Chapter 5

Contact Process

In the contact process on a graph $G$, occupied sites become vacant at rate 1, and give birth onto vacant neighbors at rate $\lambda$. Harris (1974) introduced the contact process on $G = \mathbb{Z}^d$ in 1974. The state at time $t$ is $\xi_t \subset \mathbb{Z}^d$. It is often thought of as a model for the spread of species. In this case $\xi_t$ is the set of occupied sites, and sites in $\xi_t^c$ are vacant. However, it can also be viewed as a spatial SIS epidemic model. In this case $\xi_t$ is the set of infected sites, and sites in $\xi_t^c$ are susceptible. Both interpretations are common in the literature, so the reader will see both here.

Let $\xi^0_t$ be the process starting from only the origin occupied and let $\xi^1_t$ be the process starting from all sites occupied. Harris introduced the critical value

$$\lambda_c = \inf\{\lambda : P(\xi^0_t \neq \emptyset \text{ for all } t) > 0\},$$

and proved that on $\mathbb{Z}^d$ we have $0 < \lambda_c < \infty$. He also showed that for $\lambda > \lambda_c$, $\xi^1_t$ converges to a limit that is a nontrivial stationary distribution. A rich theory has been developed for the contact process on $\mathbb{Z}^d$. See Liggett’s 1999 book for a summary of much of what is known.

5.1 Trees

Pemantle (1992) was the first to study the contact process on the tree $T^d$ in which each vertex has degree $d + 1$. Here, and in what follows, we assume $d \geq 2$ since $T_1 = \mathbb{Z}$. Let 0 be the root of the tree and let $P_0$ be the probability measure for the process starting from only the root occupied. Pemantle found that the contact process on $T^d$ has two critical values.

$$\lambda_1 = \inf\{\lambda : P_0(\xi_t \neq \emptyset \text{ for all } t) > 0\},$$

$$\lambda_2 = \inf\{\lambda : \liminf_{t \to \infty} P_0(0 \in \xi_t) > 0\}.$$

By deriving bounds on the critical values, he showed that $\lambda_1 < \lambda_2$ when $d \geq 3$. Liggett (1996) settled the case $d = 2$ by showing $\lambda_1 < 0.605 < 0.6609 < \lambda_2$. At about the same time, Stacey (1996) gave a proof that $\lambda_1 < \lambda_2$ that did not rely on bounds. The stationary
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distributions and limiting behavior of the contact process on trees is an interesting subject that has been extensively studied. See Liggett’s 1999 book for an account of the results.

Consider the contact process on \(-n, \ldots, n\) starting from all sites occupied and let \(\tau_n = \inf\{t : \xi_t = \emptyset\}\). Combining results of Durrett and Liu (1988) and Durrett and Schonmann (1988) gives the following results

(i) If \(\lambda < \lambda_c\) then there is a constant \(\gamma_1(\lambda)\) so that
\[
\frac{\tau_n}{\log n} \to \gamma_1(\lambda) \quad \text{in probability.}
\]

(ii) If \(\lambda > \lambda_c\) then there is a constant \(\gamma_2(\lambda)\) so that
\[
\frac{\log \tau_n}{n} \to \gamma_2(\lambda) \quad \text{in probability.}
\]

(iii) When \(\lambda > \lambda_c\) there is “metastability”:
\[
\frac{\tau_n}{E\tau_n} \Rightarrow \text{exponential}(1)
\]

where \(\Rightarrow\) means convergence in distribution. Intuitively, the process on the interval stays exponentially long in a state that looks like the stationary distribution for the process on \(\mathbb{Z}\), and then suddenly dies out, hence the lack of memory property of the survival time.

Results on \(\mathbb{Z}^d\) with \(d > 1\) had to wait for the work of Bezuidenhout and Grimmett (1990), who showed that in \(d > 1\) the contact process dies out at the critical value and in doing so introduced a block construction that can be used to study the supercritical process. Mountford proved the metastability result in 1993 and that \((\log \tau_n)/n^d \to \gamma(\lambda)\) in 1999.

Stacey (2001) studied the contact process on a tree truncated at height \(\ell\), \(T^d_\ell\). To be precise, the root has degree \(d\), vertices at distance \(0 < k < \ell\) from the root have degree \(d + 1\), while those at distance \(\ell\) have degree 1. Cranston, Mountford, Mourrat, and Valesin (2014) improved Stacey’s result to establish that the time to extinction starting from all sites occupied \(\tau^d_\ell\) satisfies

**Theorem 5.1.1.** (a) For any \(0 < \lambda < \lambda_2(T^d_\ell)\) there is a \(c \in (0, \infty)\) so that as \(\ell \to \infty\)
\[
\frac{\tau^d_\ell}{\log |T^d_\ell|} \to c \quad \text{in probability.}
\]

(b) For any \(\lambda_2(T^d_\ell) < \lambda < \infty\) there is a \(c \in (0, \infty)\) so that as \(\ell \to \infty\)
\[
\frac{\log(\tau^d_\ell)}{|T^d_\ell|} \to c \quad \text{in probability.}
\]

Moreover \(\tau^d_\ell/E\tau^d_\ell\) converges to a mean one exponential.

When a tree is truncated at a finite distance, a positive fraction of the sites are on the boundary. A more natural finite version of a tree is a random regular graph in which all vertices have degree \(d + 1\). In this case there is no boundary and the graph has the same distribution viewed from any point. If there are \(n\) vertices, the graph looks like \(T^d_\ell\) in neighborhoods of a point that have \(\leq n^{1/3}\) vertices. Refer to results in Chapter 1 Mourrat and Valesin (2016) have shown for a random regular graph, the time to extinction starting from all sites occupied \(\tau_n\) satisfies:
Theorem 5.1.2. (a) For any $0 < \lambda < \lambda_1(\mathbb{T}^d)$ there is a $C < \infty$ so that as $n \to \infty$

$$P(\tau_n < C \log n) \to 1,$$

(b) For any $\lambda_1(\mathbb{T}^d) < \lambda < \infty$ there is a $c > 0$ so that as $n \to \infty$

$$P(\tau_n > e^{cn}) \to c.$$

Notice that the threshold in the second result comes at $\lambda_1$, while the one in Stacey’s result comes at $\lambda_2$. The difference is that when $\lambda \in (\lambda_1, \lambda_2)$ on the infinite tree the origin is in the middle of linearly growing vacant region. On the truncated tree the system dies out when the vacant region is large enough. However, on the random regular graph the occupied sites will later return to the origin. Durrett and Jung (2017) investigated the qualitative differences between $\lambda \in (\lambda_1, \lambda_2)$ and $\lambda > \lambda_2$ on the small world graph.
5.2 Power-law Random Graphs

Pastor-Satorras and Vespigniani who we will abbreviate PSV (2001a, 2001b, 2002) have made an extensive study of the contact process on “scale-free” random networks using mean-field methods. For this and many other related results see the 2015 survey paper by Pastor-Satorras, Castellano, van Meighem, and Vespigniani in Reviews of Modern Physics.

**Mean-field theory.** Let \(\rho_k(t)\) denote the fraction of vertices of degree \(k\) that are infected at time \(t\), and \(\theta(\lambda)\) be the probability that a given link points to an infected vertex. If we make the mean-field assumption that there are no correlations then

\[
\frac{d\rho_k(t)}{dt} = -\rho_k(t) + \lambda k[1 - \rho_k(t)]\theta(\lambda)
\]

so the equilibrium frequency \(\rho_k\) satisfies

\[
0 = -\rho_k + \lambda k[1 - \rho_k]\theta(\lambda)
\]  \hspace{1cm} (5.2.1) \hspace{1cm} MFeq

Solving we have

\[
\rho_k = \frac{k\lambda\theta}{1 + k\lambda\theta}
\]

Suppose \(p_k\) is the degree distribution in the graph. The probability that a given link points to a vertex of degree \(k\) is \(q_k = kp_k/\mu\) where \(\mu = \sum_j j p_j\), so we have the following self-consistent equation for \(\theta\):

\[
\theta = \sum_k q_k\rho_k = \sum_k q_k \frac{k\lambda\theta}{1 + k\lambda\theta}
\]  \hspace{1cm} (5.2.2) \hspace{1cm} thetaeq

In the Barabási-Albert model \(p_k \sim ck^{-3}\), or in the continuous approximation \(p(x) = 2m^2/\lambda x^3\) for \(x \geq m\). The size biased distribution has \(q(x) = m/x^2\) for \(x \geq m\) and (5.2.2) becomes

\[
\theta = \int_m^\infty \frac{m\lambda}{x + \lambda\theta x} dx = m \int_m^\infty \frac{\lambda}{x + \lambda\theta x} dx - \frac{(\lambda\theta)^2}{1 + \lambda\theta x} dx
\]

The two parts of the integrand are not integrable separately, but if we replace the upper limit of \(\infty\) by \(M\) the integral is

\[
m\lambda\theta \{\log M - \log m\} - m\lambda\theta \{\log(1 + \lambda\theta M) - \log(1 + \lambda\theta m)\}
\]

The first and third terms combine to \(-m\lambda\theta \log(\lambda\theta + 1/M)\) so letting \(M \to \infty\) the integral is

\[-m\lambda\theta (\log m + \log(\lambda\theta) - \log(1 + \lambda\theta m)) = m\lambda\theta \log \left(1 + \frac{1}{m\lambda\theta}\right)\]

and the equation we want to solve is

\[1 = m\lambda \log \left(1 + \frac{1}{m\lambda\theta}\right)\]
Dividing by \( m\lambda \) and exponentiating

\[
e^{1/m\lambda} = 1 + \frac{1}{m\lambda \theta}
\]

Solving for \( \theta \) now we have

\[
\theta = \frac{1}{m\lambda(e^{1/m\lambda} - 1)} = \frac{e^{-1/m\lambda}}{m\lambda} (1 - e^{-1/m\lambda})^{-1} \tag{5.2.3}
\]

in agreement with (12) in [PSV2]. The fraction of occupied sites

\[
\rho = \sum_k p_k \frac{k\lambda\theta}{1 + k\lambda\theta} \sim \lambda \theta \mu \tag{5.2.4}
\]

as \( \lambda, \theta \to 0 \) by the dominated convergence theorem.

In the formula for \( \theta \), \( m \) and \( \lambda \) appear as the product \( m\lambda \). A little thought reveals that this will always be the case if we work with continuous variables, so we will for simplicity restrict our attention to the case \( m = 1 \). Turning to powers between 2 and 3, let \( p(x) = (1 + \gamma)x^{-2-\gamma} \) for \( x \geq 1 \) and assume \( 0 < \gamma < 1 \). In this case the size biased distribution is \( q(x) = \gamma x^{-1-\gamma} \) and (5.2.2) becomes

\[
1 = \int_1^\infty \frac{\gamma}{x^{\gamma} 1 + \lambda \theta x} dx
\]

The right-hand side is a decreasing function of \( \theta \) that is \( \infty \) when \( \theta = 0 \) and \( \to 0 \) when \( \theta \to \infty \) so we know there is a unique solution. Changing variables \( x = u/\lambda \theta, \; dx = du/(\lambda \theta) \) we have

\[
1 = \lambda^{\gamma} \theta^{\gamma-1} \int_{\lambda \theta}^\infty \frac{1}{u^{\gamma} 1 + u} du
\]

Since \( \gamma < 1 \) the integral on the right has a limit \( c_\gamma \) as \( \lambda, \theta \to 0 \). Rearranging we have

\[
\theta \sim (c_\gamma \lambda^{\gamma})^{1/(1-\gamma)} \tag{5.2.5}
\]

in agreement with (22) in PSV(2001b). Again, the fraction of occupied sites

\[
\rho = \sum_k p_k \frac{k\lambda\theta}{1 + k\lambda\theta} \sim \lambda \theta \mu \tag{5.2.6}
\]

as \( \lambda, \theta \to 0 \) by the dominated convergence theorem.

Turning to powers larger than 3, let \( p(x) = (1 + \gamma)x^{-2-\gamma} \) for \( x \geq 1 \) and assume \( \gamma > 1 \). Again the size biased distribution is \( q(x) = \gamma x^{-1-\gamma} \) and (5.2.2) is

\[
1 = \int_1^\infty \frac{\gamma}{x^{\gamma} 1 + \lambda \theta x} dx \tag{5.2.7}
\]
However, now the integral converges when \( \theta = 0 \), so for a solution to exist we must have

\[
\lambda > \lambda_c = 1/\int_1^\infty \frac{\gamma}{x^\gamma} dx = \frac{\gamma - 1}{\gamma}
\]

Letting \( F(\lambda, \theta) \) denote the right-hand side of (5.2.7), we want to solve \( F(\lambda, \theta) = 1 \). If \( \lambda > \lambda_c \), \( F(\lambda, 0) = \lambda/\lambda_c > 1 \). To find the point where \( F(\lambda, \theta) \) crosses 1 we note that

\[
\frac{\partial F}{\partial \theta} = -\int_1^\infty \frac{\gamma}{x^\gamma} \frac{\lambda^2 x}{(1 + \lambda \theta x)^2} dx
\]

When \( \gamma < 2 \), \( \partial F/\partial \theta \to \infty \) as \( \theta \to 0 \). Changing variables \( y = \theta x \) the above becomes

\[
-\int_0^\infty \gamma \theta^\gamma \frac{\lambda^2 y/\theta}{y^\gamma} \frac{dy}{\theta} \sim -\theta^{\gamma-2} \int_0^\infty \frac{\gamma}{y^{\gamma-1}} \frac{\lambda^2}{(1 + \lambda y)^2} dy
\]

Writing \( c_{\gamma,\lambda} \) for the integral (which is finite) and integrating

\[
F(\lambda, \theta) - F(\lambda, 0) \sim -c_{\gamma,\lambda} \theta^{\gamma-1}/(\gamma - 1)
\]

Rearranging

\[
\theta_c \sim \left( (\gamma - 1) \frac{F(\lambda, 0) - 1}{c_{\gamma,\lambda}} \right)^{1/(\gamma-1)}
\]

Recalling \( F(\lambda, 0) = \lambda/\lambda_c \), it follows that

\[
\theta(\lambda) \sim C(\lambda - \lambda_c)^{1/(\gamma-1)}
\]

Thus the critical exponent \( \beta = 1/(\gamma - 1) > 1 \) when \( 1 < \gamma < 2 \). When \( \gamma > 2 \), \( \partial F/\partial \theta \) has a finite limit as \( \theta \to 0 \) and \( \beta = 1 \).

The mean field calculations above will not accurately predict equilibrium densities or critical values (when they are positive). However, they suggest the following conjectures about the contact process on power law graph with degree distribution \( p_k \sim C k^{-\alpha} \).

- If \( \alpha \leq 3 \) then \( \lambda_c = 0 \)
- If \( 3 < \alpha < 4 \), \( \lambda_c > 0 \) but the critical exponent \( \beta > 1 \)
- If \( \alpha > 4 \) then \( \lambda_c = 0 \) and the equilibrium density \( \sim C(\lambda - \lambda_c) \) as \( \lambda \downarrow \lambda_c \)

The first result about the long time survival of the contact process was proved by Berger, Borggs, Chayes, and Saberi in 2005. They considered the preferential attachment model which has a power law with \( \alpha = 3 \), so when they proved that \( \lambda_c = 0 \) they confirmed the physicists’ prediction. Chatterjee and Durrett showed in 2009 that \( \lambda_c > 0 \) is not correct when \( \alpha > 3 \).
Theorem 5.2.1. Consider a graph $G_n$ with $n$ vertices generated by the configuration model with $P(d_i = k) \sim C k^{-\alpha}$ with $\alpha > 3$ and $P(d_i \leq 2) = 0$. Let $\xi^t_1$, $t \geq 0$ denote the contact process on $G_n$ starting from all sites occupied. Then for any $\lambda > 0$ there is a positive constant $p(\lambda) > 0$ so that for any $\delta > 0$

$$\inf_{t \leq \exp(n^{1-\delta})} P\left(\xi^t_1/n \leq p(\lambda)\right) \to 1 \quad \text{as } n \to \infty.$$ 

Sections 5.3 and 5.4 are devoted to the proof of this result.

Figure 5.1: Mean field critical exponents (solid line) versus rigorous results (dashed line) given in (5.2.8) as $\alpha$ varies from 2 to 4.5.

In 2013 Mountford, Valesin, and Yao extended the results of Chatterjee and Durrett to include $2 < \alpha \leq 3$ and proved upper and lower bounds that had the same dependence on $\lambda$ but different constants, showing that

$$\rho(\lambda) \sim \begin{cases} \lambda^{1/(3-\alpha)} & 2 < \alpha \leq 5/2 \\ \lambda^{2\alpha-3} \log^{2-\alpha}(1/\lambda) & 5/2 < \alpha \leq 3 \\ \lambda^{2\alpha-3} \log^{4-2\alpha}(1/\lambda) & 3 < \alpha \end{cases} \quad \text{(5.2.8)}$$

The result for $2 < \alpha \leq 5/2$ agrees with the mean-field calculations quoted above but that formula is claimed to hold for $2 < \alpha < 3$. Figure 2 gives a visual comparison of the mean-field and rigorous results for critical exponents. For more about why the change occurs at 5/2 see the 2013 paper cited above. Three years later, Mountford, Mourrat, Valesin, and Yao (2016) showed that for all $\lambda > 0$, there exists some $c > 0$ so that the survival time $\geq e^{cn}$ with high probability.
5.3 Results for the star graph

The first step in the proof of Theorem 5.2.1 is to prove a lower bound on the survival time of contact process on the star graph with \( n \) leaves. Here, and throughout this chapter, we write the state of the contact process on the star as \((j, k)\) where \( j \) is the number of occupied leaves, \( k = 1 \) if the center is occupied, and \( k = 0 \) otherwise. When the state is a subscript we often drop the parentheses. For example, we let \( T_{0,0} \) be the hitting time of the state \((0, 0)\).

We write \( P_{i,j} \) for the law of the process starting from \((i, j)\), and \( E_{i,j} \) for the expectation.

### 5.3.1 Upper bound for the survival time

To show that our lower bound in Theorem 5.3.2 is sharp, we will begin by proving an upper bound

**Lemma 5.3.1.** Suppose \( \lambda \leq 1 \) and let \( K = \lfloor \lambda n / (\lambda + 1) \rfloor \). If \( K \) is large then the contact process on the star graph with \( n \) leaves has

\[
E_{k,1} T_{0,0} \leq 6 (\log n) e^{(1+\epsilon)\lambda^2 n} \quad \text{for any } \epsilon \in [1/(\log n), 1] \text{ and } k \leq (1 + \epsilon)K.
\]

To explain the importance of \( K \), note that when the center is occupied and the number of occupied leaves is \( k \), the number of occupied leaves increases at rate \( \lambda(n - k) \) and decreases at rate \( k \). The two rates are equal when \( k = K \).

Obviously the longest survival time occurs when the process starts at \( n \). To get an upper bound on the survival time starting from \( n \), assume that the center is always occupied and let \( X_t \) be the number of occupied leaves. When \( X_t = xn \) the infinitesimal mean of \( X_t/n \) is \( \lambda(1 - x) - x \) while the infinitesimal variance is \( O(1/n) \). It is routine to show that if we let \( T_{k,1}^- = \inf\{t : X_t = k\} \) then \( X(t \wedge T_{(1+\epsilon)K}^-)/n \) converges to the solution of

\[
\frac{dx}{dt} = \lambda(1 - x) - x
\]

with \( x(0) = 1 \), stopped when it reaches \( \lambda(1+\epsilon)/(1+\lambda) \) so the time to go from \( n \) to \( (1 + \epsilon)K \) is \( O(1) \). A closer look at the right-hand side of ODE near \( x = \lambda/(1 + \lambda) \) shows that the time is \( O(\log(1/\epsilon)) \).

**Proof.** The proof is surprisingly simple and shows that the event that leads to the extinction of the contact process on the star is that the center becomes vacant and all of the leaves become vacant before the center becomes occupied again. The factor \( \log n \) comes from the fact that the time to go from \( m \) to \( m - 1 \) occupied leaves is an exponential random variable with mean \( O(1/m) \), so the journey from \( K \) to 0 takes time \( O(\log K) \).

It suffices to prove the result when \( k = (1 + \epsilon)K \). The central vertex becomes vacant at rate 1. When it is vacant and there are \( j \) occupied leaves, it becomes occupied at rate \( \lambda j \) and occupied leaves become vacant at rate \( j \). Hence if the center becomes vacant when there
are \( k \) occupied leaves, the probability the process reaches \((0,0)\) before the center is occupied is
\[
\left( \frac{1}{1 + \lambda} \right)^k = e^{-k \log(1 + \lambda)} \geq e^{-\lambda k}. \tag{5.3.1}
\]

To estimate the extinction time we will be concerned with *vacant intervals* that begin when the center becomes vacant and end when it becomes occupied or all the leaves are vacant. Let \( T \) be the length of a vacant interval and suppose that there are initially \( k \) occupied leaves where \( k \leq (1 + \epsilon)K \). Let \( \mathcal{E}(\beta) \) denote an exponential random variable with intensity \( \beta \).

\[
E_{k,0}(T) \leq \sum_{i=1}^{(1+\epsilon)K} E\mathcal{E}((1 + \lambda)i) = \sum_{i=1}^{(1+\epsilon)K} \frac{1}{(1 + \lambda)i} \leq \log(n) + C_0
\]
since \( \epsilon, \lambda \leq 1 \), and \( 1 \leq K \leq \lambda n \).

We also need to estimate the time between the end of one vacant interval and the start of another. In our proof of Lemma 5.3.1 we first bound the amount of time the process stays < \((1 + \epsilon)K\), so in this part of the proof we can suppose that the vacant interval is stopped when \( k = (1 + \epsilon)K \). The transition rates when the center is occupied are:

- \((k, 1) \rightarrow (k - 1, 1)\) at rate \( k \)
- \((k, 1) \rightarrow (k + 1, 1)\) at rate \( \lambda(n - k) \)
- \((k, 1) \rightarrow (k, 0)\) at rate 1

Each jump takes an exponential random time with mean
\[
\frac{1}{\lambda(n - k) + k + 1} \leq \frac{1}{\lambda n + 1},
\]
since we have assumed \( \lambda \leq 1 \). When \( k \leq (1 + \epsilon)K \) the next jump is \((k, 1) \rightarrow (k, 0)\) with probability
\[
\frac{1}{\lambda(n - k) + k + 1} \geq \frac{1}{\lambda(n - (1 + \epsilon)K) + (1 + \epsilon)K + 1},
\]
so the expected time until the center becomes vacant is
\[
\leq \frac{\lambda n + (1 - \lambda)(1 + \epsilon)K + 1}{\lambda n + 1} \leq 1 + 2. \tag{5.3.2}
\]

Combining this with the bound on \( ET \) shows that each cycle from the center healthy to occupied and back to vacant lasts for time \( \leq \log n + C_1 \) where \( C_1 = 3 + C_0 \). The bound in (5.3.1) implies that when \( k \leq (1 + \epsilon)K \) the probability the process reaches 0 before the center becomes occupied is \( \geq \exp(-(1 + \epsilon)\lambda K) \). Thus the expected amount of time spent in \([0, (1 + \epsilon)K]\) before the process hits \((0,0)\) is
\[
\leq (\log n + C_1)e^{(1+\epsilon)\lambda K} \leq 2(\log n)e^{(1+\epsilon)\lambda K}. \tag{5.3.3}
\]
if \( K \) is large (and hence \( n \) is large).

(5.3.2) implies that at \((1 + \epsilon)K\) there is a probability \( \geq 1/(3 + 3\lambda n) \) for the center to become vacant before the number of infected leaves changes, so using (5.3.1) there is a probability
\[
\geq \frac{1}{3 + 3\lambda n} e^{-(1+\epsilon)\lambda K}
\]
to hit \((0, 0)\) before the center becomes infected again. Thus the expected number of visits to \((1 + \epsilon)K\) before dying out is
\[
\leq (3 + 3\lambda n) e^{(1+\epsilon)\lambda K}. \tag{5.3.4}
\]

To estimate the time spent \( \geq (1 + \epsilon)K \), we again assume the center is always occupied, and define a process \( Y_t \) that dominates the number of occupied leaves when \( Y_t \in [(1 + \epsilon)K, n] \)
\[
Y_t \to Y_t + 1 \quad \text{at rate } \lambda(n - (1 + \epsilon)K) \\
Y_t \to Y_t - 1 \quad \text{at rate } (1 + \epsilon)K
\]
Let \( T_m^- = \inf \{ t \geq 0 : Y_t \leq m \} \). We want to estimate \( E_{(1+\epsilon)K} T_{(1+\epsilon)K-1}^- \). The drift of \( Y_t \) is
\[
\mu = \lambda(n - (1 + \epsilon)K) - (1 + \epsilon)K = \lambda n - (1 + \epsilon)(\lambda + 1)K = -\epsilon \lambda n.
\]
Since \( Y_t - \mu t \) is a martingale, the optional stopping theorem gives
\[
E_{(1+\epsilon)K,1}[Y(t \wedge T_{(1+\epsilon)K-1}^-) - \mu(t \wedge T_{(1+\epsilon)K-1}^-)] = (1 + \epsilon)K.
\]
Since \( Y_t \) is a random walk with negative drift, we can let \( t \to \infty \) and conclude
\[
E_{(1+\epsilon)K,1}[T_{(1+\epsilon)K-1}^-] = \frac{1}{-\mu} = \frac{1}{\epsilon \lambda n}.
\]
Therefore, an excursion \( \geq (1 + \epsilon)K \) takes expected time \( \leq 1/\epsilon \lambda n \) to get back to \((1 + \epsilon)K - 1\). This implies that the expected amount of time spent above \((1 + \epsilon)K\) before dying out is
\[
\leq (3 + 3\lambda n) e^{(1+\epsilon)\lambda K} \cdot \frac{1}{\epsilon \lambda n} \leq \frac{4}{\epsilon} e^{(1+\epsilon)\lambda K} \tag{5.3.5}
\]
since \( K = [\lambda n/(1 + \lambda)] \) is large. Using \( \epsilon \geq 1/\log n \) in (5.3.5) and adding (5.3.3) gives the desired result.

\[\square\]

### 5.3.2 Lower bounds on the survival time

There are many lower bounds for the survival time on stars. See Theorem 4.1 in Pemantle (1992), Lemma 5.3 in Berger, Borgs, Chayes, and Saberi (2005), and Lemma 1.1 in Chatterjee and Durrett (2009). These bounds can be used to show that the critical value for prolonged survival of the contact process on some random graphs is 0. Here we will prove a sharp, not only for the mathematical aesthetics but also because it is useful in some circumstances.
Theorem 5.3.2. Let \( L = (1 - 4\delta)\lambda n \) with \( \delta > 0 \). If \( \eta > 0 \) is small then

\[
P_{L,1}(T_{0,0} \geq \frac{1}{\lambda^2 n} e^{(1-\eta)\lambda^2 n}) \to 1 \quad \text{as} \quad n \to \infty.
\]

Combining this with Theorem 5.3.2 we see that if \( \lambda^n \to \infty \) then \( T_{0,0} = \exp((1 + o(1))\lambda^2 n) \).

Following the approach of Chatterjee and Durrett, we will reduce the contact process on a star to a one dimensional chain. We will only look at times when the center is occupied. When the center is vacant and there are \( j \) occupied leaves, the next event will occur after exponential time with mean \( 1/(j\lambda + j) \). The probability that it will be a birth at the center is \( \lambda/(\lambda + 1) \). The probability it will be the death of a leaf particle is \( 1/(\lambda + 1) \). Thus, the number of leaf particles \( Z \) that will be lost while the center is vacant has a shifted geometric distribution with success probability \( \lambda/(\lambda + 1) \), i.e.,

\[
P(Z = j) = \left(\frac{1}{\lambda + 1}\right)^j \cdot \frac{\lambda}{\lambda + 1} \quad \text{for} \quad j \geq 0.
\]  

(5.3.6) \( Z_{\text{dist}} \)

Note that

\[
EZ = \frac{\lambda + 1}{\lambda} - 1 = \frac{1}{\lambda}.
\]

Since we are interested in a lower bound on the survival time, we can simply ignore the time spent when the center is vacant. Here we will construct a process \( X_t \) that gives a lower bound on the number of occupied leaves in the contact process.

Let \( \delta > 0 \) and \( L = (1 - 4\delta)\lambda n \). When there are \( k \leq L \) occupied leaves and the center is occupied, new leaves become occupied at rate

\[
\lambda(n - k) \geq \lambda(n - \lambda n) \geq \lambda(1 - \delta)n
\]

for sufficiently large \( n \) since \( \lambda = \sqrt{c \log n/n} \to 0 \) as \( n \to \infty \).

\[
\begin{array}{cccccc}
  & L & (1 - \delta)\lambda n & L \\
  \downarrow & \uparrow & \downarrow & \downarrow & \downarrow & \downarrow \\
  \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow \\
  j = 1 & \quad & \quad & \quad & \quad & \quad \\
  \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow \\
  j = 0 & \quad & \quad & \quad & \quad & \quad \\
\end{array}
\]

Let \( X_t \) have the following transition rates:

- \( X_t \to X_t - 1 \quad \text{at rate} \quad (1 - \delta)\lambda n \)
- \( X_t \to \min\{X_t + 1, L\} \quad \text{at rate} \quad (1 - \delta)\lambda n \)
- \( X_t \to X_t - Z \quad \text{at rate} \quad 1 \)

Here \( Z \) is independent of \( X_t \) and has the distribution given in (5.3.6).
Lemma 5.3.3. Let $\delta > 0$. Suppose $\lambda = \sqrt{c(\log n)/n}$ and let

$$\theta = \frac{1}{1 + \lambda} \left( \frac{1}{\delta} \right)$$

If $n$ is large then $h(X_t) \equiv (1 - \theta)^{X_t}$ is a supermartingale when $X_t < L$.

Proof. Suppose the current value is $V = (1 - \theta)^{X_t}$ where $X_t \leq L = (1 - 4\delta)\lambda n$. We have

$$V \to V/(1 - \theta) \quad \text{at rate} \leq L$$
$$V \to V(1 - \theta) \quad \text{at rate} \geq (1 - \delta)\lambda n$$
$$V \to V(1 - \theta)^{-Z} \quad \text{at rate 1}$$

The changes in value due to the first two transitions are, if $\theta$ is small,

$$V \left( \frac{1}{1 - \theta} - 1 \right) \leq (1 - \delta)^{-1}\theta V \quad \text{at rate} \leq L$$
$$V[(1 - \theta) - 1] = -\theta V \quad \text{at rate} \geq (1 - \delta)\lambda n$$

We have $L = (1 - 4\delta)\lambda n < (1 - \delta)(1 - 3\delta)\lambda n$, so the first two types of jumps have a net drift

$$(1 - \delta)^{-1}L - (1 - \delta)\lambda n \theta V \leq -2(\delta\lambda n)\theta V. \quad (5.3.7)$$

In the third case, ignoring the fact that the number of occupied leaves cannot drop below 0, we have

$$E(1 - \theta)^{-Z} \leq \sum_{k=0}^{\infty} \left( \frac{1}{1 + \lambda} \right)^k \frac{\lambda}{1 + \lambda} \cdot (1 - \theta)^{-k}$$

$$= \frac{\lambda}{1 + \lambda} \sum_{k=0}^{\infty} \left( \frac{1}{(1 + \lambda)(1 - \theta)} \right)^k$$

$$= \frac{\lambda}{1 + \lambda} \cdot \frac{1}{1 - \frac{1}{(1 + \lambda)(1 - \theta)}} = \frac{\lambda(1 - \theta)}{\lambda - \theta - \theta \lambda}$$

so we have

$$V(E(1 - \theta)^{-Z} - 1) = \frac{\theta V}{\lambda - \theta(1 + \lambda)} = (\delta\lambda n)\theta V$$

for the chosen value of $\theta$. Combining this with (5.3.7) gives that for any $\delta > 0$, $h(X_t)$ is a supermartingale for large $n$.

We use $P_i$ to denote the law of the process $X_t$ starting with $X_0 = i$. The next result completes the proof of Theorem 5.3.2.

Lemma 5.3.4. Let $L = (1 - 4\delta)\lambda n$. If $\eta > 0$ is small then for large $n$

$$P_{L-1} \left( T_{\eta L}^- \geq \frac{1}{\lambda^2 n} e^{(1 - 4\eta)\lambda^2 n} \right) \geq 1 - 2e^{-\eta\lambda^2 n}$$
5.3. RESULTS FOR THE STAR GRAPH

Proof. Suppose $a < x < b$ are integers. Let $T_a^- = \inf\{ t : X_t < a \}$, let $T_b = \inf\{ t : X_t = b \}$ and note that $X(T_b) = b$ while $X(T_a^-) \leq a - 1$. Since $h(X_t)$ is a supermartingale and $h$ is decreasing

$$h(x) \geq h(a - 1)P_x(T_a^- < T_b) + h(b)\left[1 - P_x(T_a^- < T_b)\right]$$

Rearranging we have

$$P_x(T_a^- < T_b) \leq \frac{h(x) - h(b)}{h(a - 1) - h(b)}.$$ 

When $x = b - 1$ this implies

$$P_x(T_a^- < T_b) \leq \frac{h(b - 1) - (1 - \theta)h(b - 1)}{h(a - 1) - h(b - 1)} \leq \frac{\theta h(b - 1) / h(a - 1)}{1 - h(b - 1) / h(a - 1)}.$$

Let $\eta > 0$. We will apply this result with $b = (1 - 4\delta)\lambda n$ and $a = \eta b$. If $\delta$ is small $b \geq (1 - \eta)\lambda n$. If $\lambda$ is small then $1 - \theta < 1 - (1 - \eta)\lambda$. With these choices

$$h(b - 1) / h(a - 1) = (1 - \theta)^{b - a} < (1 - (1 - \eta)\lambda)^{(1 - 2\eta)\lambda n} \leq \exp\left(-(1 - 3\eta)\lambda^2 n\right).$$

If $n$ is large,

$$P_{b-1}(T_a^- < T_b) \leq 2\lambda \exp(-(1 - 3\eta)\lambda^2 n). \quad (5.3.8)$$

Let

$$G_L = \{X_t \text{ returns } M = (1/2\lambda)e^{(1-4\eta)\lambda^2 n} \text{ times to } L \text{ before going } < \eta L\}.$$ 

It follows from (5.3.8) that

$$P(G_L) \geq 1 - e^{-\eta \lambda^2 n}. \quad (5.3.9)$$

In order to return to $L$ we have to jump from $L - 1$ to $L$, a time that dominates an exponential random variable with parameter $\lambda n / 2$.

Let $t_1, t_2, \ldots$ be i.i.d. exponential with mean $2/n\lambda$ and variance $4/(n\lambda)^2$, and let $S_M = t_1 + \cdots + t_M$. Using Chebyshev’s inequality

$$P(S_M \leq M/n\lambda) \leq \frac{4M/(n\lambda)^2}{(M/n\lambda)^2} \leq \frac{4}{M}.$$ 

Since $4/m$ is much smaller than the error in (5.3.9) the proof of Lemma 5.3.4 is complete and Theorem 5.3.2 follows.
5.4 Proving prolonged persistence

A general strategy for showing long-time survival on a finite graph (or for upper bounding $\lambda_2$ on an infinite graph) is the following. Vertices of “large degree” are called hubs.

1. show long-time survival of the process on a star graph when the center is a hub,
2. develop a method for “pushing” particles from one hub to other nearby hubs,
3. estimate the probability of ignition, i.e., going from only the center occupied to a level close to equilibrium.

The first point was taken care of in the previous section. The second and third will be covered in this one.

5.4.1 Pushing the infection

Lemma 5.4.1. Let $v_0, v_1, \ldots, v_r$ be a path in a graph and suppose that $v_0$ is infected at time 0. Then there is a $\gamma > 0$ so that the probability that $v_r$ will become infected by time $2r$ is

$$\geq \left(\frac{\lambda}{\lambda + 1}\right)^r (1 - \exp(-\gamma r)).$$

If $\epsilon > 0$ and we let $\hat{\lambda} = (1 - \epsilon)\lambda / (\lambda + 1)$ then for large $r$ this probability is $\geq \hat{\lambda}^r$.

Proof. The probability that $v_{i-1}$ infects $v_i$ before it is cured is $\lambda / (1 + \lambda)$. When this transfer of infection occurs the amount of time is $t_i$ exponential with rate $1 + \lambda$. By large deviations for the exponential distribution $P(t_1 + \cdots + t_r > 2r) \leq e^{-\gamma r}$ for some $\gamma > 0$.

Lemma 5.4.2. Run the contact process on a graph consisting of a star with $k$ leaves, to which there has been added a single chain $v_1, \ldots, v_r$ of length $r$ where $v_1$ is a neighbor of 0, the center of the star. Suppose that at time 0 there are $L$ infected leaves. For large $r$ the probability that $v_r$ will not be infected before time $T = m(2r + 1)$ is

$$\leq (1 - \hat{\lambda}^r)^m.$$

Proof. Consider a sequence of times $t_i = (2r + 1)i$ for $i \geq 1$. The center 0 may not be infected at time $t_i$ but since the star at 0 is good the number of infected neighbors is $\geq \epsilon L$ and it will with high probability be infected by time $t_i + 1$. By Lemma 5.4.1 the probability $v_r$ is successfully infected in $[t_i, t_{i+1})$ is $\geq \hat{\lambda}^r$ when 0 is good, even if we condition on the events up to time $t_i$. The desired result follows.
5.4. PROVING PROLONGED PERSISTENCE

5.4.2 Hub ignition

We treat $L$ and $K$ in the following lemma as integers for simplicity.

**Lemma 5.4.3.** Suppose $\lambda = \sqrt{c_0 \log n}/n$. Let $T_{0,0}$ be the first time the star is vacant and $T_i$ be the first time the star has $i$ occupied leaves. For any small $\delta > 0$ if $K = \lambda n/\sqrt{\log n}$ and $L = (1 - 4\delta)\lambda n$, then for large $n$

\[(i) P_{0,1}(T_K > T_{0,0}) \leq \frac{3}{\sqrt{\log n}}, \]
\[(ii) P_{K,1}(T_{0,0} < T_L) \leq 2 \exp(-c_0/3\sqrt{\log n}) \]
\[(iii) E_{0,1} \min\{T_{0,0}, T_L\} \leq (1 + \log n)/2\delta \]

**Proof.** Let $p_0(t)$ be the probability a leaf is occupied at time $t$ when there are no occupied leaves at time 0 and the central vertex has been occupied for all $s \leq t$. $p_0(0) = 0$ and

\[\frac{dp_0(t)}{dt} = -p_0(t) + \lambda(1 - p_0(t)) = \lambda - (\lambda + 1)p_0(t)\]

Solving gives $p_0(t) = \lambda(1 - e^{-t})/(\lambda + 1)$. As $t \to 0$, $p_0(t) \sim \lambda t$ so if $t$ is small $p_0(t) \geq \lambda t/2$

Taking $t = 2/\sqrt{\log n}$ it follows that if $B = \text{Binomial}(n, \lambda/\sqrt{\log n})$

\[P_{0,1}(T_K < T_{0,0}) \geq P(B > K) \exp(-2/\sqrt{\log n})\]

The second factor is the probability that the center stays occupied until time $2/\sqrt{\log n}$, and

\[\exp(-2/\sqrt{\log n}) \geq 1 - 2/\sqrt{\log n}.\]

$B$ has mean $\lambda n/\sqrt{\log n}$ and variance $\leq \lambda n/\sqrt{\log n}$ so Chebyshev’s inequality implies

\[P(B < \lambda n/(2\sqrt{\log n})) \leq \frac{\lambda n/\sqrt{\log n}}{\lambda n/(2\sqrt{\log n})} = \frac{4\log n}{\lambda n} \leq \frac{1}{\sqrt{\log n}}\]

For (ii) we use the supermartingale $h(X_t)$ from Lemma 5.5.2.

\[P_{K,1}(T_{0,0} < T_L) \leq 2(1 - \lambda/3)^{\lambda n/\sqrt{\log n}} \leq 2 \exp(-\lambda^2 n/3\sqrt{\log n}) = 2 \exp(-(c_0/3)\sqrt{\log n}).\]

For (iii) we compare with the process $X_t$ in which we ignore the time spent when the center is vacant. To bound the time for the process $X_t$ to reach $L$ or die out we note that $EZ = (\lambda + 1)/\lambda - 1 = 1/\lambda$ so when $n$ is large

\[\mu = (1 - \delta)\lambda n - (1 - 4\delta)\lambda n - 1/\lambda = 3\delta \lambda n - 1/\lambda \geq 2\delta \lambda n\]

gives a lower bound on the drift. Let $\hat{T}_{0,0}$ be the first time $X_t$ hits 0 and $\hat{T}_L$ be the first time $X_t$ hits $L$. $X_t - \mu t$ is a submartingale before time $V_L = \hat{T}_{0,0} \wedge \hat{T}_L$. Stopping the submartingale $X_t - \mu t$ at the bounded stopping time $V_L \wedge s$

\[EX(V_L \wedge s) - \mu E(V_L \wedge s) \geq EX_0 = 0.\]
Since $EX(V_L \land s) \leq L$, it follows that $E(V_L \land s) \leq L/\mu$.

Letting $s \to \infty$ we have $EV_L \leq L/\mu \leq 1/2\delta$ since $L = (1 - 4\delta)\lambda n$ and $\mu \geq 2\delta \lambda n$. Note that the above calculation is for $X_t$ which ignores the time when the center is vacant. To bound the time when the center is vacant, we note that the most extreme excursion that starts at $n$ and goes to 0 takes a time with mean $(\log n)/(1 + \lambda)$. During time $[0, V_L]$ the excursions occur at rate 1, that is, $E_{0,1} \min \{T_{0,0}, T_L\} \leq (1 + \log n)EV_L \leq (1 + \log n)/2\delta$. \qed
5.5. GALTON-WATSON TREES

5.5 Galton-Watson trees

Given an offspring distribution $p_k$, we construct a Galton-Watson tree as follows. Starting with the root, each individual has $k$ children with probability $p_k$. Pemantle has shown that

**Theorem 3.2 in Pemantle (1992).** There are constants $c_2$ and $c_3$ so that if $\mu$ is the mean of the offspring distribution, then for any $k > 1$, if we let $r_k = \max\{2, c_2 \log(1/kp_k)/\mu\}$.

$$\lambda_2 < c_3 \sqrt{r_k \log r_k \log(k)/k}. \tag{5.5.1}$$

If the offspring distribution in the Galton-Watson tree is a stretched exponential $p_k = c_\gamma \exp(-k^\gamma)$ with $\gamma < 1$ then $\log(1/kp_k) \sim k^\gamma$ and hence $\lambda_2 = 0$.

Given this result, it is natural to ask about the critical values $\lambda_1$ and $\lambda_2$ when degrees have a geometric distribution. $p_k = (1 - p)^k p$ for $k \geq 1$. The most interesting problem is to prove $\lambda_1 > 0$. Here, we prove upper bounds.

**Theorem 5.5.1.** $\lambda_1 \leq p/(1 - p)$.

**Proof.** Modify the contact process so that births from a site can only occur on sites further from the root. Each vertex $x$ will be occupied at most once. If $x$ is occupied then it will give birth with probability $\lambda/(\lambda + 1)$ onto each neighbor $y$. The birth events are not independent but that is not important. If we let $Z_n$ be the number of sites at distance $n$ that are ever occupied, $Z_n$ is a branching process in which the offspring distribution has mean $\lambda/((\lambda+1)p)$ which is $> 1$ if $\lambda > p/(1 - p)$.

When $p_k = (1 - p)^k p$, $\log(1/kp_k) \sim c_p k$, so (5.5.1) gives a finite upper bound on $\lambda_2$. It is difficult to trace through all the calculations to get an explicit lower bound. However, Pemantle uses $e^{-1}/5 = 0.0735$ as the lower bound for the probability of long time survival starting with only the center of a large degree star graph occupied, while Lemma 5.4.3 gives $1 - 3k^{-1/3}$ when the degree is $k$. This probability $e^{-1}/5$ appears cubed near the end of his proof, so we think that his bound is much worse than the following:

**Theorem 5.5.2.** If $p_k = 2^{-k}$ for $k \geq 1$, then $\lambda_2 \leq 2.5$.

This result is proved by combining our new estimates for the contact process on stars with the mysterious Lemma 2.4 in Pemantle’s paper (see Lemma 5.5.4 below).

The proof works for a general geometric $p_k = (1 - p)^k p$, $k \geq 1$. We cannot get a nice formula for the upper bound as a function of $p$ but the upper bounds can easily be computed numerically and graphed. These upper bounds are only interesting for small $p$. A Galton-Watson tree with $p_0 = 0$ and $p_1 < 1$ contains a copy of $\mathbb{Z}$ (start with a vertex with two children and follow their descendants) so using Liggett’s bound on $\lambda_c(\mathbb{Z})$ proved in Liggett (1995) we conclude $\lambda_2 \leq 2$ for all $0 < p < 1$.

In addition, the proof of Theorem 5.5.2 yields an improvement of Pemantle’s result for stretched exponential distributions. We say that $p_k$ is subexponential if

$$\limsup_{k \to \infty} (1/k) \log p_k = 0.$$

\[\text{ubglam2} \]
Theorem 5.5.3. If the offspring distribution $p_k$ for a Galton-Watson tree is subexponential and has mean $\mu > 1$ then $\lambda_2 = 0$.

Remark. Due to the way the proof is done, if we condition on 0 being good then successes on two different chains are independent events.

To prepare for the proof of the main results we need the next lemma, which is Lemma 2.4 from Pemantle (1992). Let $\varphi(x) = \sum_{n=0}^{\infty} p_n x^n$ be the generating function of the Galton-Watson tree. We will apply Lemma 5.5.4 to

$f(t) = P(0 \in \xi_0^t) \geq p_k P(0 \in \xi_0^t | 0 \text{ has at least } k \text{ children})$.

Lemma 5.5.4. Let $H$ be any nondecreasing function on the nonnegative reals with $H(x) \geq x$ when $x \in [0, x_0]$. If $f$ satisfies (i) $\inf_{0 \leq s \leq t-L} f(s) > 0$ and (ii) $f(t) \geq H(\inf_{0 \leq s \leq t-L} f(s))$ for $t \geq L$ some $L > 0$ then $\liminf_{t \to \infty} f(t) > 0$.

Proof. For any $t_0$ and $\epsilon > 0$, (ii) implies that there is a decreasing sequence $t_i$ with $t_{i+1} \leq t_i - L$ and $t_k < L$ for some $k$

$$f(t_i) \geq H(f(t_{i+1})) - \epsilon 2^{-i}.$$ 

If $f(t_i) < x_0$ for all $1 \leq i \leq k$ then

$$f(t_i) \geq f(t_{i+1}) - \epsilon 2^{-i}$$

and summing gives $f(t_0) > f(t_k) - \epsilon$ which gives the desired result. Suppose now that $j$ is the smallest index with $f(t_j) > x_0$. If $j = 0$ we have $f(t_0) > x_0$. If $j = 1$ we have $f(t_0) \geq H(x_0)$. If $j \geq 2$ we have

$$f(t_0) \geq f(t_{j-1}) - \epsilon \geq H(x_0) - \epsilon$$

so in all cases we get the desired conclusion. \qed

Proof for $p_n = 2^{-n}$, $n \geq 1$. Our proof follows the outline of the proof of Theorem 3.2 in Pemantle (1992), see pages 2109–2110. We can suppose without loss of generality that the root has degree $k$. Otherwise examine the children of the root until we find one with degree $k$ and apply the argument to the children of this vertex. There are two steps in the proof.

1. Push the infection to vertices at a distance $r = k$ that have degree $k$.

2. Bring the infection back to the root at time $t$ using Lemma 5.5.4.

Step 1. The mean of the offspring distribution 2. Let $Z_r$ be the number of vertices at distance $r$ from 0 and let $v_1^r, \ldots, v_J^r$ be the subset of those that have exactly $k$ children, where $J$ is a random variable that represents the number of such vertices.
Since the root has degree \( k \) and \( p_k = 2^{-k} \) if we set \( r = k \)
\[
EJ \geq k\mu^{r-1}p_k = k/2,
\]
where \( \mu = 2 \) is the mean offspring number.

If we condition on the value of \( W = Z_r/(k\mu^{r-1}) \) and let \( \tilde{J} = (J|W) \) be the conditional distribution of \( J \) given \( W \) then
\[
\tilde{J} = \text{Binomial}(k2^{r-1}W, 2^{-k}).
\]

Let \( M \) be the random number of vertices among \( v_1^r, \ldots, v_J^r \) that are infected before time
\[
S = \frac{1}{2k(2 + \lambda)}(1 + \lambda/2)^{L(1-\epsilon)}
\]
defined in Lemma ???. By Lemma 5.4.2 the probability a given vertex will not become infected by time \( S \) is
\[
p_{noi} \leq (1 - \hat{\lambda}^m)
\]
where \( \hat{\lambda} = (1 - \epsilon) \frac{\lambda}{\lambda + 1} \) and
\[
m = \frac{S}{2k + 1} = \frac{(1 + \lambda/2)^{L(1-\epsilon)}}{2k(2k + 1)(2 + \lambda)} \quad \text{with} \quad L = \frac{\lambda k}{1 + 2\lambda}.
\]

Combining the definitions and using \( (1 - x) \leq e^{-x} \) we have
\[
p_{noi} \leq \exp \left( -\frac{\Gamma^k}{2k(2k + 1)(2 + \lambda)} \right)
\]
where \( \Gamma = \hat{\lambda}(1 + \lambda/2)^{(1-2\epsilon)\lambda/(1+2\lambda)} \).

When \( \lambda = 2.5 \)
\[
\frac{\lambda}{\lambda + 1} (1 + \lambda/2)^{\lambda/(1+2\lambda)} = 1.0014 > 1, \quad (5.5.2)
\]
so \( \Gamma > 1 \) when \( \epsilon \) is small and \( p_{noi} \to 0 \) as \( k \to \infty \). From this we see that if \( \delta > 0 \) then for large \( k \)
\[
EM \geq (1 - \delta)EJ.
\]

The remark after Lemma 5.4.2 implies that if we condition on the value of \( W \) and let \( \tilde{M} = (M|W) \) then
\[
\tilde{M} \geq \text{Binomial}(k2^{r-1}W, 2^{-k}(1 - \delta)).
\]

To prepare for the following two generalizations of the result for Geometric(1/2) offspring distribution we ask the reader to verify that in Step 2, all we use is the fact that (5.5.2) implies the bounds on \( EM \) and \( \tilde{M} \).

**Step 2.** Let \( H_1(t) = P(v_i^r \in \xi_{t-s} \text{ for some } 1 \leq i \leq J) \) and
\[
H_2(t) = P(0 \in \xi_t | v_i^r \in \xi_{t-s} \text{ for some } 1 \leq i \leq J),
\]
so that \( f(t) \geq H_1(t)H_2(t) \). Fix \( t > 2S \) and let
\[
\chi(t) = \inf\{f(s) : s \leq t - S\}.
\]
Since \( t \) is fixed, we simplify the notation and write \( \chi(t) \) as \( \chi \).

Ignore all but the first infection of each \( v_i^r \) by its parent. any of these will evolve independently from the time \( s < S \) it is first infected, and will be infected at time \( t - S \) with probability at least \( \chi \). Thus given \( M \) the number of infected at time \( t - S \) will dominate \( N = \text{binomial}(M, \chi) \). If we let \( \tilde{N} = \text{binomial}(M, \chi) \) and let \( \delta > 0 \), then by Lemma 2.3 in Pemantle (1992) we see that there exists a \( \varepsilon > 0 \) such that
\[
P(\tilde{N} \geq 1) \geq (1 - \delta)\chi \text{EM} \wedge \varepsilon
\]
Therefore \( H_1(t) \geq (1 - \delta)\chi \text{EM} \wedge \varepsilon \) when \( t > 2S \).

Finally, if some \( v_i^r \) is infected at time \( t - S \) then the probability of finding 0 infected at time \( t \) is bounded below by \( \rho_1 \rho_2 \) where

- \( \rho_1 \) is the probability that the contact process starting with only \( v_i^r \) infected at time \( t - S \) infects 0 at some time \( s \) with \( t - S \leq s \leq t \). By Lemmas 5.4.3, ??, and 5.4.2, \( \rho_1 \geq 1 - \delta \).

- \( \rho_2 \) is the probability 0 is infected at time \( t \) given the infection of 0 at such a time \( s \). For any \( \varepsilon > 0 \), by Lemma 5.4.2 the probability that 0 have not been infected by time \( S/2 \) is less than \( \varepsilon \) when \( k \) is sufficiently large. By Lemma ??, with probability \( \geq 1 - (2 + 2\lambda)k^{-1/3} \) there should be at least \( \epsilon L \) infected leaves at time \( t - \epsilon \). Hence 0 is infected at \( t \) with probability at least \( (1 - e^{-\lambda^2 t})e^{-\epsilon} \), where the second term guarantees that the root is infected at time \( t \). Choosing \( \epsilon \) is sufficiently small and \( k \) sufficiently large gives \( \rho_2 \geq 1 - \delta \).

Thus
\[
f(t) \geq \begin{cases} 
\chi(t)\text{EM}(1 - \delta)^3 \wedge \varepsilon & t > 2S, \\
\inf_{0 \leq s \leq 2S} f(s) & S \leq t \leq 2S.
\end{cases}
\]

We can take \( \varepsilon < \inf_{0 \leq s \leq 2S} f(s) \) so that \( f(t) \geq \chi(t)\text{EM}(1 - \delta)^3 \wedge \varepsilon \) for all \( t \geq S \). The result now follows from Lemma 5.5.4 with \( L = S \) and \( H(x) = (1 - \delta)^3(\text{EM})x \wedge \varepsilon \).

**Proof for** \( p_n = (1 - p)^{n-1}p \). It is now straightforward to replace 1/2 by \( p \). We only have to pick \( k \) and \( r \) so that we can prove the analogue of (5.5.2). The mean of the offspring distribution is \( 1/p \). Let \( Z_r \) be the number of vertices at distance \( r \) from 0 and let \( v_i^1, \ldots, v_i^r \) be those that have exactly \( k \) children. Since the root has degree \( k \) and \( p_k = (1 - p)^{k-1}p \)
\[
EJ \geq k(1/p)^{r-1}(1 - p)^{k-1}p.
\]
(5.5.3)

In this case we want to pick \( r \) so that \( (1/p)^r(1 - p)^k \approx 1 \). Hence \( EJ \) can be large when \( k \) is large. Ignoring the fact that \( r \) and \( k \) must be integers this means
\[
r/k = \log(1 - p)/\log p.
\]
Let $M$ be the random number of vertices among $v_1^r, \ldots, v_r^J$ that are infected before time $S$. By Lemma 5.4.2 the probability a given vertex will not become infected is

$$\leq (1 - \hat{\lambda}^r)^{[S/(2r+1)]} \leq \exp\left(-\frac{-\Gamma^k}{2k(2r+1)(2+\lambda)}\right)$$

where $\Gamma = \hat{\lambda}^r/k(1 + \lambda/2)^{(1-2\epsilon)/(1+2\lambda)}$. That is, if we choose $\lambda$ such that

$$\left(\frac{\lambda}{\lambda + 1}\right)^{r/k} \cdot (1 + \lambda/2)^{(1+2\lambda)} > 1$$

then we have $\Gamma > 1$ for large $k$. By the same reasoning as before this choice of $\lambda$ gives an upper bound on $\lambda_2$.

If we want to graph the bound as a function of $p$ it is better to work backwards. Given $\lambda$ the second factor is $> 1$ so we can easily find the value of $r/k$ that makes this 1. Having done this we can easily compute the value of $p$ for which $\lambda$ gives the upper bound on $\lambda_2$.

**Proof for subexponential distributions.** We suppose that the mean of the offspring distribution is $\mu > 1$. If $p_k$ is subexponential, i.e.,

$$\limsup_{k \to \infty} (1/k) \log p_k = 0,$$

then for any $\delta$ there is a $k$ with $p_k \geq (1 - \delta)^k$. It follows from the same reasoning as in (5.5.3) that we can take $r$ such that

$$\frac{r}{k} = -\frac{\log(1 - \delta)}{\log \mu}.$$
References


Chatterjee, S., and Durrett, R. (2009) Contact processes on random graphs with power law degree distributions have critical value 0. *Ann. Probab.* 37, 2332-2356


Huang, X., and Durrett, R. The Contact Process on Periodic Trees arXiv:1808.01863


