Around tilings of a hexagon

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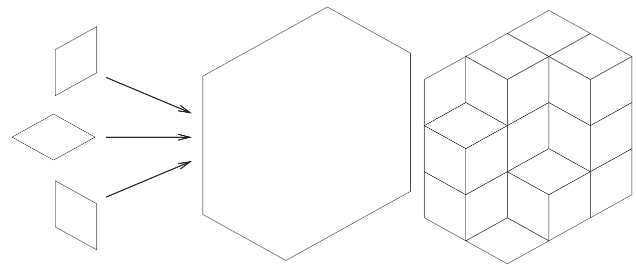
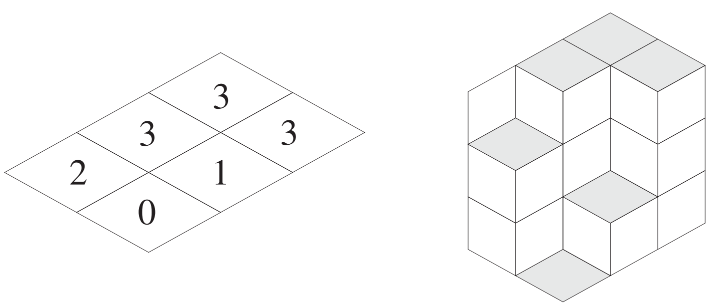
July, 2009

Consider an equi-angular hexagon of side lengths a, b, c, a, b, c. We are interested in tilings of such hexagon by rhombi with angles $\pi/3$ and $2\pi/3$ and side lengths 1 (lozenges).

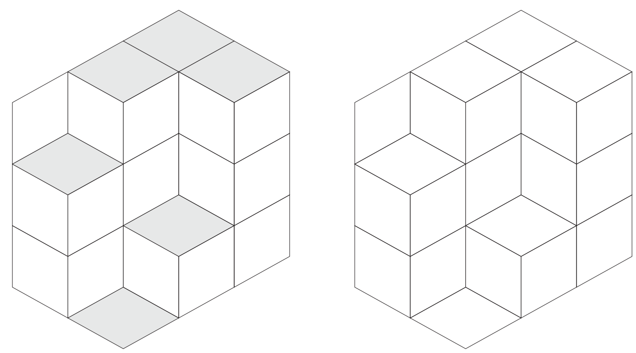


A tiling can be identified with a 3d Young diagram (equivalently a boxed plane partition).



We put k unit cubes on a cell with number k.

Each 3d Young diagram corresponds to its border — stepped surface in $\mathbb{R}^3$.



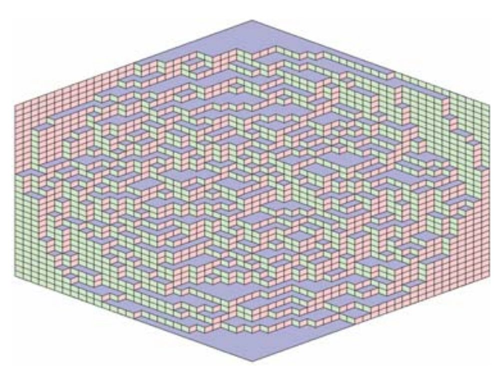
Projection of stepped surface = lozenge tiling.

We want to study random stepped surfaces (random tilings). There are several ways to introduce a randomness here.

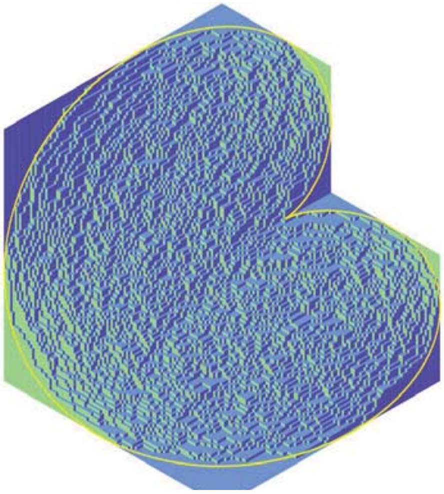
Simplest case: Let us fix an $a \times b \times c$ hexagon. We consider all tilings of this hexagon and equip the set of tilings with uniform measure.

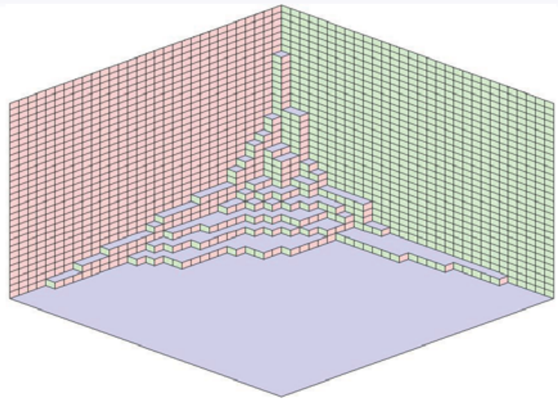
Why is this model interesting? Limit shape phenomena.

In 1998 Cohn, Larsen and Propp proved that scaled random surface corresponding to $Ma \times Mb \times Mc$ hexagon (in other words, one corresponding to tilings by rhombi with side length equal to $1/M$) converges to the deterministic limit shape as M tends to infinity.

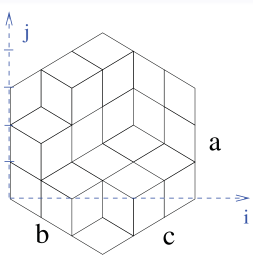


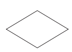
One can also use more general boundary conditions (Kenyon-Okounkov).





Free Young diagrams with weight q^{volume}.
[Cerf-Kenyon; Okounkov-Reshetiknin]





Probability of a tiling is proportional to the product of weights of horizontal lozenges

$$w(\diamond) = \zeta q^j - \frac{1}{\zeta q^j}$$

[There are certain restrictions on q and ζ to ensure positivity]

$$w(\diamond) = \zeta q^j - \frac{1}{\zeta q^j}$$

Degenerations:

1. If $\zeta = 0$, then we get the weight $q^{-volume}$. If $\zeta = \infty$, then we get the weight q^{volume}
2. weight $\alpha j + \beta$
3. weight 1, i.e. uniform measure on tilings

We want to study big random tilings.

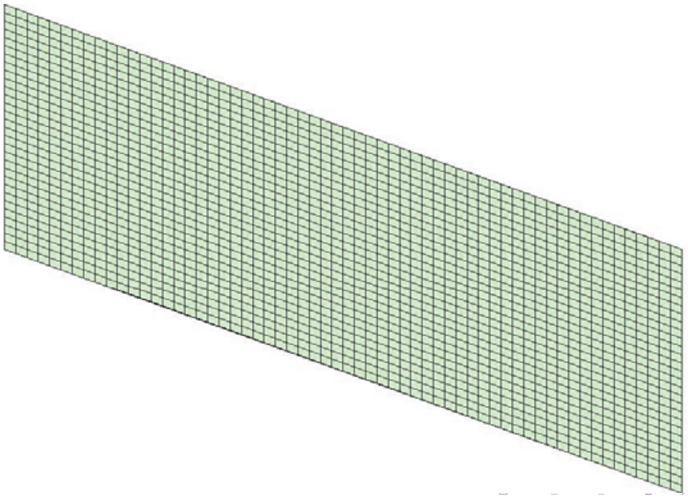
How can we sample from such distribution?

First, note that if $c = 0$ then there is exactly one tiling.

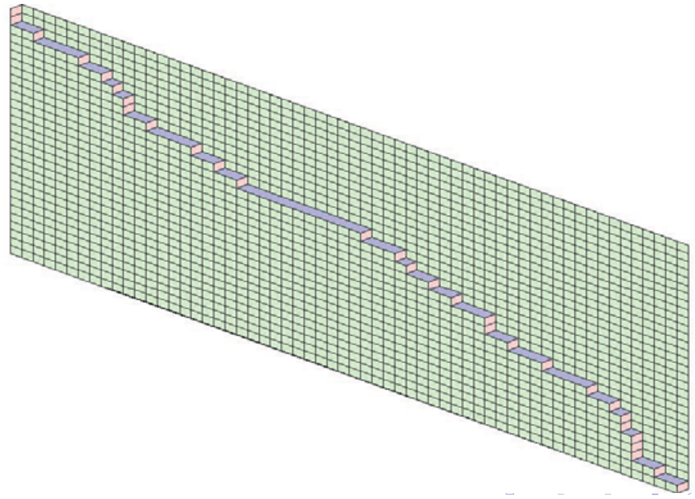
Theorem. (A. Borodin, V.G., 2008 - uniform measure case, A. Borodin, V.G., E. Rains,2009 - general case) There exists a simple discrete time Markov chain which relates random (distributed according to the measurtes above) tilings of hexagons of various sizes. Elementary step of this chain changes the size of hexagon from $a \times b \times c$ to $a \times (b-1) \times (c+1)$.

Algorithmically, one step involves generating some independent one-dimensional random variables. It takes $O(a(b+c))$ arithmetic operations to construct a tiling of $a \times (b-1) \times (c+1)$ hexagon using a tiling of $a \times b \times c$ hexagon

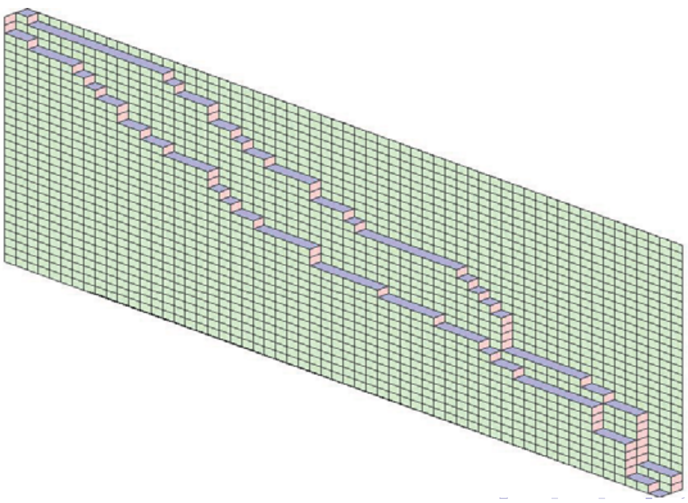
$30 \times 60 \times 0$



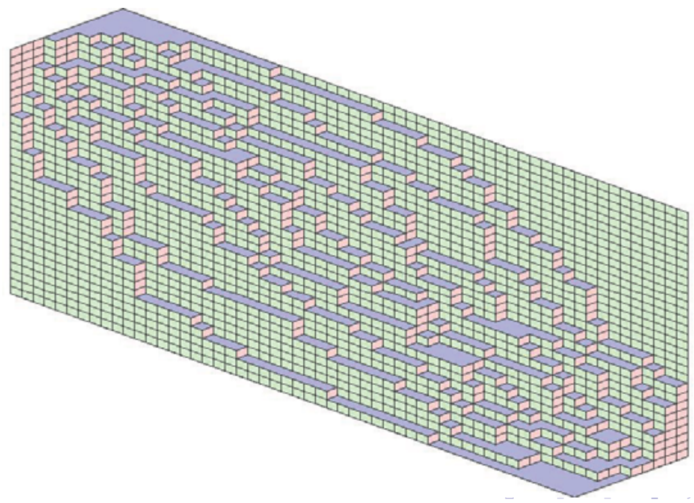
$30 \times 59 \times 1$



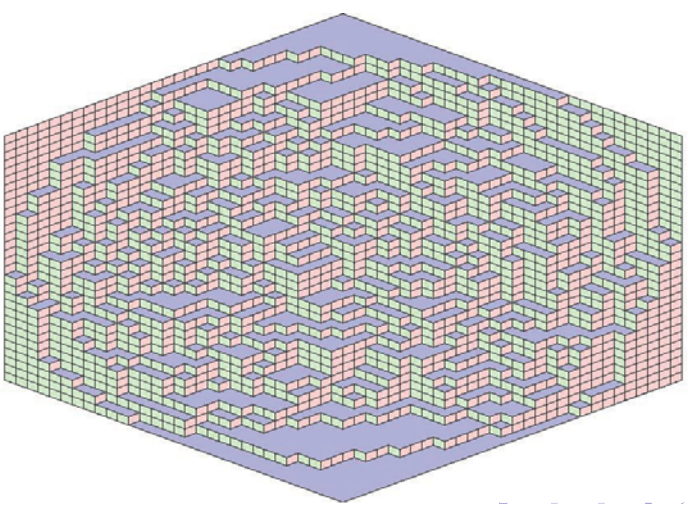
$30 \times 58 \times 2$



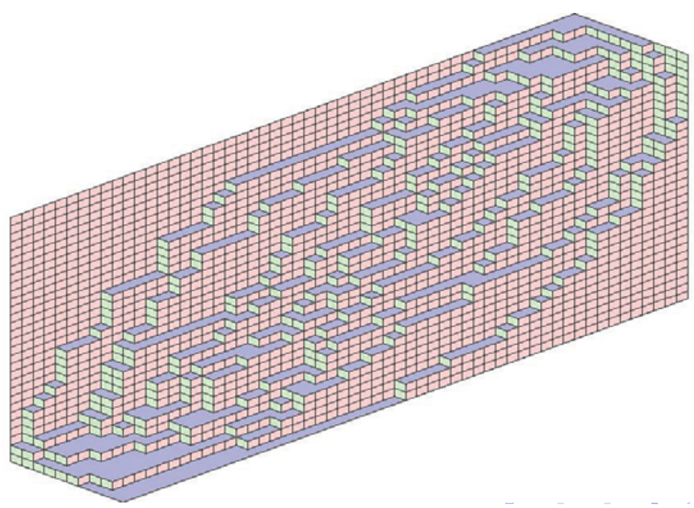
$30 \times 50 \times 10$



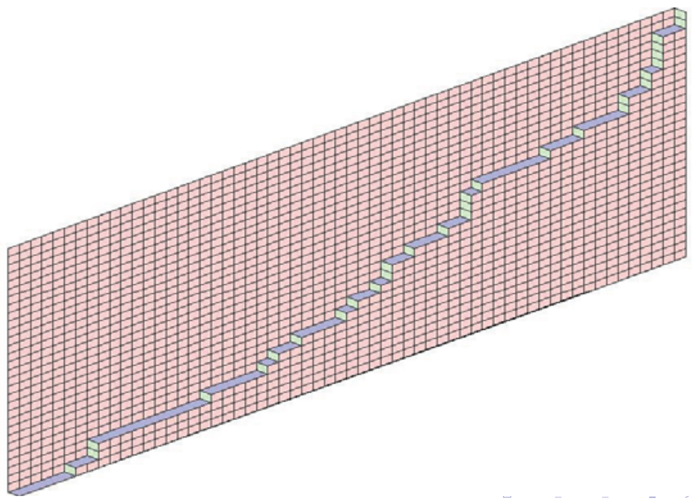
$30 \times 30 \times 30$



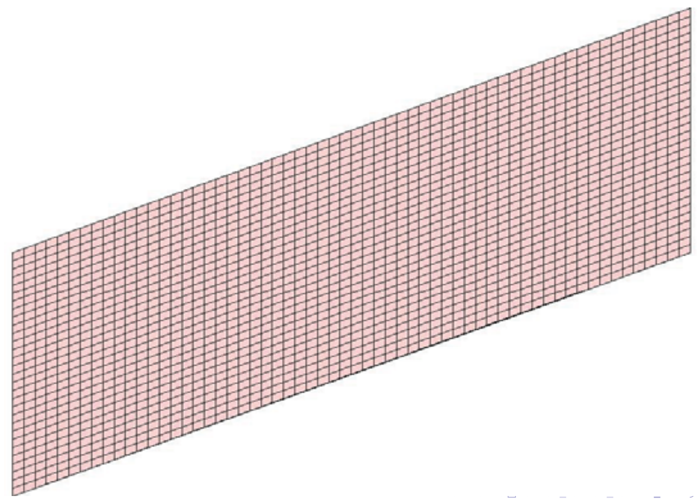
$30 \times 10 \times 50$



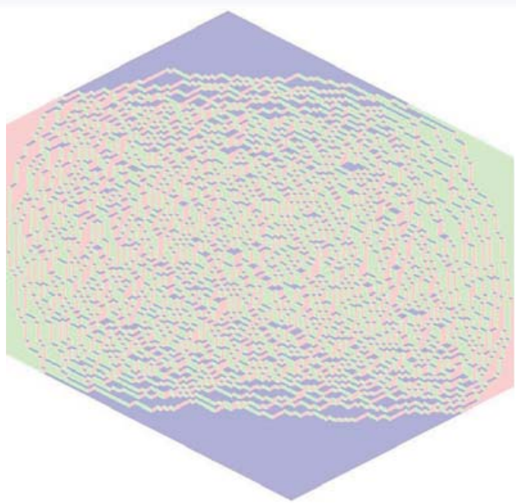
$30 \times 1 \times 59$



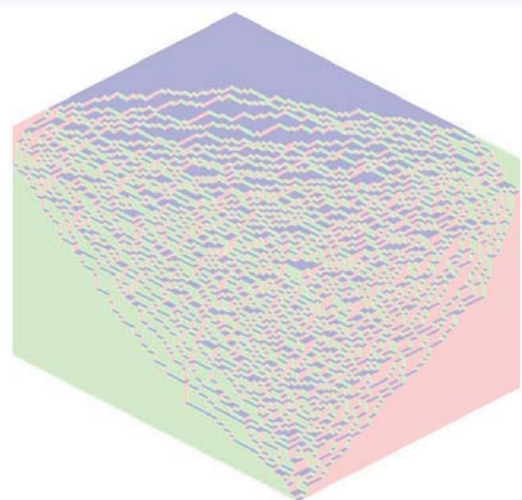
$30 \times 0 \times 60$



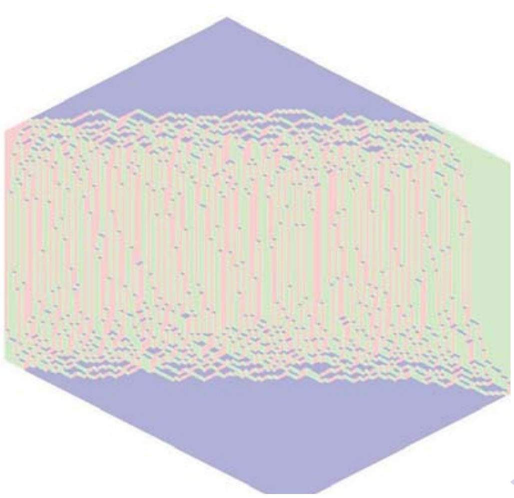
$70 \times 90 \times 70$ box; $q = 0.97$, $\zeta = q^{-71/2}$.



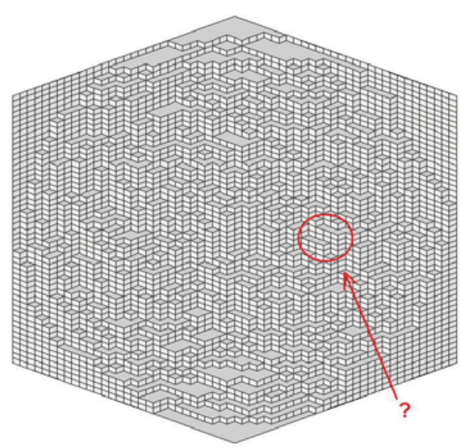
$70 \times 90 \times 70$ box, Linear weight with zero at the corner.



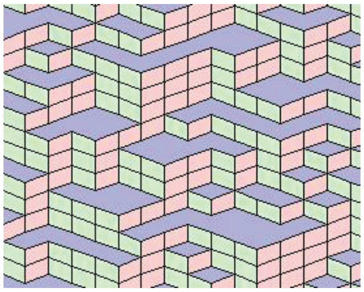
Different limit regime: waterfall.
$70 \times 90 \times 70$ box; $q = 0.9$, $\zeta = q^{-71/2}$



Zoom in



In the limit we see some measure on tilings of the plane



[we erased borders of horizontal lozenges]

Slope of the limit shape $\leftrightarrow$ mean concentrations of lozenges of 3 types.

Limit regime:

$$\rho \to \infty$$

$$a(\rho) = \mathrm{a}\rho + o(\rho)$$

$$b(\rho) = \mathrm{b}\rho + o(\rho)$$

$$c(\rho) = \mathrm{c}\rho + o(\rho)$$

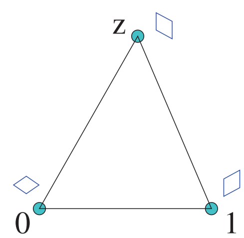
$$\ln q(\rho) = \frac{\ln \mathrm{q}}{\rho} + o(1/\rho)$$

$$\zeta(\rho) = \zeta q(\rho)^{c(\rho)/2}$$

Theorem. (V.G, 2007 - uniform measure, A. Borodin, V.G., E. Rains, 2009 - general case) In the limit we obtain a translation-invariant ergodic Gibbs measure on tilings of the plane. Densities of 3 lozenges (slope of the measure) depend on the coordinates of a point inside the hexagon through certain explicit formula.

Remark 1. Due to S. Sheffield there is a unique ergodic translation invariant Gibbs measure of a given slope.
Remark 2. Correlation functions of these measures have determinantal form.
Remark 3. This theorem allows us to predict the limit shape.

Explicit formula for the slope of the measure is very complicated. It is convenient to encode the slope of the measure (equivalently the slope of the limit shape) with coordinate z:



Angles are proportional to the densities of lozenges of 3 types.

R. Kenyon, A. Okounkov. Case of the measure q^{-vol} (z=0): [Limit shapes]
There exists a second degree polynomial $Q(u,v)$, such that the slope of the limit shape $z(x,y)$ can be found as a solution of:

$$Q(u,v) = 0,$$

$$u = z\mathrm{q}^{\mathrm{x}}, \quad v = (1-z)\mathrm{q}^{\mathrm{y}}.$$

[Here we parameterize the limit shape by its projection on the plane $x_1 + x_2 + x_3 = 0$, and x, y are normalized coordinates $x_2 - x_1$ and $x_3 - x_1$ on this plane.]

A. Borodin, V.G., E. Rains. $w(\diamond) = \zeta q^j - \frac{1}{\zeta q^j}$
[Local measures]
There exists a second degree polynomial $Q(u,v)$, such that the slope of the limit shape (slope of the limit measure) $z(x,y)$ can be found as a solution of:

$$Q(u,v) = 0,$$

$$u = \frac{z\mathrm{q}^{\mathrm{x}} - \zeta^2\mathrm{q}^{2\mathrm{y}}}{1 - z\zeta^2\mathrm{q}^{2\mathrm{y}-\mathrm{x}}}, \qquad v = \frac{(1-z)\mathrm{q}^{\mathrm{y}}}{1 - z\zeta^2\mathrm{q}^{2\mathrm{y}-\mathrm{x}}}$$

How to find $Q(u,v)$? The procedure is the same in both cases.

We consider the frozen regions. In terms of z it means that z is real.

The boundary of the frozen region should be tangent to the boundary of the hexagon (except for the case when the frozen boundary has a node at some vertex of the hexagon). This condition uniquely defines $Q(u,v)$.

What is the number of plane partitions inside $a \times b \times c$ box?

$$\prod_{1 \le i \le a, 1 \le j \le b, 1 \le k \le c} \frac{i+j+k-1}{i+j+k-2} = \prod_{1 \le i \le a, 1 \le j \le b} \frac{i+j+c-1}{i+j-1}.$$

q–partition function?

$$Z(q) = \sum_{\Pi \subset a \times b \times c} q^{vol(\Pi)}$$

$$= \prod_{1 \le i \le a, 1 \le j \le b, 1 \le k \le c} \frac{1-q^{i+j+k-1}}{1-q^{i+j+k-2}} = \prod_{1 \le i \le a, 1 \le j \le b} \frac{1-q^{i+j+c-1}}{1-q^{i+j-1}}.$$

What happens if we replace q^{vol} by the weight function considered above, i.e.

$$w(\Pi) = \prod w(\Diamond),$$

$$w(\Diamond) = \zeta q^j - \frac{1}{\zeta q^j}$$

We get

$$\sum_{\Pi \subset a \times b \times c} q^{vol(\Pi)} \prod_{1 \le i \le a, 1 \le j \le b} \frac{\zeta^2 - q^{i+j-2\Pi_{ij}-2}}{\zeta^2 - q^{i+j-c-2}}$$

$$= \prod_{1 \le i \le a, 1 \le j \le b} \frac{1-q^{i+j+c-1}}{1-q^{i+j-1}}.$$