_G < C \log n) = 1.$$

(b) For any $\lambda \in (\lambda_c(\mathbb{Z}), \infty)$ there is a constant $c > 0$ so that

$$\lim_{n\to\infty} \inf_{G \in \mathcal{G}_{n,d}} P(\tau_G > e^{cn}) = 1.$$

Note that the constants are uniform over $G \in \mathcal{G}_{n,d}$. The form of the result in (b) suggests the method of proof: we are going to find a large path in the graph and compare with the one dimensional contact process.

5.3.1 Proofs of Theorems 5.3.3 and 5.3.4

The two proofs are independent of each other. Both are from Mourrat and Valesin (2016).

Proof of part (a) of Theorem 5.3.4

We prove the more general result by comparing the contact process on the graph with the contact process on the tree. In the statement of the first lemma we use a notation for the projection π from $\mathcal{T}$ to G that is defined in the proof. The parenthetical names of results refer to their (2016) paper published in EJP. As usual τ^C are the extinction times for the contact process starting from sites in C occupied.

Lemma 5.3.5. *(Lemma 4.2.) For any $G \in \mathcal{G}_{n,d}$, $A \subset V$, and $B \subset \mathbb{T}^d$ for which $\pi(B) \supset A$*

$$P_G(\tau^A > t) \leq P_{\mathbb{T}^d}(\tau^B > t).$$

Proof. To prove this we introduce the **universal cover** of the graph G. Fix a reference vertex $x \in V$. Given an oriented edge $\vec{e}$ let $v_0(\vec{e})$ be the vertex that the edge e points away from, and $v_1(\vec{e})$ be the vertex points that the edge points towardd. We say that a sequence of oriented edges $(\vec{e_1}, \vec{e_2}, \dots \vec{e_m})$ is a nonbacktracking path from x if $v_0(\vec{e_1}) = x$, for $2 \leq i \leq m$, $v_1(\vec{e_{i-1}}) = v_0(\vec{e_i})$, and $v_0(\vec{e_{i-1}}) \neq v_1(\vec{e_i})$. Let $\mathcal{V}$ be the set of non-backtracking paths, including the empty path which is denoted by o. For any $\gamma = (\vec{e_1}, \vec{e_2}, \dots \vec{e_m})$ and

$\gamma' = (\vec{e_1}, \vec{e_2}, \dots \vec{e_m}, \vec{e_{m+1}})$ there is an edge from γ to γ'. This defines the edge set $\mathcal{E}$ of the tree $\mathcal{T}$, which has degree $\leq d+1$.

To map the tree into the graph put $\psi(o) = x$ (where x is the reference vertex). If $\gamma == (\vec{e_1}, \vec{e_2}, \dots \vec{e_m})$ then $\psi(\gamma) = v_1(\vec{e_m})$. For $y \in V$, let $\psi^{-1}(v)$ be the set of vertices mapped to v by ψ. Define the set of configurations that have at most one particle per fiber by

$$\Omega_{\mathcal{T}} = \left\{ \zeta \in \{0,1\}^{\mathcal{V}} : \sum_{y \in \phi^{-1}(v)} \zeta(y) \in \{0,1\} \right\}.$$

Define the projection $\pi : \Omega_{\mathcal{T}} \to \{0,1\}^{\mathcal{V}}$ by

$$[\pi(\zeta)](v) = \sum_{y \in \phi^{-1})v)} \zeta(y).$$

In the usual way we will identify configurations $\zeta \in \{0,1\}^{\mathcal{T}}$ with the set of sites that are in state 1, $\{y \in \mathcal{T} : \zeta(v) = 1\}$.

Coupling. Suppose we have $A \subset V$ and $B \subset \mathcal{T}$ with $B \supset A$. and let A_t and B_t be the contact process with these initial states. To couple the two processes we let $\kappa : A_t \to B_t$ so that $\pi(\kappa(a)) = a$. κ can change over time but there is no reason to let it change between jumps of the process.

We use a graphical presentation to couple the two processes. We use the deaths associated with particles in $\mathcal{T}$ and the births associated with particles on G and use independent Poisson processes to complete the graphical representation on $\mathcal{T}$.

(i) If a death occurs at a b which is $= \kappa(a)$ for some a then we also kill the particle at a.

(ii) Suppose a birth occurs from a to a neighbor a'. If the corresponding neighbor of $b = \kappa(a)$ which we will call b' is vacant then b will give birth onto b' and we will let $\kappa(a') = b'$. If b' is already occupied then since there are no particles in $\phi^{-1}(a')$ we can again let $\kappa(a') = b'$.

The coupling shows that if $A_t \neq \emptyset$ then $B_t \neq \emptyset$ which proves the desired result. $\square$

To complete the proof we use their formula (4.1)

Lemma 5.3.6. *For any $\lambda < \lambda_1(\mathbb{T}^d)$ there are constants c_0, C_0 so that*

$$E|\xi_t^0| \leq C_0 e^{-c_0 t}.$$

They prove this result by combining several facts from Section I.4 of Liggett(1999). The details are not useful for other developments here, so we refer the reader to the paper for details. The additivity property of the contact process implies that for $B \subset \mathbb{T}^d$

$$E|\xi_t^B| \leq |B| \cdot C_0 e^{-c_0 t}.$$

which implies the desired conclusion.

Proof of part (b) of Theorem 5.3.4

We begin with a classical large deviations result for the binomial.

MV3_1 **Lemma 5.3.7.** *(Lemma 3.1.) For every $\delta \geq 0$*

$$P(\textit{binomial}(m,p) \geq (p+\delta)m) \leq e^{-m\psi_p(\delta)},$$

where $\psi_p(\delta) = \sup_\lambda[\lambda(p+\delta) - \log(1-p+pe^\lambda)]$

$$= (p+\delta)\log\left(\frac{p+\delta}{p}\right) + (1-p-\delta)\log\left(\frac{1-p-\delta}{1-p}\right) \quad (5.3.1)$$ MVL2.2

The next ingredient is a lemma from Salzano and Schonmann (1998). Throughout this section vertices have degree $d \geq 3$

MV3_2 **Lemma 5.3.8.** *(Lemma 3.2) For every $\lambda > \lambda_1(\mathbb{T}^d)$ there are constants S, $p_0 > 0$ and $\alpha > 1$ so that*

$$P_{\mathbb{T},\lambda}(|\{x \in \xi^0_{tS} : \textit{dist}(0,x) = \ell\}| \geq \alpha^\ell) \leq p_0.$$

which they generalize as follows

MV3_3 **Lemma 5.3.9.** *(Lemma 3.3.) For every $\lambda > \lambda_1(\mathbb{T}^d)$ there are R, $\sigma > 0$ so that for every ℓ large enough the following holds. For any graph G with vertices x, y so that $\textit{dist}(x,y) \leq r$ and (y, G) embeds $(0, \mathbb{T}^d_\ell)$*

$$P_{G,\lambda}\left(|\xi^x_{R\ell}| \geq \alpha^\ell\right) \geq \sigma.$$

Proof. This follows from Lemma 5.3.8 and the next two facts

$$P_{G,\lambda}(\xi^x_t(x) = 1) \geq e^{-t}$$

$$P_{G,\lambda}\left(\xi^x_{\text{dist(x,y)}}(y) = 1\right) \geq (e^{-2}(1-e^{-\lambda}))^{\text{dist}(x,y)}$$

The first estimate is trivial. In order for x to be 1 at time t it is enough that the initial 1 at x survives for time t. For the second result suppose $x_0 = x, x_1, \ldots x_k = y$ is a path of length k from x to y. We have $\xi^x_k(y) = 1$ if in each time interval $[j-1.j]$ with $1 \leq j \leq k$ there is no death at $j-1$ or j and there is a birth from $x_{j-1} \to x_j$. □

We say that a set of vertices $W \subset V_n$ is **(ℓ,r)-regenerative** if there is a family G'_v, $v \in W$ of subgraphs of G_n that are pairwise disjoint have $v \in G'_v$ and there is x so that

the distance in G'_v between x and v is r and (x, G'_v) embeds $(0, \mathbb{T}^d_\ell)$

MV3_4 **Theorem 5.3.10.** *(Theorem 3.4.) For k and r sufficiently large and for every ℓ there is an ϵ_0 so that for all $\epsilon \leq \epsilon_0$ the following holds with high probability: from every $W \subset V_n$ of cardinality at least $k\epsilon n$ one can extract a (ℓ,r)-regenerative set of cardinality at least ϵn.*

Proof of (b) of Theorem 5.3.4 assuming Theorem 5.3.10. Fix $\lambda > \lambda_1(\mathbb{T}^d)$ and choose constants as follows

(i) fix r and k large, as required by Theorem **?? ref?**
(ii) let α, R, σ correspond to λ and r as in Lemma 5.3.9.
(iii) take ℓ large enough, as required by Lemma 5.3.9 and so that $\alpha^\ell > 2k/\sigma$
(iv) take $\epsilon < \epsilon_0$ where ϵ_0 corresponds to k, r, ℓ as in Theorem 5.3.10.

Assume G_n has the property stated in Theorem 5.3.10 for every $W \subset V_n$ with $|W| \geq k\epsilon n$ one can extract a (ℓ,r)-regenerative set $W' = \{v_1, \dots v_{\epsilon n}\}$ of cardinality at least ϵn. Let $G'_{v_1} \dots G'_{v_\epsilon n}$ be the disjoint subgraphs so that (x_i, G'_{v_i}) embeds $\mathbb{T}^d_\ell$. We will now show that

$$P_{G_n}(|\xi^W_{R\ell}| \geq k\epsilon n) \geq 1 - e^{-cn}, \qquad (5.3.2)$$ suffT12

which when iterated gives (b).

For each i Let ζ^i_t be the contact process on G'_{v_i} starting with only v_i infected. Clearly

$$\xi^W_t \supset \xi^{W'}_t \supset \cup_{i=1}^{\epsilon n} \zeta^i_t,$$

and the ζ^i_t are independent. Let $E_i = \{|\zeta^i_{R\ell}| \geq \alpha^\ell\}$. Lemma 2.3 implies that $P_{G_n}(E_i) \geq \sigma$. Therefore by the large deviations result in Lemma 2.1

$$P_{G_n}\left(\sum_{i=1}^{\epsilon n} 1_{E_i} \geq \epsilon n \sigma/2\right). \geq 1 - \exp(-c(\epsilon, \sigma)n)$$

Finally if the event above occurs we have

$$|\xi^W_{R\ell}| \geq \alpha^\ell \cdot \epsilon n \sigma/2 > k\epsilon n,$$

by the assumption in (iii). For more details see Section 3 of Mourrat and Valesin (2013) □

5.3.2 Work of Schapira and Valesin

They extends the work of Mountford, Mourrat, Valesin and Yao (2012), or MMVY for short, in two ways. They proved without any restriction in the graph G that if the infection rate is larger than the critical value of the one-dimensional contact process, $\lambda_c(\mathbb{Z})$ then the extinction timeτ_G starting from all sites occupied grows faster than $\exp(|G|/(\log|G|)^\kappa)$ for any $\kappa > 1$. Also for general graphs they showed that the extinction time divided by its expectation converges to the exponential distribution with mean 1. Even though this is a subsection of a section titled bounded degrees, they do not make that assumption. To state their results formally using the numbers in their paper.

Theorem 1.2. For any $\lambda > \lambda_c(\mathbb{Z})$ there is a constant c_ϵ such that for any connected graph G that has at least 2 vertices

$$E_\lambda[\tau_G] \geq \exp\left(\frac{c_\epsilon|G|}{|G|^{1+\epsilon}}\right).$$

Clearly there is no hope when there is only one vertex since in that case it is not possible to give birth.

Theorem 1.3. For any $\lambda > \lambda_c(\mathbb{Z})$ and any sequence of graphs G_n with $|G_n| \to \infty$

$$\frac{\tau_{G_n}}{E\tau_{G_n}} \Rightarrow \text{exponential}(1).$$

There are a large number of details involved in carrying out these proofs, so we will content ourselves to describe the main new ideas

Proposition 2.7. There is a $c_{comp} > 0$ so that for any $n \geq 2$ and any tree G with n vertices

$$P\left(\xi_t^A \neq \emptyset, \xi_t^A \neq \xi_t^1\right) \leq \exp\left(-c_{comp}\left\lfloor \frac{t}{n(\log n)^3} \right\rfloor\right),$$

for all $t > 0$ and $A \neq \emptyset$.

Once this result is established Theorem 1.3 follows fairly easily, since it is a key step in showing that limits of $\tau_{G_n}/E\tau_{G_n}$ have the lack of memory property, and hence must be exponential(1). This idea already appeared as Lemma A.1 in MMVY.

Proposition 2.7 is an immediate consequence of

Lemma 2.8 There is a $c_1 < 1$ so that for all G with $n \geq 2$ vertices and $t \geq n(\log n)^3$

$$P\left(\xi_t^A \neq \emptyset, \xi_t^A \neq \xi_t^1\right) < c_1.$$

Proposition 2.7 leads to the second main ingredient.

Proposition 2.9. There exists a constant c_{split} so that for any tree G containing N connected and disjoint subtrees $G_1, \ldots G_N$

$$E[\tau_G] \geq \frac{c_{split}}{(|G|^3)^{N+1}} \prod_{i=1}^N E[\tau_{G_i}].$$

This result is interesting in its own right. The authors use it to give a simple proof of part (b) of Theorem 5.3.4 without the assumption of bounded degree. For more information about the proof the reqder will have to consult Schapira and Valesin (2015). In their (2018) they use these methods to analyze the behavior of eight different equences of graphs G_n.

5.4 Erdős-Rényi et al

In the last section we considered bounded degree distributions, i.e., treees and random regular graphs. In next several sections we will consider random graphs with degree distributions that are power-law, subexpoentital, and finally with exponential tails. The Erdős-Rényi graphs we consider are unbounded but the tails decay faster than exponential. As in the previous section we will be concerned with showing that if the contact process is sufficiently supercritical tnen starting from all sites occupied the process survives for time $\exp(cn)$, which as remarked earlier, see (5.2.8) is the best possible result for graphs with n vertices and $\leq Bn$ vertices.

Proving survival for time $\exp(cn)$ in the Erdős-Rényi case is easy thanks to Theorem 1.4.7 which shows that if the mean degree is $\lambda > 1$ then there is a path with length $\geq n(\sigma_\lambda - Li_2(\sigma_\lambda))$ where σ_λ is the suvival probability and Li_2 is the dilogarithm function. Comparing the contact process on the path with the contact process on an interval, see Theorem 5.3.1, it follows that if we start with all sites occupied then survival occur for time $\geq \exp(cn)$.

A new approach due to Cator and Don (2021)

The method relies on uniformly bounding the infection rate from below for all sets with a fixed number of vertices k. Once the minimum infection rate for sets of size k has been estimated and it is $\geq \rho k$ for a range of values, a simple comparison with a birth and death chain will show exponential growth of the infection time.

To illustrate this method we consider the Erdös-Renyi(n, p) graph with $np = \sigma$. Their method will clearly fail if $n\sigma < 1$, but as they explain it will also fail if the fraction of sites in the giant component is $< 1/2$, because in that case there will be sets of size $n/2$ that have no neighbors (take the complement of the giant component). The probability ρ that a vertex does not belong to the giant component satisfies $\rho = \exp(-\sigma(1-\rho))$ so recalling that the generating function $\phi(z) = \exp(\sigma(1-z)$ is convex and has $\phi'(1) = \sigma$ we see that $\rho > 1/2$ if $1/2 < \exp(-\sigma/2)$, i.e., $\sigma < 2\log 2 = 1.3862$. For technical reasons that will become clear later they have to choose $\sigma > 4\log 2$.

Large deviations for the number of connections to a set

Suppose we have a graph with n vertices, i.e., $|V| = n$, and a set $S \subset V$ of size k. The number of potential connections from S to $V - S$ is $k(n-k)$ and each is independently present with probability σ/n so the number of links from from one fixed set S, $L_S = \text{binomial}(k(n-k), \sigma/n)$. The number of sets of vertices size k is $\binom{n}{k}$ so it is a simple exercise in large deviations to find a lower bound $M_{n,k}$ on the minimum number of connections from S to $V - S$ for a set S of size k.

Fix n and $k = \gamma n$, $p = \sigma/n$. L_S is binomial$(k(n-k), p)$. To begin we compute

$$EL_S = k(n-k)\sigma/n \quad \text{and}$$
$$E\exp(-\theta L_S) = \left(1 - \frac{\sigma}{n} + \frac{\sigma}{n}e^{-\theta}\right)^{k(n-k)} = \exp\left(-(\sigma/n)(1-e^{-\theta})k(n-k)\right).$$

Thus if $\theta > 0$, Markov's inequality implies

$$\exp(-\theta\rho EL_S)P(L_S \leq \rho E(L_S)) \leq \exp(-(\sigma(1-e^{-\theta})k(n-k)/n),$$

Filling in the formula for EL_S and rearranging

$$P(L_S \leq \rho E(L_S)) \leq \exp\left(-\frac{\sigma}{n}k(n-k)\cdot(-\theta\rho + 1 - e^{-\theta})\right).$$

The next step is to optimize this bound over θ. Differentiating gives

$$(-\theta\rho + 1 - e^{-\theta})' = -\rho + e^{-\theta},$$

so we let $\theta_0 = -\log \rho$. In this case

$$(-\theta_0\rho + 1 - e^{-\theta_0}) = \rho \log \rho + (1-\rho) \equiv G(\rho).$$

If we let $\gamma = k/n$ then we have

$$P(L_S \le \rho E(L_S)) \le \exp(-\sigma G(\rho)\gamma(1-\gamma)) \tag{5.4.1}$$ 3.21

This is (3.12) in Cator and Don (2021) which they prove by invoking the "Chernoff-Hoeffding inequality," and cite Hoeffding (1963).

Before we move on note that $G(1) = 0$, which is natural since it gives us $P(L_S \le EL_S)$ which by the central limit theorem has limit 1/2.

$$\text{As } \rho \to 0,\ G(\rho) \to 1. \tag{5.4.2}$$ G01

At first it may be surprising that $G(\rho) \not\to \infty$as $\rho \to 0$ but then you remember than if X has a Poisson(bn) distribution then $P(X = 0) = e^{-bn}$.

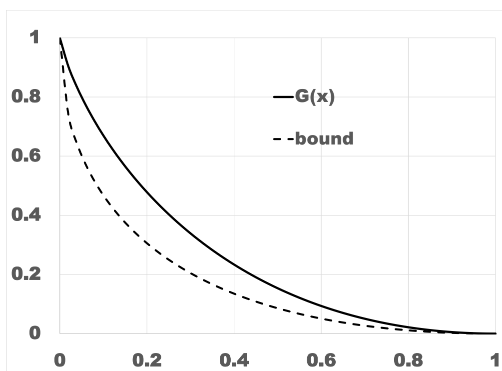

Figure 5.5: Picture proof of $G(\rho) > (1-\sqrt{\rho})^2$ fig:Gbdd

Later we will want to know that $G(\rho) > (1-\sqrt{\rho})^2$ for $0 < \rho < 1$. We content ourslves with a proof by picture.

Counting the number of sets of size k, $\gamma = k/n$

Lemma 5.4.1. *If $H(\gamma) = -\gamma\log(\gamma) - (1-\gamma)\log(1-\gamma)$ then for large n*

$$\binom{n}{\gamma n} \le e^{nH(\gamma)} \tag{5.4.3}$$ 3.22

Proof. To prove this, we use Stirling's formula $m! \sim m^m e^{-m}\sqrt{2\pi m}$ to conclude that (the e^{-m} terms in Stirling's formla cancel)

$$\begin{aligned}\binom{n}{\gamma n} &= \frac{n!}{(\gamma n)!((1-\gamma)n)!} \\ &\sim \frac{n^n}{(\gamma n)^{\gamma n}((1-\gamma)n)^{(1-\gamma)n}} \cdot \frac{\sqrt{2\pi n}}{\sqrt{2\pi\gamma n}\sqrt{2\pi(1-\gamma)n}} \\ &= \frac{1}{\gamma^{\gamma n}(1-\gamma)^{(1-\gamma)n}} \cdot \frac{1}{\sqrt{2\pi\gamma(1-\gamma)n}}\end{aligned}$$

The last term is < 1 for large n which completes the proof. □

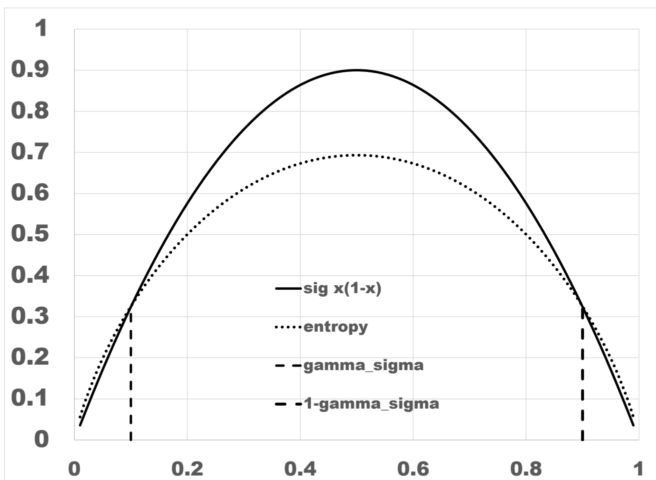

Figure 5.6: Comparison of $0.9\gamma(1-\gamma)$ and the entropy $H(\gamma)$. fig:mftphi

Combining (5.4.1) and (5.4.3) we have

$$P\{\exists S : |S| = k, L_S \le \rho E L_S\} \le e^{nH(\gamma)} \cdot Ce^{-\sigma G(\rho)\gamma(1-\gamma)}$$

The exponent is negative if

$$G(\rho) \cdot \sigma\gamma(1-\gamma) > H(\gamma) \tag{5.4.4}$$ negexp

The functions $\sigma\gamma(1-\gamma)$ and $H(\gamma)$ are symmetric about 1/2. It is easy to see that $\gamma(1-\gamma)$ is maximized at 1/2 where the value is 1/4. A little caclulus shows that $H(\gamma)$ is maximized at 1/2 where the value is $\log 2$. If we suppose that $\sigma > 4\log 2$ then $\gamma(1-\gamma) > H(\gamma)$ when $\gamma = 1/2$ so using (5.4.2) it is possible to pick $\rho > 1$ so that (5.4.4) holds at $\lambda = 1/2$

If we consider sets of size $k = n/2$ and know that $L_S \ge \rho E L_S$ then the birth rate $b_k = \lambda\rho(\sigma/n)k(n-k) = \lambda\rho\sigma n/4$

death rate $d_k = k = n/2$

Note that once we are able to choose $\rho > 0$ then we can get $b_k > d_k$ by choosing λ large.

In our simple calculation once we knoe we can choose $\rho > 0$ at 1/2 then we can widen this to $(1/2-\delta, 1/2+\delta)$ for some $\delta > 0$ and we can get the exponential survial result by coupling the contact to a birth and death chain and using calculations at the end of Section 5.2. Cator and Don (2003) have a better way of generating an interval of values with $\rho > 0$.

Lemma 3.2. Fix $\sigma > 4\log 2$ and consider a Erdős-Rényi graphs with n vertices and edge probabilities σ/n. Choose

$$\gamma_\sigma = \frac{1}{2} - \sqrt{\frac{1}{4} - \frac{\log 2}{\sigma}} \in (0, 1/2) \quad \text{and} \quad \alpha_\sigma = \frac{2\log(1 - 2\sqrt{(log2)/\sigma}}{\log(1/4 - (log2)/\sigma)} \in (0, 2)$$

For $\gamma_\sigma < \gamma < 1 - \gamma_\sigma$ let

$$\rho(\gamma) = (\gamma(1-\gamma) - (\log 2)/\sigma)^{\alpha_\sigma} \in (0,1) \quad \text{and} \quad M_{n,k} = \rho(k/n)\cdot\sigma k(n-k)$$

Then with high probability $L_S \geq M_{n,k}$ for all $\gamma_\sigma n < k < (1-\gamma_\sigma)n$ and all $S \in \mathcal{S}_k$.

Proof. A little calculus shows that for $\gamma_\sigma < (1-\gamma_\sigma)$

$$\alpha_\sigma \geq \frac{2\log\left(1 - 2\sqrt{\frac{H(\gamma)}{\sigma\gamma(1-\gamma)}}\right)}{\log(\gamma/(1-\gamma) - (\log 2)/\sigma} \in (0,2) \tag{5.4.5}$$ alphasineq

Since $G(\rho) > (1-\sqrt{\rho})^2$ we have

$$\sigma G(\rho)\gamma(1-\gamma) > \sigma\left(1 - \sqrt{(\gamma(1-\gamma) - (\log 2)/\sigma)^{\alpha_\sigma}}\right)^2 \gamma(1-\gamma)$$
$$\geq \sigma\left(1 - \left(1 - \sqrt{\frac{H(\gamma)}{\sigma\gamma(1-\gamma)}}\right)\right)^2 \gamma(1-\gamma) = H(\gamma)$$

where the second inequality follows from (5.4.5). It follows that

$$\sup_{\gamma_\sigma<\gamma<1-\gamma_\sigma)} H(\gamma) - \sigma G(\rho)\gamma(1-\gamma) < 0$$

which gives the desired result □

Using this result with results from their Section 2 on birth and death proicesses, they are able to prove

Theorem 3.2. Consider a Erdős-Rényi graphs $G_n$with n vertices and edge probabilities σ/n where $\sigma > 4\log 2$. There are functions $\lambda_0(\sigma) = (1+o(1))/\sigma$ and $\epsilon(\sigma) = o(1)$. so that if $\lambda > \lambda_0(\sigma)$ then T_n the survial time of contact process starting from all sites occupied satisfies

$$\frac{1}{n}\log ET_n \geq (1 - \epsilon(\sigma)\log(\lambda\sigma) + \frac{1-\epsilon(\sigma}{\lambda\sigma} - 1$$

with high probability.

5.5 Power-law random graphs

:CPpowerlaw

Pastor-Satorras and Vespigniani who we will abbreviate PSV (2001a, 2001b, 2002) have made an extensive study of the contact process on "scale-free" random networks using mean-field methods. For this and many other related results see the survey paper by Pastor-Satorras, Castellano, van Meighem, and Vespigniani (2015).

Mean-field theory. To be be precise we will use what is called **degree based mean-field theory**. The results here are from PSV (2001b) although the arguments will be written differently. Let $\rho_k(t)$ denote the fraction of vertices of degree k that are infected at time t, and $\theta(\lambda)$ be the probability that a given link points to an infected vertex. If we make the assumption that there is no correlation between the degree of a site and the state of the vertex pointed to then

$$\frac{d\rho_k(t)}{dt} = -\rho_k(t) + \lambda k[1-\rho_k(t)]\theta(\lambda).$$

This will turn out to be a bad assumption but it is needed to conclude that the equilibrium frequency ρ_k satisfies

$$0 = -\rho_k + \lambda k[1-\rho_k]\theta(\lambda) \tag{5.5.1}$$ MFeq

Solving and writing $\theta(\lambda)$ simply as θ we have

$$\rho_k = \frac{k\lambda\theta}{1+k\lambda\theta}$$

Suppose p_k is the degree distribution in the graph. The probability that a given link points to a vertex of degree k is the size-biased degree distribution $q_k = kp_k/\mu$ where $\mu = \sum_j jp_j$, so we have the following self-consistent equation for θ:

$$\theta = \sum_k q_k\rho_k = \sum_k q_k \frac{k\lambda\theta}{1+k\lambda\theta} \tag{5.5.2}$$ thetaeq

Once θ is computed we can compute the fraction of occupied sites from

$$\rho = \sum_k p_k \frac{k\lambda\theta}{1+k\lambda\theta} \tag{5.5.3}$$ BArho

To reduce the problem to computing θ we begin by noting that in all of the examples we will consder $\mu = \sum_k kpk < \infty$. In examples the critical value λ_c may be 0 or positive, but in all cases $\theta(\lambda) \downarrow 0$ as $\lambda \to \lambda_c$. Using the dominated convergence theorem

$$\rho(\lambda) = \lambda\theta \sum_k kp_k \frac{1}{1+k\lambda\theta} \sim \lambda\theta(\lambda)\mu \tag{5.5.4}$$ thtorho

Analysis of (5.5.2), which we are about to describe in some detail, suggests the following conjectures about the contact process on power law graph with degree distribution $p_k \sim Ck^{-\alpha}$. Here β is the critical exponent for the equilibrium density $\rho(\lambda) \approx (\lambda-\lambda_c)^\beta$ as $\lambda \downarrow \lambda_c$.

- If $\alpha \le 3$ then $\lambda_c = 0$
- If $3 < \alpha < 4$, $\lambda_c > 0$ but the critical exponent $\beta > 1$
- If $\alpha > 4$ then $\lambda_c > 0$ and $\rho(\lambda) \sim C(\lambda - \lambda_c)^1$ so $\beta = 1$.

In Section 1.8 we defined β and other exponents associated iwith the phase transition in Erdős-Rényi graphs. In Section 2.7 we computed the exponent β associated with the percolation phase transition in power-law graphs generated by the configuration model. The results presented there in the table after Theorem 2.7.1 are closely related to the ones we will find here. See Figure 5.3.

The results depend on the power α in the degree distribution so our discussion s organized by its value. For reasons that will become clear in the discussion we have abandoned our usual γ and will denote the power law by α. We begin in the middle of the range of values.

$\alpha = 3$

Since we are following in the footsteps of physicists, we will use the continuous approximation $p(x) = 2/x^3$ for $x \ge 1$, and enjoy the fact that it simplifies computations. The size biased distribution has $q(x) = 1/x^2$ for $x \ge 1$ and (5.5.2) becomes

$$\begin{aligned}\theta &= \int_1^\infty \frac{1}{x} \cdot \frac{\lambda\theta}{1+\lambda\theta x}\, dx = \int_1^\infty \frac{\lambda\theta}{x} + \frac{\lambda\theta}{x}\left(\frac{1}{1+\lambda\theta x} - 1\right) dx \\ &= \int_1^\infty \frac{\lambda\theta}{x} - \frac{(\lambda\theta)^2}{1+\lambda\theta x}\, dx\end{aligned}$$

The two parts of the last integrand are not integrable separately, but if we replace the upper limit of ∞ by M the integral is

$$\begin{aligned}&\lambda\theta \log M - \lambda\theta\{\log(1+\lambda\theta M) - \log(1+\lambda\theta)\} \\ &= -\lambda\theta\log(\lambda\theta + 1/M) + \lambda\theta\log(1+\lambda\theta)\end{aligned}$$

so letting $M \to \infty$ the integral is

$$-\lambda\theta\big[(\log(\lambda\theta) - \log(1+\lambda\theta)\big] = \lambda\theta\log\left(1 + \frac{1}{\lambda\theta}\right)$$

The equation we want to solve is $1 = \lambda\log(1 + 1/\lambda\theta)$. Dividing by λ and exponentiating

$$e^{1/\lambda} = 1 + \frac{1}{\lambda\theta}$$

Solving for θ now we have

$$\theta(\lambda) = \frac{1}{\lambda(e^{1/\lambda} - 1)} = (1/\lambda)e^{-1/\lambda}(1 - e^{-1/\lambda})^{-1} \tag{5.5.5}$$ BAtheta

Using (5.5.4) and dropping constants

$$\rho(\lambda) \sim \lambda\theta(\lambda) \approx e^{-1/\lambda}$$

Notice that $\rho(\lambda) \to 0$ exponentially fast. This agrees with the result for percolaton on graphs but with $p_k \sim Ck^{-3}$ in Section 2.7 but as we will see later in Figure 5.3 it is not accurate for the contact process.

$2 < \alpha < 3$

In this and future examples, we will let

the degree distribution be $p(x) = (1+\gamma)x^{-2-\gamma}$ for $x \geq 1$
so the size biased distributon is $q(x) = \gamma x^{-1-\gamma}$.

In the new notation $2 < \alpha < 3$ is $0 < \gamma < 1$ and (5.5.2) becomes

$$1 = \int_1^\infty \frac{\gamma}{x^\gamma} \cdot \frac{\lambda}{1+\lambda\theta x}\, dx \equiv F(\lambda, \theta)$$

$\theta \to F(\lambda, \theta)$ a decreasing function of θ that is ∞ when $\theta = 0$ and $\to 0$ when $\theta \to \infty$,so we know there is a unique solution. Changing variables $x = u/\lambda\theta$, $dx = du/(\lambda\theta)$ we have

$$1 = \lambda^\gamma \theta^{\gamma-1} \int_{\lambda\theta}^\infty \gamma u^{-\gamma} \frac{1}{1+u}\, du$$

Since $\gamma < 1$ the integral on the right has a limit c_γ as $\lambda\theta \to 0$. Rearranging we have

$$\theta \sim C\lambda^{\gamma/(1-\gamma)} \tag{5.5.6}$$ smexpth

Using (5.5.4) the fraction of occupied sites

$$\rho(\lambda) \sim C'\lambda^{1/(1-\gamma)} \tag{5.5.7}$$ smexprho

$\alpha > 3$

In the new notation this case is $\gamma > 1$ and (5.5.2) becomes

$$1 = \int_1^\infty \frac{\gamma}{x^\gamma} \cdot \frac{\lambda}{1+\lambda\theta x}\, dx \tag{5.5.8}$$ thetaeq3

However, now the integral converges when $\theta = 0$, so for a solution to exist we must have

$$F(\lambda, 0) = \int_1^\infty \frac{\lambda\gamma}{x^\gamma}\, dx > 1 \quad \text{or} \quad \text{or} \quad \lambda > \lambda_c = 1 \Big/ \int_1^\infty \frac{\gamma}{x^\gamma}\, dx = \frac{\gamma - 1}{\gamma}$$

For fixed $\lambda > \lambda_c$ we want to solve $F(\lambda, \theta(\lambda)) = 1$. If $\lambda > \lambda_c$, $F(\lambda, 0) = \lambda/\lambda_c > 1$. To find the point where $F(\lambda, \theta)$ crosses 1 we begin by noting that

$$\frac{\partial F}{\partial \theta} = -\int_1^\infty \frac{\gamma}{x^\gamma} \frac{\lambda^2 x}{(1+\lambda\theta x)^2}\, dx. \tag{5.5.9}$$ Fderiv

When $\gamma \le 2$, $\partial F/\partial\theta \to \infty$ as $\theta \to 0$ so we will begin with the case $\gamma > 2$ where

$$\frac{\partial F}{\partial \theta}(0) = -b_{\gamma,\lambda}$$

so we have

$$1 - \frac{\lambda}{\lambda_c} = F(\lambda, \theta(\lambda)) - F(\lambda, 0) \sim -b_{\gamma,\lambda_c}\theta(\lambda)$$

and it follows that

$$\theta(\lambda) \sim \frac{\lambda - \lambda_c}{\lambda_c b_{\gamma,\lambda_c}} \tag{5.5.10}$$

Using (5.5.4) we conclude

$$\rho(\lambda) \sim C(\lambda - \lambda_c) \quad \text{as } \lambda \to \lambda_c. \tag{5.5.11}$$

Turning now to $\gamma < 2$, changing variables $y/\theta = x$ (5.5.9) becomes

$$-\int_\theta^\infty \frac{\gamma\theta^\gamma}{y^\gamma} \frac{\lambda^2 y/\theta}{(1+\lambda y)^2} \frac{dy}{\theta} \sim -\theta^{\gamma-2} \int_0^\infty \frac{\gamma}{y^{\gamma-1}} \frac{\lambda^2}{(1+\lambda y)^2}\, dy$$

Writing $c_{\gamma,\lambda}$ for the integral (which is finite) and integrating

$$1 - \frac{\lambda}{\lambda_c} = F(\lambda, \theta(\lambda)) - F(\lambda, 0) \sim -c_{\gamma,\lambda}\theta^{\gamma-1}/(\gamma-1)$$

Rearranging

$$\theta(\lambda) \sim C(\lambda - \lambda_c)^{1/(\gamma-1)}$$

Using(5.5.4) we conclude the for $3 < \lambda < 4$

$$\rho(\lambda) \sim C(\lambda - \lambda_c) \quad \text{as } \lambda \to \lambda_c. \tag{5.5.12}$$

Rigorous results

The first result about the long time survival of the contact process was proved by Berger, Borgs, Chayes, and Saberi in 2005. They considered the preferential attachment model which has a power law with $\alpha = 3$, so when they proved that $\lambda_c = 0$ they confirmed the physicists' prediction. Chatterjee and Durrett showed in 2009 that $\lambda_c > 0$ is not correct when $\alpha > 3$.

CDpower **Theorem 5.5.1.** *Consider a graph G_n with n vertices generated by the configuration model with $P(d_i = k) \sim Ck^{-\alpha}$ with $\alpha > 3$ and $P(d_i \le 2) = 0$. Let ξ_t^1, $t \ge 0$ denote the contact process on G_n starting from all sites occupied. Then for any $\lambda > 0$ there is a positive constant $p(\lambda) > 0$ so that for any $\delta > 0$*

$$\inf_{t \le \exp(n^{1-\delta}} P\left(\xi_t^1|/n \le p(\lambda)\right) \to 1 \qquad \text{as } n \to \infty.$$

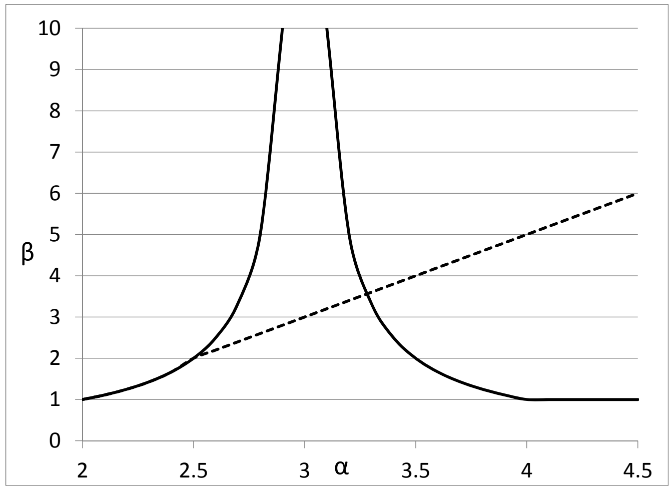

Figure 5.7: Mean field critical exponents (solid line) versus rigorous results (dashed line) given in (5.5.13) as α varies from 2 to 4.5. fig:crexp

Sections 5.4 and 5.5 are devoted to the proof of this result.

In 2013 Mountford, Valesin, and Yao extended the results of Chatterjee and Durrett to include $2 < \alpha \le 3$ and proved upper and lower bounds that had the same dependence on λ but different constants, showing that

$$\rho(\lambda) \sim \begin{cases} \lambda^{1/(3-\alpha)} & 2 < a \le 5/2 \\ \lambda^{2\alpha-3} \log^{2-\alpha}(1/\lambda) & 5/2 < \alpha \le 3 \\ \lambda^{2\alpha-3} \log^{4-2\alpha}(1/\lambda) & 3 < \alpha \end{cases} \tag{5.5.13}$$ cvcon

The result for $2 < \alpha \le 5/2$ agrees with the mean-field calculations quoted above but that formula is claimed to hold for $2 < \alpha < 3$. Figure 5.7 gives a visual comparison of the mean-field and rigorous results for critical exponents. For more about why the change occurs at 5/2 see the 2013 paper cited above. Three years later, Mountford, Mourrat, Valesin, and Yao (2016) showed that for all $\lambda > 0$, there exists some $c > 0$ so that the survival time $\ge e^{cn}$ with high probability.

5.6 Results for the star graph

sec:star

Let G_k be a star graph with center 0 and leaves $1, 2, \dots, k$ and let ξ_t be set of vertices infected in the contact process at time t. Write the state ξ_t as (i, j) where i is the number of infected leaves and $j = 1$ if the center is infected and $j = 0$ otherwise. We write $P_{i,j}$ for the law of the process starting from (i, j).

Here, following the approach in Chatterjee and Durrett (2009), we will reduce to a discrete time one dimensional chain, and we will only look at times when $j = 1$. When the state is $(i, 0)$ with $i > 0$, the next event will occur after exponential time with mean $1/(i\lambda + i)$. The probability that it will be the reinfection of the center is $\lambda/(\lambda + 1)$. The probability it will be the recovery of a leaf is $1/(\lambda + 1)$. Thus, the number of leaf infections N that will be lost while the center is healthy has a shifted geometric distribution with success probability $\lambda/(\lambda + 1)$, i.e.,

$$P(N = j) = \left(\frac{1}{\lambda+1}\right)^j \cdot \frac{\lambda}{\lambda+1} \quad \text{for } j \geq 0.$$

Note that since this version of the geometric counts the number of failures before the first success

$$EN = \frac{\lambda+1}{\lambda} - 1 = \frac{1}{\lambda}.$$

The next step is to modify the chain so that the infection rate is 0 when the number of infected leaves is at least

$$L = pk \quad \text{where} \quad p = \lambda/(1 + 2\lambda). \tag{5.6.1}$$ Ldef

To explain the choice of p note that the number of infected leaves in the modified chain is always $\leq pk$ and the number of uninfected leaves is $\geq (1 - p)k$. Thus if we look at the embedded discrete time process for the contact process on the star and only look at times when the center is infected, the process dominates Y_n where

jump	with prob
$Y_n \to Y_n - 1$	pk/D
$Y_n \to \min\{Y_n + 1, pk\}$	$\lambda(1-p)k/D$
$Y_n \to Y_n - N$	$1/D$

Here N is independent of Y_n and the denominator

$$D = pk + \lambda(1 - p)k + 1 \leq k + \lambda k + 1 \leq (2 + \lambda)k.$$

The fact that Y_n has a reflecting barrier at pk will simplify computations. We will use the process to lower bound survival times.

super **Lemma 5.6.1.** *Let $L = pk$ where $p = \lambda/(1+2\lambda)$. Let $e^\theta = 1/(1+\lambda/2)$. If k is large enough $e^{\theta Y_n}$ is a supermartingale while $Y_n \in (0, pk)$.*

Proof. We begin by noting that

$$E(\exp(\theta Y_{n+1}) - \exp(\theta Y_n)|Y_n = y) = e^{\theta y}(e^\theta - 1)\lambda(1-p)k/D \tag{5.6.2}$$ smeq

$$+e^{\theta y}(e^{-\theta} - 1)pk/D + \frac{e^{\theta y}}{D}\left[\sum_{j=0}^{\infty}\left(\frac{e^{-\theta}}{1+\lambda}\right)^j\left(\frac{\lambda}{1+\lambda}\right) - 1\right].$$

The term in square brackets is

$$\frac{1}{1-e^{-\theta}/(1+\lambda)}\cdot\frac{\lambda}{1+\lambda}-1=\frac{\lambda}{1+\lambda-e^{-\theta}}-1=\frac{e^{-\theta}-1}{1+\lambda-e^{-\theta}}\geq 0.$$

Note that this implies we must take $e^{-\theta}<1+\lambda$.

The first two terms are

$$\frac{e^{\theta y}k}{D}\left((e^{\theta}-1)\lambda(1-p)+(e^{-\theta}-1)p\right),$$

so we begin by solving

$$(e^{\theta}-1)\lambda(1-p)+(e^{-\theta}-1)p=0.$$

Rearranging and setting $x=e^{\theta}$ we want

$$x^2\lambda(1-p)-[\lambda(1-p)+p]x+p=0.$$

Factoring we have

$$(\lambda(1-p)x-p)(x-1)=0.$$

Since $p=\lambda/(1+2\lambda)$ the smaller root is

$$\frac{p}{\lambda(1-p)}=\frac{\lambda/(1+2\lambda)}{\lambda(1+\lambda)/(1+2\lambda)}=\frac{1}{1+\lambda}.$$

We let $e^{\theta}=1/(1+\lambda/2)\in(1/(1+\lambda),1)$ so that there is a $\delta>0$ with

$$e^{\theta}\lambda(1-p)+e^{-\theta}p=[\lambda(1-p)+p]-\delta$$

and hence

$$(e^{\theta}-1)\lambda(1-p)k+(e^{-\theta}-1)pk+\frac{e^{-\theta}-1}{1+\lambda-e^{-\theta}}=-\delta k+\frac{e^{-\theta}-1}{1+\lambda-e^{-\theta}}.$$

From this we see that if k is large enough $e^{\theta Y_n}$ is a supermartingale while $Y_n\in(0,pk)$. □

Let $T_\ell^-=\inf\{n:Y_n\leq\ell\}$ and let $T_m^+=\inf\{n:Y_n\geq m\}$.

exit **Lemma 5.6.2.** *Let $a,b\in(0,L)$. If $b<a$ then*

$$P_a(T_b^-<T_L^+)\leq(1+\lambda/2)^{b-a}.$$

Proof. To estimate the hitting probability let $\phi(x)=\exp(\theta x)$ where we take $e^{\theta}=1/(1+\lambda/2)$ and note that if $\tau=T_b^-\wedge T_L^+$ then $\phi(Y(t\wedge\tau))$ is a supermartingale. Let $q=P_a(T_b^-<T_L^+)$. Using the optional stopping theorem we have

$$q\phi(Y_b^-)+(1-q)\phi(Y_L^+)\leq\phi(a).$$

It is possible that $Y_b^-<b$. Note that since $\theta<0$, we have $\phi(x)\geq\phi(b)$ for $x\leq b$. Hence,

$$q\phi(b)+(1-q)\phi(L)\leq\phi(a).$$

Dropping the second term on the left, $q\leq\phi(a)/\phi(b)=(1+\lambda/2)^{b-a}$, which completes the proof. □

return **Lemma 5.6.3.** *If* $R_L = \inf\{n > T_{L-1}^- : Y_n = L\}$ *and* $b \in [0, L)$ *then*

$$P_L(T_b^- < R_L) \le (2+\lambda)(1+\lambda/2)^{b-L}.$$

Remark. Here, and in later lemmas, the computation of explict constants is somewhat annoying. However, when we consider asymptotics for critical values, λ will go to 0, so we will need to know how the constants depend on λ.

Proof. To compute the left-hand side we break things down according to the first jump. The definition of R_L allows us to ignore the attempted upward jumps that do nothing. Recall that $L = pk$. The jump is to $L-1$ with probability $pk/(pk+1)$ and to $L-j$ with probability $\frac{\lambda}{(1+\lambda)^{j+1}} \cdot \frac{1}{1+pk}$. In the first case the probability of returning to L before going below b is

$$\le (1+\lambda/2)^{b-(L-1)} = (1+\lambda/2) \cdot (1+\lambda/2)^{b-L}.$$

In the second we have to sum over the possible values of $L-j$. Using Lemma 5.6.2

$$\begin{aligned}
&\le (1+\lambda/2)^{b-L} \sum_{j=1}^{\infty} \frac{\lambda}{(1+\lambda)^{j+1}} (1+\lambda/2)^j + \frac{\lambda}{1+\lambda} P_L(T_b^- < R_L) \\
&\le (1+\lambda/2)^{b-L} \frac{\lambda}{\lambda+1} \cdot \sum_{j=0}^{\infty} \left(\frac{1+\lambda/2}{1+\lambda}\right)^j + \frac{\lambda}{1+\lambda} P_L(T_b^- < R_L) \\
&= 2(1+\lambda/2)^{b-L} + \frac{\lambda}{1+\lambda} P_L(T_b^- < R_L).
\end{aligned}$$

Noting that $\max\{2, 1+\lambda/2\} \le 2(1+\lambda/2) - \delta$ for some small $\delta < \lambda$, we have the following relation,

$$P_L(T_b^- < R_L) \le \frac{\lambda}{(1+\lambda)(1+pk)} P_L(T_b^- < R_L) + (2+\lambda-\delta)(1+\lambda/2)^{b-L}.$$

Hence for k sufficiently large, we have

$$P_L(T_b^- < R_L) \le (2+\lambda)(1+\lambda/2)^{b-L}.$$

□

life **Lemma 5.6.4.** *Let* $b = \epsilon L$ *and* $S = \frac{1}{(2+\lambda)2k}(1+\lambda/2)^{L(1-2\epsilon)}$

$$P_{L,1}\left(\inf_{t \le S} \xi_t \le b\right) \le (3+\lambda)(1+\lambda/2)^{-L\epsilon}.$$

Proof. Let $M = (1+\lambda/2)^{L(1-2\epsilon)}$. By Lemma 5.6.3 the probability that the chain fails to return M times to L before going below ϵL is

$$\le (2+\lambda)(1+\lambda/2)^{-L\epsilon}.$$

Using Chebyshev's inequality on the sum S_M of M exponentials with mean 1 (and hence variance 1),

$$P(S_M < M/2) \le 4/M.$$

When the number of infected leaves is $\le L$ maximum jump rate is $D \le (2+\lambda)k$ so

$$P\left(\frac{S_M}{(2+\lambda)k} \le \frac{(1+\lambda/2)^{L(1-2\epsilon)}}{2(2+\lambda)k}\right) \le 4(1+\lambda/2)^{-L(1-2\epsilon)}.$$

Adding up the error probabilities completes the proof. □

Up to this point we have shown that if a star has L infected leaves it will remain infected for a long time. To make this useful, we need estimates about what happens when the star starts with only the center infected.

ignite **Lemma 5.6.5.** *Let $\lambda > 0$ be fixed and $K = \lambda k^{1/3}$. Then for large k*

$$P_{0,1}(T_K^+ > T_{0,0}) \le 2\lambda k^{-1/3},$$
$$P_{K,1}(T_{0,0} < T_L^+) \le k^{-1/3},$$
$$E_{0,1}(T_L^+ | T_L^+ < T_{0,0}) \le 2/\lambda.$$

Proof. Clearly

$$P_{0,1}(T_K^+ < T_{0,0}) \ge \prod_{j=0}^{K-1} \frac{(k-j)\lambda}{1+(k-j)\lambda+j}$$

so subtracting the last inequality from $1 = \prod_{j=0}^{K-1} 1$ and using Lemma 3.4.3 from PTE5

$$P_{0,1}(T_K^+ > T_{0,0}) \le \sum_{j=0}^{K-1} \frac{1+j}{(k-j)\lambda} \le \frac{\lambda^2 k^{2/3}}{(k-\lambda k^{1/3})\lambda} \le 2\lambda k^{-1/3}.$$

For the second result we use the supermartingale $e^{\theta Y_n}$ from Lemma 5.7.2. If $q = P_{K,1}(T_{0,0} < T_L^+)$, using optional stopping theorem we have

$$q \cdot 1 + (1-q)e^{\theta L} \le e^{\theta K}.$$

Dropping the second term on the left,

$$q \le e^{\theta K} = (1+\lambda/2)^{-K} \le k^{-1/3}.$$

To bound the time we return to continuous time

jump	at rate
$Y_t \to Y_t - 1$	pk
$Y_t \to \min\{Y_t+1, pk\}$	$\lambda(1-p)k$
$Y_t \to Y_t - N$	1

Before time $V_L = T_{0,0} \wedge T_L^+$ the drift of Y_t is at least

$$\mu = \lambda(1-p)k - pk - 1/\lambda = \lambda pk - 1/\lambda \tag{5.6.3}$$ Ydrift

so $Y_t - \mu t$ is a submartingale. Stopping this martingale at the bounded stopping time $V_L \wedge t$

$$EY(V_L \wedge t) - \mu E(V_L \wedge t) \geq EY_0 \geq 0.$$

Since $EY(V_L \wedge t) \leq L$, it follows that

$$E(V_L \wedge t) \leq \frac{L}{\mu} = \frac{pk}{\lambda pk - 1/\lambda},$$

where $p = \lambda/(1+2\lambda)$, so if λ is fixed and k is large

$$E(V_L \wedge t) \leq 2/\lambda$$

which completes the proof. □

Combining Lemmas 5.6.4 and 5.6.5 we have the following

good **Lemma 5.6.6.** *Let A_t denote the number of infected leaves at time t and take S as in Lemma 5.6.4. Define $G = \{\inf_{k^{2/3} \leq t \leq S} |A_t| \geq \epsilon L\}$. If $\lambda > 0$ is fixed*

$$P_{0,1}(G) \geq 1 - C_\lambda k^{-1/3} \tag{5.6.4}$$ from0

for some constant C_λ.

When G occurs, we say **the star at 0 is good**.

5.7 Subexponential degree distributions

ec:CPsubexp

Given an offspring distribution p_k, we construct a Galton-Watson tree as follows. Starting with the root, each individual has k children with probability p_k. Pemantle (1992) has shown in his Theorem 3.2 that

Theorem 5.7.1. *There are constants c_2 and c_3 so that if $\mu > 1$ is the mean of the offspring distribution, then for any $k > 1$, if we let $r_k = \max\{2, c_2 \log(1/kp_k)/\mu\}$.*

$$\lambda_2 < c_3 \sqrt{r_k \log r_k \log(k)/k}. \tag{5.7.1}$$ Pemub

If the offspring distribution is a stretched exponential $p_k = c_\gamma \exp(-k^\gamma)$ with $\gamma < 1$ then $\log(1/kp_k) \sim k^\gamma$ and hence $\lambda_2 = 0$.

Huang and Durrett (2020a) extended the last result to subexponential distributions, which satisify

$$\limsup_{k\to\infty} (1/k) \log p_k = 0.$$

subexp **Theorem 5.7.2.** *If the offspring distribution p_k for a Galton-Watson tree is subexponential and has mean $\mu > 1$ then $\lambda_2 = 0$.*

This result was proved as a consequence of their study of the case in which degrees have a geometric distribution. $p_k = (1-p)^{k-1}p$ for $k \ge 1$. The goal of their paper was to prove that $\lambda_1 > 0$. The solution of that problem will be described in the next section. The upper bound on λ_1 is easy.

ubglam1 **Theorem 5.7.3.** *Under the assumptions of Theorem 5.7.2 $\lambda_1 \le p/(1-p)$.*

Proof. Modify the contact process so that births from a site can only occur on sites further from the root. Each vertex x will be occupied at most once. If x is occupied then it will give birth with probability $\lambda/(\lambda+1)$ onto each neighbor y. The birth events are not independent but that is not important. If we let Z_n be the number of sites at distance n that are ever occupied, Z_n is a branching process in which the offspring distribution has mean $\lambda/((\lambda+1)\cdot p)$ which is > 1 if $\lambda > p/(1-p)$. □

When $p_k = (1-p)^{k-1}p$, $\log(1/kp_k) \sim c_p k$, so (5.7.1) gives a finite upper bound on λ_2. However, the resulting bound is much worse than the following:

ubglam2 **Theorem 5.7.4.** *If $p_k = 2^{-k}$ for $k \ge 1$, then $\lambda_2 \le 2.5$.*

The proof works for a general geometric $p_k = (1-p)^{k-1}p$, $k \ge 1$. Huang and Durrett (2020a) could not get a nice formula for the upper bound as a function of p but the upper bounds can easily be computed numerically and graphed. These upper bounds are only interesting for small p. A Galton-Watson tree with $p_0 = 0$ and $p_1 < 1$ contains a copy of $\mathbb{Z}$ (start with a vertex with two children and follow their descendants) so using the bound on $\lambda_c(\mathbb{Z})$ proved in Liggett (1995) we conclude $\lambda_2 \le 2$ for all $0 < p < 1$.

Proof for $p_n = 2^{-n}$, $n \ge 1$. Our proof follows the outline of the proof of Theorem 3.2 in Pemantle (1992), see pages 2109–2110. We can suppose without loss of generality that the root has degree k. Otherwise examine the children of the root until we find one with degree k and apply the argument to the children of this vertex. There are two steps in the proof.

1. Push the infection to vertices at a distance $r = k$ that have degree k.

2. Bring the infection back to the root at time t.

To push the infection in either direction we use the following results.

transfer **Lemma 5.7.5.** *Let $v_0, v_1, \dots v_r$ be a path in a graph. Suppose that v_0 is infected at time 0 and that it is good in the sense of Lemma 5.6.6. Then there is a $\gamma > 0$ so that the probability that v_r will become infected by time $2r$ is*

$$\ge \left(\frac{\lambda}{\lambda+1}\right)^r (1 - \exp(-\gamma r)).$$

If $\epsilon > 0$ and we let $\hat\lambda = (1-\epsilon)\lambda/(\lambda+1)$ then for large r this probability is $\ge \hat\lambda^r$.

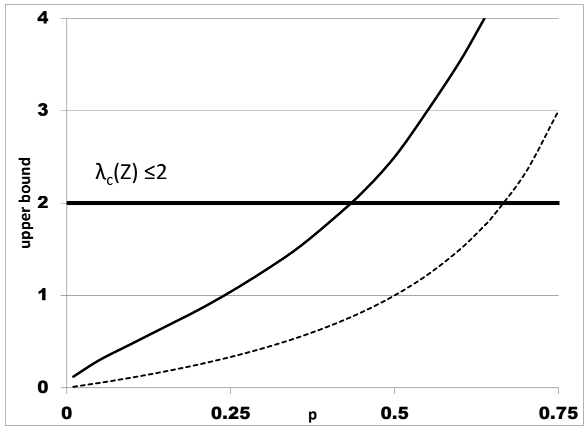

Figure 5.8: Upper bounds on λ_1 (dotted line) from Theorem 5.7.4 and on λ_2 (solid line) from (5.7.4) as a function of p for the geometric degree distribution. The horizontal line is the bound that comes from using the existence of a copy of $\mathbb{Z}$ in the graph. fig:lam2lb

Proof. The probability that v_{i-1} infects v_i before it is cured is $\lambda/(1+\lambda)$. When this transfer of infection occurs the amount of time is t_i, which is an independent exponential with rate $1+\lambda$. By large deviations for the exponential distribution $P(t_1 + \cdots + t_r > 2r) \le e^{-\gamma r}$ for some $\gamma > 0$. □

infect **Lemma 5.7.6.** *Run the contact process on a graph consisting of a star with k leaves, to which there has been added a single chain $v_1, \ldots v_r$ of length r where v_1 is a neighbor of 0, the center of the star. Suppose that at time 0 there are L infected leaves. For large r the probability that v_r will not be infected before time $T = m(2r+1)$ is*

$$\le (1 - \hat{\lambda}^r)^m.$$

Proof. Consider a sequence of times $t_i = (2r+1)i$ for $i \ge 1$. The center 0 may not be infected at time t_i but since the star at 0 is good the number of infected neighbors is $\ge \epsilon L$ and it will with high probability be infected by time $t_i + 1$. By Lemma 5.7.5 the probability v_r is successfully infected in $[t_i, t_{i+1})$ is $\ge \hat{\lambda}^r$ when 0 is good, even if we condition on the events up to time t_i. The desired result follows. □

To use the two lemmas to prove the main result we need the next weird but wonderful result, which is Lemma 2.4 from Pemantle (1992). Let $\varphi(x) = \sum_{n=0}^{\infty} p_n x^n$ be the generating function of the degree distribution. We will apply Lemma 5.7.7 to

$$f(t) = P(0 \in \xi_t^0) \ge p_k P(0 \in \xi_t^0 |\, 0 \text{ has at least } k \text{ children}).$$

magic **Lemma 5.7.7.** *Let H be any nondecreasing function on the nonnegative reals with $H(x) \geq x$ when $x \in [0, x_0]$. If f satisfies (i) $\inf_{0 \leq t \leq L} f(t) > 0$ and (ii) $f(t) \geq H(\inf_{0 \leq s \leq t-L} f(s))$ for $t \geq L$ some $L > 0$ then $\liminf_{t\to\infty} f(t) > 0$.*

Proof. For any t_0 and $\epsilon > 0$, (ii) implies that there is a decreasing sequence t_i with $t_{i+1} \leq t_i - L$ and $t_k < L$ for some k

$$f(t_i) \geq H(f(t_{i+1})) - \epsilon 2^{-i}.$$

If $f(t_i) < x_0$ for all $1 \leq i \leq k$ then

$$f(t_i) \geq f(t_{i+1}) - \epsilon 2^{-i}$$

and summing gives $f(t_0) > f(t_k) - \epsilon$ which gives the desired result. Suppose now that j is the smallest index with $f(t_j) > x_0$. If $j = 0$ we have $f(t_0) > x_0$. If $j = 1$ we have $f(t_0) \geq H(x_0)$. If $j \geq 2$ we have

$$f(t_0) \geq f(t_{j-1}) - \epsilon \geq H(x_0) - \epsilon$$

so in all cases we get the desired conclusion. □

Step 1. The mean of the offspring distribution is 2. Let Z_r be the number of vertices at distance r from 0 and let $v_r^1, \dots v_r^J$ be the subset of those that have exactly k children, where J is a random variable that represents the number of such vertices.

Since the root has degree k and $p_k = 2^{-k}$ if we set $r = k$

$$EJ \geq k\mu^{r-1} p_k = k/2,$$

where $\mu = 2$ is the mean offspring number.

If we condition on the value of $W = Z_r/(k\mu^{r-1})$ and let $\bar{J} = (J|W)$ be the conditional distribution of J given W then

$$\bar{J} = \text{Binomial}(k2^{r-1}W, 2^{-k}).$$

Let M be the random number of vertices among $v_r^1, \dots v_r^J$ that are infected before time

$$S = \frac{1}{2k(2+\lambda)}(1 + \lambda/2)^{L(1-2\epsilon)}$$

By Lemma 5.7.6 the probability a given vertex will not become infected by time S is

$$p_{noi} \leq (1 - \hat{\lambda}^k)^m \quad \text{where} \quad \hat{\lambda} = (1-\epsilon)\frac{\lambda}{\lambda+1} \quad \text{and}$$
$$m = \frac{S}{2k+1} = \frac{(1+\lambda/2)^{L(1-2\epsilon)}}{2k(2k+1)(2+\lambda)} \quad \text{with} \quad L = \frac{\lambda k}{1+2\lambda}.$$

Combining the definitions and using $(1-x) \le e^{-x}$ we have

$$p_{noi} \le \exp\left(-\frac{\Gamma^k}{2k(2k+1)(2+\lambda)} \right) \quad \text{where} \quad \Gamma = \hat{\lambda}(1+\lambda/2)^{(1-2\epsilon)\lambda/(1+2\lambda)}.$$

When $\lambda = 2.5$

$$\frac{\lambda}{\lambda+1}(1+\lambda/2)^{\lambda/(1+2\lambda)} = 1.0014 > 1, \tag{5.7.2}$$

super

so $\Gamma > 1$ when ϵ is small and $p_{noi} \to 0$ as $k \to \infty$. From this we see that if $\delta > 0$ then for large k

$$EM \ge (1-\delta)EJ.$$

The remark after Lemma 5.7.6 implies that if we condition on the value of W and let $\bar{M} = (M|W)$ then

$$\bar{M} \ge \text{Binomial}(k2^{r-1}W, 2^{-k}(1-\delta)).$$

To prepare for the following two generalizations of the result for Geometric(1/2) offspring distribution we ask the reader to verify that in Step 2, all we use is the fact that (5.7.2) implies the bounds on EM and $\bar{M}$.

Step 2. Let $H_1(t) = P(v_r^i \in \xi_{t-S}$ for some $1 \le i \le J)$ and

$$H_2(t) = P(0 \in \xi_t | v_r^i \in \xi_{t-S} \text{ for some } 1 \le i \le J),$$

so that $f(t) \ge H_1(t)H_2(t)$. Fix $t > 2S$ and let

$$\chi(t) = \inf\{f(s) : s \le t-S\}.$$

Since t is fixed, we simplify the notation and write $\chi(t)$ as χ.

Ignore all but the first infection of each v_r^i by its parent. Any of these will evolve independently from the time $s < S$ it is first infected, and will be infected at time $t-S$ with probability at least χ. Thus given M the number of infected sites at time $t-S$ will dominate $N = \text{binomial}(M, \chi)$. If we let $\bar{N} = \text{binomial}(\bar{M}, \chi)$ and let $\delta > 0$, then by Lemma 2.3 in Pemantle (1992) we see that there exists a $\varepsilon > 0$ such that

$$P(\bar{N} \ge 1) \ge (1-\delta)\chi EM \wedge \varepsilon$$

Therefore $H_1(t) \ge (1-\delta)\chi EM \wedge \varepsilon$ when $t > 2S$.

Finally, if some v_r^i is infected at time $t-S$ then the probability of finding 0 infected at time t is bounded below by $\rho_1\rho_2$ where

- ρ_1 is the probability that the contact process starting with only v_r^i infected at time $t-S$ infects 0 at some time s with $t-S \le s \le t$. By Lemmas 5.6.5, 5.6.6, and 5.7.6, $\rho_1 \ge 1-\delta$.

- ρ_2 is the probability 0 is infected at time t given the infection of 0 at such a time s. For any $\epsilon > 0$, by Lemma 5.7.6 the probability that 0 has not been infected by time $S/2$ is less than ϵ when k is sufficiently large. By Lemma 5.6.6, with probability $\geq 1-(2+2\lambda)k^{-1/3}$ there should be at least ϵL infected leaves at time $t-\epsilon$. Hence 0 is infected at t with probability at least $(1-e^{-\lambda\epsilon^2 L})e^{-\epsilon}$, where the second term guarantees that the root is infected at time t. Choosing ϵ sufficiently small and k sufficiently large gives $\rho_2 \geq 1-\delta$.

Thus

$$f(t) \geq \begin{cases} \chi(t)EM(1-\delta)^3 \wedge \varepsilon & t > 2S, \\ \inf_{0\leq s\leq 2S} f(s) & S \leq t \leq 2S. \end{cases}$$

We can take $\varepsilon < \inf_{0\leq s\leq 2S} f(s)$ so that $f(t) \geq \chi(t)EM(1-\delta)^3 \wedge \varepsilon$ for all $t \geq S$. The result now follows from Lemma 5.7.7 with $L = S$ and $H(x) = (1-\delta)^3(EM)x \wedge \varepsilon$.

Proof for $p_n = (1-p)^{n-1}p$. It is now straightforward to replace 1/2 by p. We only have to pick k and r so that we can prove the analogue of (5.7.2). The mean of the offspring distribution is $1/p$. Let Z_r be the number of vertices at distance r from 0 and let $v_r^1, \dots v_r^J$ be those that have exactly k children. Since the root has degree k and $p_k = (1-p)^{k-1}p$

$$EJ \geq k(1/p)^{r-1}(1-p)^{k-1}p. \tag{5.7.3}$$ meanoff

In this case we want to pick r so that $(1/p)^r(1-p)^k \approx 1$. Hence EJ can be large when k is large. Ignoring the fact that r and k must be integers this means

$$r/k = \log(1-p)/\log p.$$

Let M be the random number of vertices among $v_r^1, \dots v_r^J$ that are infected before time S. By Lemma 5.7.6 the probability a given vertex will not become infected is

$$\leq (1-\hat{\lambda}^r)^{\lceil S/(2r+1)\rceil} \leq \exp\left(-\frac{-\Gamma^k}{2k(2r+1)(2+\lambda)}\right)$$

where $\Gamma = \hat{\lambda}^{r/k}(1+\lambda/2)^{(1-2\epsilon)\lambda/(1+2\lambda)}$. That is, if we choose λ such that

$$\left(\frac{\lambda}{\lambda+1}\right)^{r/k} \cdot (1+\lambda/2)^{\lambda/(1+2\lambda)} > 1 \tag{5.7.4}$$ suff

then we have $\Gamma > 1$ for large k. By the same reasoning as before this choice of λ gives an upper bound on λ_2.

If we want to graph the bound as a function of p it is better to work backwards. Given λ the second factor is > 1 so we can easily find the value of r/k that makes this 1. Having done this we can easily compute the value of p for which λ gives the upper bound on λ_2.

Proof for subexponential distributions. We suppose that the mean of the offspring distribution is $\mu > 1$. If p_k is subexponential, i.e.,

$$\limsup_{k\to\infty}(1/k)\log p_k = 0,$$

then for any δ there is a k with $p_k \geq (1-\delta)^k$. It follows from the same reasoning as in (5.7.3) that we can take r such that

$$\frac{r}{k} = -\frac{\log(1-\delta)}{\log \mu}.$$

Given any $\lambda > 0$, (5.7.4) will hold if δ is small enough, which implies local survival of the process. Therefore $\lambda_2 = 0$.

5.8 Exponential tails

sec:CPexpt

In the decade after my 2009 work with Shirshendu Chatterjee on the contact process on power law graphs, one of my favorite open problems was proving that the critical value for the contact process on a Galton-Watson tree is positive if the degree distribution D has an **exponential tail**, i.e., $Ee^{\theta D} < \infty$ for some $\theta > 0$. I was so excited by this question that when I gave a talk at the Northeast Probability Seminar held in November 2018 I followed in Erdös's footsteps and offered $1000 for a proof.

Danny Nam and Oanh Nguyen, two students of Allan Sly at Princeton, were in the audience and it was not long until in joint work with their advisor and Shankar Bhamidi of UNC the problem was solved. It is somewhat surprising that the proof is not very complicated. Ménard and Singh (2016) studied random geometric graphs in d-dimensions, graphs that are built by connecting points in a spatial Poisson process that are within distance R. The proof that $\lambda_c > 0$ for these graphs, which was done by studying "cumulative merging on a weighted graph," was arduous (and an impressive achievement).

In contrast, the BNNS proof is short and sweet. As the paper explains, there are two important new ideas:

(i) They modified the process by adding a new vertex above the root that is always infected. Recoveries at the root are not allowed unless all of the descendants of the root are healthy. One might worry that this assumption gives away too much, but the approach has the advantage that while the root is infected then the subtrees below its descendants are independent and this greatly facilitates recursion.

(ii) The second, somewhat more technical, idea is to prove that the probability that the infection goes deeper than depth h decays exponentially fast in h. See theirTheorem 3.4 (Theorem 5.8.3 below). To see why this is useful the reader will need to wait to see it used in the proof. Here, and in what follows we use a dual numbering system which includes the numbers used in the paper.

Turning to the details, let D be a distribution on the positive integers with an exponential tail, as defined above. Suppose that $ED > 1$ so that the Galton-Watson tree with this offspring distribution $\mathbb{GW}(D)$ has positive probability of surviving forever.

BNNSTh1 **Theorem 5.8.1.** *(Theorem 1) Suppose the degree distribution D has an exponential tail. Then there is a λ_0 so that for $\lambda < \lambda_0$ the contact process starting from a single infection at the root dies out with probability 1.*

5.8.1 Expected survival time

Let L be a large positive integer, let $\mathcal{T}_L$ denote the depth L Galton-Watson tree and let $\mathbb{CP}^\lambda(\mathcal{T}_L; 1_\rho)$ be the contact process on $\mathcal{T}_L$ in which initially there is one infected at ρ and the other sites are healthy. .The first of two main steps in the proof of Theorem 5.8.1 is to show:

BNNSTh3_1 **Theorem 5.8.2.** *(Theorem 3.1) Let R_L be the first time that $\mathbb{CP}^\lambda(\mathcal{T}_L; 1_\rho)$ reaches $\mathbf{0}$. There are constants C, which is $= e$, and $\lambda_0 > 0$ so that for any $\lambda \le \lambda_0$, $ER_L \le C$ for all integers L.*

A recursive equation. As mentioned in (i) we add a new vertex ρ^+ as a parent of the root of the tree ρ and set ρ^+ to be permanently infected. Since ρ^+ is always in state 1, we define the state space of the process to be $\{0,1\}^{\mathcal{T}_L}$. Let X_t be the contact process modified to have permanent infection at ρ^+ and let $\tilde{X}_t$ be X_t further modified to not allow recovery at ρ until none of its descendants are infected. Let S_L be the time for X_t to reach the all healthy state $\mathbf{0}$ and let $\tilde{S}_L$ be the time for $\tilde{X}_t$ to reach the all healthy state. The first event in $\tilde{X}_t$ is either

A. ρ recovers
B. ρ infects one of its children v_i

In case A, $\tilde{S}_L$ is the exponential(1) time it took for ρ to recover. If the degree of the root ρ is D the probability of event A is $1/(1+\lambda D)$. In case B, the evolution of the contact process on the subtrees $\mathcal{T}_{v_i}$ are independent of each other until the time $\tilde{S}_L^\otimes$ at which all of them are completely healthy. Let $X_t^\otimes$ be the contact process on the union of the subtrees, which has initial state 1_{v_i} run until the time $\tilde{S}_L^\otimes$ they are all completely healthy.

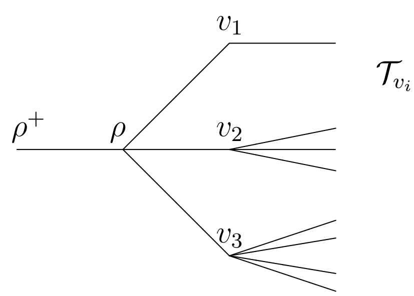

Figure 5.9: A picture of notation used in the proof.

At time $\tilde{S}_L^\otimes$ the process $\tilde{X}_t$ is again in state 1_ρ. After a geometric number of returns to this state, we will finally achieve outcome A so

$$E(\tilde{S}_L|\mathcal{T}_L) = \sum_{k=0}^{\infty} \left(\frac{\lambda D}{1+\lambda D}\right)^k \frac{1}{1+\lambda D} \cdot \left[(k+1)\cdot \frac{1}{1+\lambda D} + kE\left(\tilde{S}^\otimes \middle| \mathcal{T}_{v_i}\right)\right] \tag{5.8.1}$$ BSrec

where $|\mathcal{T}_{v_i})$ is short for conditioning on all of the subtrees $1 \le i \le D$. The mean of the geometric(p) distribution $\sum_{k=0}^{\infty}(1-p)^k p(k+1) = 1/p$ so the above is

$$= (1+\lambda D)\cdot\frac{1}{1+\lambda D} + (1+\lambda D - 1)E(\tilde{S}^{\otimes}|D)$$

which implies

$$E(\tilde{S}_L|D) = 1 + \lambda D E(\tilde{S}^{\otimes}|D) \tag{5.8.2}$$ BSgoal1

Estimating $E(\tilde{S}^{\otimes}|D)$

We do this by relating this expected value to the stationary distribution of the root-added contact process, which, conditional on $\mathcal{T}_L$, is irreducible on a finite state space and hence a unique stationary distribution. Let π^D be the stationary distribution of the product chain $\mathbb{CP}^{\otimes}_{\rho}(\mathcal{T}_L)$. If we let π_i be the stationary distribution of $\mathbb{CP}^{\lambda}_{\rho}(\mathcal{T}_i^+)$, where $\mathcal{T}_i^+$ is $\mathcal{T}_{v_i}$ with ρ added as a permanently infected added root, then

$$\pi^D = \otimes_{i=1}^D \pi_i$$

For any state η on $\mathcal{T}_L - \{\rho\}$, $\pi^D(\eta)$ is proportional to the time that the chain $X_t^{\otimes} \sim \mathbb{CP}^{\otimes}_{\rho}(\mathcal{T}_L)$ stays at state η. The expected time to stay at $\mathbf{0}$ (conditional on D) is $(\lambda D)^{-1}$. After escaping from $\mathbf{0}$ it spends expected time $E(\tilde{S}_L^{\otimes}|\mathcal{T}_L)$ before returning to $\mathbf{0}$. Therefore

$$\pi^D(\mathbf{0}) = \frac{(\lambda D)^{-1}}{(\lambda D)^{-1} + E(\tilde{S}_L^{\otimes}|\mathcal{T}_L)} = \frac{1}{1+\lambda D E(\tilde{S}_L^{\otimes}|\mathcal{T}_L)}$$

Similarly we have

$$\pi_i(\mathbf{0}) = \frac{1}{1+\lambda E(\tilde{S}_{L-1}|\mathcal{T}_{v_i})}$$

where S_{L-1} is the first time $X_t^i \sim \mathbb{CP}^{\lambda}_{\rho}(\mathcal{T}_{v_i}^+, 1_{v_i})$ reaches state $\mathbf{0}$. Therefore we obtain

$$1 + \lambda D E(\tilde{S}^{\otimes}|\mathcal{T}_L) = \prod_{i=1}^{D}[1 + \lambda E(\tilde{S}_{L-1}|\mathcal{T}_{v_i})] \tag{5.8.3}$$ LvsL1

Since T_{v_i}, $i \ge 1$ are i.i.d. $\mathbb{GW}(D)_{L-1}$ integrating out the randomness of T_{v_i}, $i \ge 1$ we have

$$1 + \lambda D E(\tilde{S}^{\otimes}|D) = (1 + \lambda E(\tilde{S}_{L-1}))^D$$

Using $1 + x \le e^x$ this becomes

$$\le \exp(\lambda E(\tilde{S}_{L-1}))$$

Using this with (5.8.2) we get

$$E(\tilde{S}_L|D) \le \exp(\lambda E(\tilde{S}_{L-1}) \cdot D))$$

and it follows that

$$ES_L \le E\tilde{S}_L \le E_D[\exp(\lambda E(S_{L-1}|D))] \tag{5.8.4}$$ BSgoal2

To complete the proof now let $c > 0$ be so that $M = E\exp(cD) < \infty$. Define K and λ so that

$$K = e \cdot \max\{\log M, 1\}, \qquad \lambda_0 = c/K$$

To prove Theorem 5.8.2 by induction note that $ES_0 = 1$ and suppose that $ES_{L-1} \le e$. If $\lambda < \lambda_0$ we have

$$\gamma \equiv \frac{\lambda ES_{L-1}}{c} \le \frac{\lambda_0 e}{c} = \frac{e}{K} < 1$$

so using Jensen's inequality

$$E\exp(\gamma cD) \le (E\exp(cD))^\gamma = M^\gamma = \exp(\log M \cdot (e/K)) \le e$$

which completes the proof of Theorem 5.8.2.

5.8.2 Exponential decay of infection depth

We continue to use the notation introduced previously. Recall that $\mathcal{T}_L = \mathbb{GW}(D)_L$ and $\mathcal{T}_L^+$ is the graph obtained by adding a permanently infected vertex ρ^+ above the root. For each state η of the contact process on $\mathcal{T}_L$ define the depth of η to be

$$r(\eta) = \max\{d(\rho^+, v) : \eta(v) = 1\}$$

with $r(\mathbf{0}) = 0$. This definition is natural since ρ^+ will be permanently occupied. Consider the root added process $X_t \sim \mathbb{CP}^\lambda_{\rho^+}(\mathcal{T_L}^+, 1_\rho)$ and let S_l be the first time the process reaches $\mathbf{0}$. Let $H = \max\{r(X_t) : t \in [0, S_L]\}$ be the maximum depth reached. The second main step in the proof of Theorem 5.8.1 is

BNNSTh3_4 **Theorem 5.8.3.** *(Theorem 3.4) Let $L > 0$ be an integer. There are constants K, $\lambda_0 > 0$ depending only on the degree distribution D so that for all $\lambda \le \lambda_0$, $h > 0$ and $m > 0$ we have*

$$P(H > h|\mathcal{T}_L) \le 2m(K\lambda)^h$$

for a collection of values of $\mathcal{T}_L$.with probability $\ge 1 - m^{-1}$.

Delayed contact process

Let $\mathcal{S}^+$ be a finite graph rooted at ρ^+. We assume that the graph is finite so that the stationary distributions we define later in this paragraph will exist. Here we will only consider $\mathcal{S} = \mathcal{T}_L^+$, but the general notation saves some typing. Let $\mathcal{S} = \mathcal{S}^+ - \{\rho^+\}$. For any two states $\eta, \zeta \in \{0,1\}^\mathcal{S}$ let $Q_{\eta,\zeta}$ be the transition rate from η to ζ in $\mathbb{CP}^\lambda_{\rho^+}(\mathcal{S}^+)$. For a fixed constant $\theta \in (0,1)$, the delayed contact process, denoted by $\mathbb{DP}^{\lambda,\theta}_{\rho^+}(\mathcal{S}^+, \eta_0)$ is the continuous time Markov process on $\{0,1\}^\mathcal{S}$ with initial state η_0 and transition rate

$$Q^\theta_{\eta,\zeta} = \theta^{r(\eta)} Q_{\eta,\zeta}$$

If π_S and ν_S^θ are the stationary distribution for the ordinary and delayed contact processes then it is immediate from the definitions that

$$\nu_{\mathcal{S}}^\theta(\eta) = \frac{\theta^{-r(\eta)}\pi_{\mathcal{S}}(\eta)}{\sum_\zeta \theta^{-r(\zeta)}\pi_{\mathcal{S}}(\zeta)} \tag{5.8.5}$$ tceqc4

BNNSL3_6 **Lemma 5.8.4.** *(Lemma 3.6) Let $L > 0$ be an integer. There are constants K and $\lambda_0 > 0$ so that for all $\lambda \le \lambda_0$ and L*

$$E\left[\nu_{\mathcal{T}_L}^\theta(\mathbf{0})^{-1}\right] \le 2$$

The proof is similar to that of Theorem 5.8.2, but requires more work, so we refer the reader to the paper for details.

Proof of Theorem 5.8.3. To simplify notation write π_L for $\pi_{\mathcal{T}_L}$ and ν_L for $\nu_{\mathcal{T}_L}^\theta$ with $\theta = K\lambda$ where K is the constant from Lemma 5.8.4. Let $A = \{\eta : r(\eta) \ge h\}$ and note that (5.8.5) implies

$$\frac{\pi_L(A)}{\pi_L(\mathbf{0})} \le \theta^h \frac{\nu_L(A)}{\nu_L(\mathbf{0})}$$

Lemma 5.8.4 and Markov's inequality imply that, for a collection of $\mathcal{T}_L$ with probability $\ge 1 - m^{-1}$, we have $\nu_L(\mathbf{0})^{-1} \le 2m$ and hence

$$\frac{\pi_L(A)}{\pi_L(\mathbf{0})} \le 2m(K\lambda)^h \tag{5.8.6}$$ p34bd

Now if $X_t \sim \mathbb{CP}_{\rho^+}^\lambda(\mathcal{T}_L^+)$ hits A before 0 then the time needed to escape form A is at least exponential with mean 1, since escape can only happen when there is exactly one infected at distance h from ρ^+. Thus if $\gamma(h) = |\{t \in [0, S_L] : X_t \in A\}|$, where $|\cdot|$ denotes Lebesgue measure then

$$E(\gamma(h)|H \ge h, \mathcal{T}_L) \ge 1$$

Combining this with (5.8.6) and noting that $\gamma(h) = 0$ on $H < h$

$$\begin{aligned} P(H \ge h|\mathcal{T}_L) &\le E(\gamma(h)|H \ge h, \mathcal{T}_L) \cdot P(H \ge h|\mathcal{T}_L) \\ &\le E(\gamma(h)|\mathcal{T}_L) \le \frac{\pi_L(A)}{\pi_L(\mathbf{0})} \le 2m(K\lambda)^h \end{aligned}$$

which completes the proof. □

At this point the rest is routine.

Proof of Theorem 5.8.1. Let $\mathcal{T} \sim \mathbb{GW}(D)$, ρ be its root, and let $X_t \sim \mathbb{CP}(\mathcal{T}, 1_\rho)$. Let K, λ_0 be given in Theorem 5.8.3 and let $\lambda \le \lambda_0$ so that $K\lambda \le 1$. Let $\delta > 0$ be small and pick h so that $(K\lambda)^h = \delta^2/8$. Let $E(h)$ be the event that the infection inf X_t does not go deeper than h before dying out. Taking $m = 2/\delta$ in Theorem 5.8.3 we see that

$$P(H \ge h|\mathcal{T}_L) \le (4/\delta)(\delta^2/8) = \delta/2$$

for a collection of values of $\mathcal{T}_L$ with probability $\geq 1-\delta/2$ so $P(E(h)) \geq 1-\delta$.

Let $\mathcal{T}_h$ be $\mathcal{T}$ truncated at depth h and couple the processes $\mathbb{CP}(\mathcal{T}, 1_\rho)$ and $\mathbb{CP}(\mathcal{T}_h, 1_\rho)$ by identifying the recoveries and infections inside $\mathcal{T}_h$. Let R and R_h be the times $\mathbb{CP}(\mathcal{T}, 1_\rho)$ and $\mathbb{CP}(\mathcal{T}_h, 1_\rho)$ reach $\mathbf{0}$. Then Theorem 5.8.2 tells us that

$$E(R|E(h)) = E(R_h|E(h)) \leq \frac{ER_h}{P(E(h))} < \infty$$

Thus $P(X_t \neq \mathbf{0}$ for all $t \geq 0) \leq \delta$. Since δ is arbitrary the desired result follows. □

5.8.3 Bounds on survival time

BNNSTh3 **Theorem 5.8.5.** *(Theorem 3) Suppose that G_n is a graph on n vertices generated by the configuration model with degree distribution D with (i) $ED(D-2) > 0$, so there is a giant component, and (ii) $E(e^{cD}) < \infty$ for some $c > 0$. Consider the contact process on G_n starting from all sites infected. There are constants $0 < \lambda_i \leq \lambda_{ii} < \infty$ so that*

(i) For $\lambda < \lambda_i$ the process survives for time at most $n^{1+o(1)}$
(ii) for $\lambda > \lambda_{ii}$ the process survives for time at least $e^{\Theta(n)}$.

The proof of this result requires hard work, so we will only explain a few ideas from the proof.

Proof of (i). It is enough to prove

BNNSTh4_1 **Theorem 5.8.6.** *(Theorem 4.1) Fix an arbitrary vertex $v \in G$ and let T_v be the time when $\mathbb{CP}^\lambda(G; 1_v)$ reaches the root 0. There is an event H so that $P(G \in H) = 1 - o(1)$ and constants B, $\lambda_0 > 0$ depending on μ so that for all $\lambda \in (0, \lambda_0)$ we have*

$$E(T_V|G \in H) \leq B$$

The result follows from Markov's inequality. Using this approach we have to estimate the probability the survival time $> n^{1+o(1)}$ so that n times the error probability goes to 0.

To prove Theorem 5.8.6, they show that the graph looks locally like a Galton-Watson tree with an augmented distribution defined as follows. Suppose for simplicity that $k_{max} = \max\{k : p_k > 0\}$ and let

$$k_0 = \max\{k : \sum_{j \geq k} \sqrt{p_j} \geq 1/2\}$$

They define the augmented distribution by

$$\hat{\mu}(j) = \begin{cases} p_j/2Z & \text{if } j \leq k_0 \\ \sqrt{p_j}/Z & \text{if } j > k_0 \end{cases}$$

where Z is chosen to make the sum of the $\hat{\mu}(j) = 1$. In addition to changing the tails of the distribution, one cannot assume that the graph is tree like, but one must allow an extra edge that may form a cycle.

Proof of (ii). To prove prolonged persistence they show that in Section 7 the random graph contains a large (α, R)-embedded expander. A subset W_0 of vertices has this property if for every set $A \subset W_0$ with $|A| \leq \alpha|W_0|$

$$|N(A,R) \cap W_0| \geq 2|A|$$

where $N(A,R)$ is the collection of vertices in G_n within distance at most R of A.

Once this is done they can prove

BNNSL5_4 **Lemma 5.8.7.** *(Lemma 5.4) There are constants λ_0, C, C', and β so that for all $\lambda \geq \lambda_0$ and integer $a \in (0, \alpha\beta n]$*

$$P(|X^0_{t+C}| \leq 5a/4 | X^0_t = a) \leq 2\exp(-a/C')$$

This leads easily to the lower bound on the survival times. For more details see the paper.

Further Reading. Huang (2020) shows that if a contact process on a Galton-Watson tree survies then it dominates a Crump-Mode-Jagers branching process. This implies that thee surival probability $p(\lambda)$ is conitnuous at $\lambda = \lambda_1$

Nam, Nguyen, and Sly (2022) have studied the asymptotics of critical values λ_1 and λ_2 for the contact process on Galton-Watson trees and finite graphs when the mean degree $ED \to \infty$ and degree distribution satisfies a concentration condition for D/ED. Theorem 1 shows that $\lambda_1/ED \to 1$. Theorem 2 shows that the same result holds for the short and long time survival thresholds.

5.9 Threshold-θ contact process

ec:CPthresh

Reproduction of particles in the contact process is asexual. An individual at x gives birth to a new individual at a neighboring site y at rate λ. As a consequence, the process is additive: when built on the graphical representation

$$xi^{A\cup B}_t = \xi^A_t \cup \xi^B_t$$

A number of particle systems have been defined that have sexual reproduction: at least two particles are needed to create a new one. Here we will describe results for one non-additive process with nonlinear flip rates that has been studied on random regular graphs: the threshold $\theta \geq 2$ contact process of Chatterjee and Durrett (2013). Along the way we will mention some of the other models. The introduction of CD2013 has examples that have been studied on $\mathbb{Z}^d$. Before we get lost in the details of what is in CD2013, we should note that Danny Nam (2019) has proved results for graphs that have more general degree distributions. Since our proof of Proposition 5.9.8 does not work when the degree is not constant his proof required some substatial new ideas.

Let G_n be a random r-regular graph with $r \geq 3$ on n vertices, constructed for example by the algorithm used in Section 2.1 for the configuration model. Let $\mathbb{P}$ denote the distribution of G_n, which is the first of several probability measures we will define. We condition on the event E_n that the graph is **simple**, i.e., it does not contain a self-loop at any vertex, or more than one edge between two vertices. Theorem 2.1.1 implies that $\mathbb{P}(E_n)$ converges to a positive limit c_r as $n \to \infty$, and hence

$$\text{if } \tilde{\mathbb{P}} = \mathbb{P}(\cdot|E_n), \text{ then } \tilde{\mathbb{P}}(\cdot) \le c_r \mathbb{P}(\cdot) \text{ for some constant } c_r > 0. \tag{5.9.1}$$ Ptilde

Thus conditioning on the event E_n will not have much effect on the distribution of G_n. It is easy to see that the distribution of G_n under $\tilde{\mathbb{P}}$ is uniform over the collection of all r-regular (simple) graphs on the vertex set V_n. (We put simple in parentheses since it is redundant: graphs by definition are simple.) We choose G_n according to the distribution $\tilde{\mathbb{P}}$ on simple graphs, and once chosen the graph remains fixed through time.

Having defined the graph, the next step is to introdcue the dynamics on the graph. For the proofs it is crucial that we work in discrete time. We write $x \sim y$ to mean that x is a neighbor of y, and let

$$\mathcal{N}_y = \{x \in V_n : x \sim y\} \tag{5.9.2}$$ Nbrdef

be the set of neighbors of y. The distribution $P^{G_n}_{p,\theta}$ of the (discrete time) threshold-θ contact process $\xi_t \subseteq V_n$ with parameters p and θ conditioned on G_n can be described as follows:

$$\begin{aligned} &P^{G_n}_{p,\theta}\left(x \in \xi_{t+1} \mid |\mathcal{N}_x \cap \xi_t| \ge \theta\right) = p \text{ and} \\ &P^{G_n}_{p,\theta}\left(x \in \xi_{t+1} \mid |\mathcal{N}_x \cap \xi_t| < \theta\right) = 0, \end{aligned}$$

where the decisions for different vertices at time $t+1$ are made independently. Let $\xi^A_t \subseteq V_n$ denote the threshold-θ contact process starting from $\xi^A_0 = A$, and let ξ^1_t denote the special case when $A = V_n$.

Being an attractive processes, the threshold-θ contact process on an r-regular tree has a translation invariant upper invariant measure, ξ^1_∞, that is the limit as $t \to \infty$ for the system starting from all 1's. There are three basic questions for our models.

Q1. Let ξ^p_t be the system starting from product measure with density p, i.e., $\xi^p_0(x)$ are independent and equal 1 with probability p. Does ξ^p_t die out for small p? That is, do we have $P(\xi^p_t(x) = 1) \to 0$ as $t \to \infty$ if $p \le p_0(\lambda)$?

Q2. Let $\rho(\lambda) = P(\xi^1_\infty(x) = 1)$ and let $\lambda_c = \inf\{\lambda : \rho(\lambda) > 0\}$. Is $\rho(\lambda)$ discontinuous at λ_c? If so, then soft results imply that $P(\xi^1_\infty(x) = 1) > 0$ when $\lambda = \lambda_c$. (As $\lambda \downarrow \lambda_c$ the upper invariant measures ξ^1_∞ decrease to a limit which must be the upper invariant measure at λ_c.)

Q3. Let $\xi^{0,\beta}_\infty$ be the limit as $t \to \infty$ for the system starting from all 0's when sites become occupied spontaneously at rate β along with the original dynamics. Is $\lim_{\beta\to 0} P(\xi^{0,\beta}_\infty(x) = 1) = 0$? If so, we say that **0 is stable under perturbation**, and it follows that there are two nontrivial stationary distributions when $\beta > 0$ is small. (To see this note $\xi^{1,\beta}_\infty$ is larger that $\xi^{1,0}_\infty$, so we have two stationary distributions if the density of particles in $\xi^{0,\beta}_\infty$ is less than the density in $\xi^{1,0}_\infty$.

One of the first processes with sexual reproduction that was studied is Toom's (1974) NEC (north-east-center) rule on $\mathbb{Z}^2$. See also Toom (1980). In the original formulation it was an example of a stochastic perturbation of a cellular automaton. a 0 at x is changed to 1 if $x+e_1$ and $x+e_2$ are both in state 1. This cellular automaton is an **eroder**. If the initial configuration has only finitely many 0's then after a finite number of iterations the configuration is all 1's. By **Toom's eroder theorem** if we add errors that change 1's to 0's with probability ϵ and $\epsilon < \epsilon_0$ then there is a nontrivial stationary distribution

More relevant for us, is the reformulation of Toom's rule as a growth model, where the state of x changes

$$\begin{aligned} &1 \to 0 \quad \text{at rate } 1, \\ &0 \to 1 \quad \text{at rate } \lambda \text{ if } x+e_1 \text{ and } x+e_2 \text{ are both in state 1.} \end{aligned} \tag{5.9.3}$$ Toom

For the model in (5.9.3), Durrett and Gray (1985) have proved (a)–(d) below. Until recently the only source was the announcement of results in Durrett (1985), but now a pdf of the original preprint is avaiable (see references).

(a) if we let ξ_t^A denote the set of all 1's at time t starting from $\xi_0^A = A$, and

$$\lambda_f = \inf\{\lambda : P(\xi_t^A \neq \emptyset \text{ for all } t\ ) > 0\}$$

be the critical value for survival from a finite set, then $\lambda_f = \infty$, because if all the 1's in the initial configuration are inside a rectangle, then there will never be any birth of 1's outside that rectangle.

(b) Durrett and Gray used a contour argument to prove $\lambda_c \leq 110$. Swart, Szabo, and Toninelli (2022) have further developed the method of contours, Bramson and Gray have proved a version of Toom's eroder theorem in continuous time when implies $\lambda_c < \infty$

(c) if p^* is such that $1-p^*$ equals the critical value for oriented bond percolation on $\mathbb{Z}^2$, then for any $p < p^*$ the process starting from product measure with density p dies out. This is trivial to prove if there is an oriented path of 0's that only moves up and to the right then these 0's are permanent. They can never be changed to 1.

(d) Suppose that sites become occupied spontaneously at rate β along with the original dynamics. If $\lambda > \lambda_c$ and $6\beta^{1/4}\lambda^{3/4} < 1$, then there are two stationary distributions.

Chen (1992, 1994) has generalized Toom's growth model. He begins by defining the following diagonally adjacent pairs for each site x.

pair 1	pair 2	pair 3	pair 4
$x-e_1, x-e_2$	$x+e_1, x-e_2$	$x+e_1, x+e_2$	$x-e_1, x+e_2$

His models are numbered by the pairs that can give birth: Type I (pair 1 = South-West-Corner rule); Type IV (any pair); Type III (pairs 1, 2, and 3); Type 2A (pairs 1 and 2); and Type 2B (pairs 1 and 3). Chen (1992) proves for model IV that if $0 < p < p(\lambda)$, then

$$P(0 \in \xi_t^p) \leq t^{-c \log_{2\lambda}(1/p)}.$$

He also shows for the same model that

$$\lim_{\beta\to 0} P(0 \in \xi_\infty^{0,\beta}) > 0$$

for large λ, so 0 is unstable under perturbation. In contrast, Chen (1994) shows that 0 is stable under perturbation in model III (and hence in modes IIA, IIB, and I).

5.7.1. Results

The first step is to prove that threshold-θ contact process dies out for small values of p and survives for p close to 1. It is easy to see that on any graph in which all vertices have degree r the threshold-θ contact process dies out rapidly if $p < 1/r$, because an occupied site has at most r neighbors that it could cause to be occupied at the next time step suggesting $E^{G_n}_{p,\theta}\xi^1_t \le n(rp)^t$.

Survival from initial density close to 1

Our next result shows that if $\theta \ge 2, r \ge \theta + 2$ and p is sufficiently close to 1, then with high probability the fraction of occupied vertices in the threshold-θ contact process on G_n starting with all 1's stays above $1 - \epsilon_1$ for an exponentially long time.

p_c **Theorem 5.9.1.** *Suppose $\theta \ge 2$ and $r \ge \theta + 2$. There are constants $\epsilon_1, \gamma_1 > 0$, and a good set of graphs $\mathcal{G}_n$ with $\tilde{\mathbb{P}}(G_n \in \mathcal{G}_n) \to 1$ so that if $G_n \in \mathcal{G}_n$ and $p \ge p_1 = 1 - \epsilon_1/(3r - 3\theta)$, then*

$$P^{G_n}_{p,\theta}\left(\inf_{t \le \exp(\gamma_1 n)} \frac{|\xi^1_t|}{n} < 1 - \epsilon_1\right) \le \exp(-\gamma_1 n).$$

Here and in what follows, all constants will depend on the degree r and threshold θ. If they depend on other quantities, that will be indicated.

The reason for the restriction to $r \ge \theta+2$ comes from Proposition 5.9.8 (with $j = r-\theta+1$) below. When $r \le \theta + 1$, it is impossible to pick $\eta > 0$ so that $(1 + \eta)/(r - \theta) < 1$. There may be more than algebra standing in the way of constructing a proof. We conjecture that the result is false when $r \le \theta + 1$. To explain our intuition in the special case $\theta = 2$ and $r = 3$, consider a rooted binary tree in which each vertex has two descendants and hence, except for the root, has degree three. If we start with a density u of 1's on level k and no 1's on levels $m < k$, then at the next step the density will be $g(u) = pu^2 < u$ on level $k - 1$. When each vertex has three descendants instead of two, then

$$g(u) = p(3u^2(1 - u) + u^3),$$

which has a nontrivial fixed point for $p \ge 8/9$ (divide by u and solve the quadratic equation).

As the next result shows, there is a close relationship between the threshold-θ contact process ξ_t on a random r-regular graph and the corresponding process ζ_t on the homogeneous r-tree. Following the standard recipe for attractive interacting particle systems, if we start with all sites on the tree occupied, then the sequence $\{\zeta_t^1\}$ of sets of occupied vertices decreases in distribution to a limit ζ_∞^1, which is called the *upper invariant measure*, since it is the stationary distribution with the most 1's. Here and later we denote by $\mathbf{0}$ any fixed vertex of the homogeneous tree. Writing $P_{p,\theta}$ for the distribution of ζ_t with parameters p and θ, the critical value is defined by

$$p_c(\theta) := \sup\{p : P_{p,\theta}(\zeta_\infty^1(\mathbf{0}) = 1) = 0\},$$

pctree **Corollary 5.9.2.** *Suppose $\theta \geq 2$, $r \geq \theta + 2$ and that p_1 and ϵ_1 are the constants in Theorem 5.9.1. If $p \geq p_1$, then there is a translation invariant stationary distribution for the threshold-θ contact process on the homogeneous r-tree in which each vertex is occupied with probability $\geq 1 - \epsilon_1$.*

Fontes and Schonmann (2008a) have considered the continuous time threshold-θ contact process on a tree in which each vertex has degree $b+1$, and they have shown that if b is large enough, then $\lambda_c < \infty$. Our result improves their result by removing the restriction that b is large.

Dying out from small initial density

If we set the death rate $= 0$ in the threshold-θ contact process, then we can without loss of generality set the birth rate equal to 1 and the process reduces to bootstrap percolation (with asynchronous updating). Balogh and Pittel (2007) have studied bootstrap percolation on random regular graphs. They have identified an interval $[p_-(n), p_+(n)]$ so that the probability that all sites end up active goes sharply from 0 to 1. The limits $p_\pm(n) \to p_*$ and $p_+ - p_-$ is of order $1/\sqrt{n}$. If bootstrap percolation cannot fill up the graph, then it seems that our process with deaths will be doomed to extinction. The next result proves this, and more importantly extends the result to arbitrary initial conditions with a small density of occupied sites.

Here, since processes with larger θ have fewer survivals, it is enough to prove the result when $\theta = 2$.

th1 **Theorem 5.9.3.** *Suppose $\theta \geq 2$ and $p_2 < 1$. There are constants $0 < \epsilon_2(p_2), C_2(p_2) < \infty$, and a good set of graphs $\mathcal{G}_n$ with $\tilde{\mathbb{P}}(G_n \in \mathcal{G}_n) \to 1$ so that if $G_n \in \mathcal{G}_n$, then for any $p \leq p_2$, and any subset $A \subset V_n$ with $|A| \leq \epsilon_2 n$,*

$$P_{p,\theta}^{G_n}\left(\xi_{\lceil C_2 \log n \rceil}^A \neq \emptyset\right) \leq 2/n^{1/6} \text{ for large enough } n.$$

The density of 1's $\rho(p,\theta) := P_{p,\theta}(\zeta_\infty^1(\mathbf{0}) = 1)$ in the stationary distribution on the homogeneous r-tree is a nondecreasing function of p. The next result shows that the threshold-θ contact process on the r-tree has a discontinuous phase transition.

disco **Corollary 5.9.4.** *Suppose $\theta \geq 2$, let p_1 be the constant from Theorem 5.9.1, and let $\epsilon_2(\cdot)$ be as in Theorem 5.9.3. $\rho(p,\theta)$ never takes values in $(0, \epsilon_2(p_1))$.*

This result, like Theorem 5.9.3 does not require the assumption $r \geq \theta+2$. On the other hand, if $\rho(p,\theta) \equiv 0$ for $r \leq \theta+1$, the result is not very interesting in that case. Again Fontes and Schonmann (2008a) have proved that the threshold-θ contact process has a discontinuous transition when the degree $b+1$ is large enough.

Fontes and Schonmann (2008b) have studied θ-bootstrap percolation on trees in which each vertex has degree $b+1$ and $2 \leq \theta \leq b$. They have shown that there is a critical value p_f so that if $p < p_f$, then for almost every initial configuration of product measure with density p, the final bootstrapped configuration does not have any infinite component. This suggests that we might have $\epsilon_2(p)$ bounded away from 0 as $p \to 1$.

Stability of 0

The previous pair of results are the most difficult in the paper. From their proofs one easily gets results for the process with sponataneous births with probability β, i.e., after the threshold-θ dynamics has been applied to the configuration at time t, we independently make vacant sites occupied with probability β. For this new process, we denote the set of occupied vertices at time t starting with all 0's by $\hat\xi^0_t$ and its distribution conditioned on the graph G_n by $P^{G_n}_{p,\theta,\beta}$ to have the following:

th3 **Theorem 5.9.5.** *Suppose $\theta \geq 2$. There is a good set of graphs $\mathcal{G}_n$ with $\tilde{\mathbb{P}}(G_n \in \mathcal{G}_n) \to 1$ so that if $G_n \in \mathcal{G}_n$ and $p < 1$, then there are constants $C_3(p), \beta_3(p), \gamma_3(p,\beta) > 0$ so that for $\beta < \beta_3$,*

$$P^{G_n}_{p,\theta,\beta}\left(\sup_{t \leq \exp(\gamma_3 n)} \frac{|\hat\xi^0_t|}{n} > C_3\beta \right) \leq 2\exp(-\gamma_3 n).$$

Let $\hat\zeta^0_\infty$ be the limiting distribution for the process on the homogeneous tree, which exists because of monotonicity.

c3 **Corollary 5.9.6.** *If $\theta \geq 2$ and $p < 1$, then $\lim_{\beta\to 0} P_{p,\theta,\beta}(\hat\zeta^0_\infty(\mathbf{0}) = 1) = 0$.*

5.7.2. Key ideas for the proof

We now describe the "isoperimetric inequalities" that are the keys to the proofs of our results. Let $\partial U := \{y \in U^c : y \sim x \text{ for some } x \in U\}$ be the boundary of U, and given two sets U and W, let $e(U,W)$ be the number of edges having one end in U and the other end in W. Given an $x \in V_n$ let $n_U(x)$ be the number of neighbors of x that are in U, and let

$$U^{*j} = \{x \in V_n : n_U(x) \geq j\}.$$

The estimation of the sizes of $e(U,U^c)$ is key to the study of random walks on graphs, see Chapter 6, and especially Section 6.3. Here we are interested in studying the sizes of U^{*j} and having better constants by restricting to small sets. The last remark will make more sense after reading Section 6.3.

isoper0 **Proposition 5.9.7.** *Let $E^{*1}(m, \leq k)$ be the event that there is a subset $U \subset V_n$ with size $|U| = m$ so that $|U^{*1}| \leq k$. There are constants C_0 and Δ_0 so that for any $\eta > 0$, there is an $\epsilon_0(\eta)$ which also depends on r so that for $m \leq \epsilon_0(\eta)n$,*

$$\mathbb{P}\left[E^{*1}(m, \leq (r-1-\eta)m)\right] \leq C_0 \exp\left(-\frac{\eta^2}{4r} m \log(n/m) + \Delta_0 m\right).$$

This result yields the next proposition which we need to prove Theorems 5.9.1 and 5.9.3. For Theorem 5.9.1, note that if $W = V_n \backslash \xi_t$ is the set of vacant vertices at time t, then at time $t+1$ the vertices in $W^{*(r-\theta+1)}$ will certainly be vacant and the vertices in its complement will be vacant with probability $1-p$. So having an upper bound for $|W^{*(r-\theta+1)}|$ will be helpful. On the other hand for Theorem 5.9.3, if U is the set of occupied vertices at time t, then at time $t+1$ the vertices in $U^{*\theta}$ will be occupied with probability p and the vertices in its complement will certainly be vacant. So having an upper bound for $|U^{*\theta}|$ will be helpful.

Keeping these in mind, it is easy to see from the definitions that if $j > 1$ and $|Z| = m$, then

$$rm \geq \sum_{y \in Z^{*1}} e(\{y\}, Z) \geq |Z^{*1} \setminus Z^{*j}| + j|Z^{*j}| = |Z^{*1}| + (j-1)|Z^{*j}|.$$

So for any set Z of size m, if $|Z^{*j}| \geq k$, then $|Z^{*1}| \leq rm-(j-1)k$. Taking $k = m(1+\eta)/(j-1)$ so that $rm-(j-1)k = (r-1-\eta)m$ and using Proposition 5.9.7 we get

isoper2 **Proposition 5.9.8.** *Let $E^{*j}(m, \geq k)$ be the event that there is a subset $Z \subset V_n$ with size $|Z| = m$ so that $|Z^{*j}| \geq k$. For the constants C_0, Δ_0, and $\epsilon_0(\eta)$ given in Proposition 5.9.7, if $j > 1$ and $m \leq \epsilon_0(\eta)n$, then*

$$\mathbb{P}\left[E^{*j}\left(m, \geq \left(\frac{1+\eta}{j-1}\right) m\right)\right] \leq C_0 \exp\left(-\frac{\eta^2}{4r} m \log(n/m) + \Delta_0 m\right).$$

The reasoning that led to Proposition 5.9.8 depends on the fact that all vertices have the same degree. Danny Nam has developed new results to cover the case of variable degree.

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