Brownian couplings – my favorite open problems. I. Shy couplings

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**Motivation**


**Definition of shy coupling**

We say that $X$ and $Y$ form a shy coupling if $X$ and $Y$ have the same transition probabilities, $(X,Y)$ is a Markov process and for some starting points, with positive probability,

$$\text{dist}(X_t,Y_t) > c(\omega) > 0 \quad \forall t \geq 0.$$  

**Theorem**

Suppose that $D \subset \mathbb{R}^2$ is open, bounded, strictly convex, and $C^2$-smooth. Then there are no shy couplings of reflected Brownian motions in $D$.

**Conjectures**

There are no shy couplings for reflected Brownian motions in $D$ if $D$ is open, bounded,

- $D \subset \mathbb{R}^n$ and $D$ is convex
- $D \subset \mathbb{R}^2$ and $D$ is simply connected

There exists a shy coupling of reflected Brownian motions in the annulus.
Open problem

Does there exist a shy coupling of reflected Brownian motions in a disc with off-center hole?

Brownian motion on graphs


Theorem

If all vertices of a graph have degree greater than 2 then there exists a shy coupling of Brownian motions on this graph.

Theorem

If the graph is a finite tree then there are no shy couplings of Brownian motions on this graph.

Graphs with symmetries

If there exists an isometry $S : G \rightarrow G$ with $\inf_x \text{dist}(x, S(x)) > 0$ then there exists a shy coupling of reflected Brownian motions on $G$.

There exists a shy coupling of Brownian motions on this graph. All edges have the same length.
Open problem

Characterize graphs for which shy couplings of Brownian motions exist.

Proofs - outlines

Reflected Brownian motions in convex planar domains.

Extra assumption - there exists a strictly increasing \( \phi: (0, \infty) \to (0, \infty) \) such that

\[
\frac{d}{dt}\left(\sqrt{X_t - Y_t}\right) \geq \phi(|X_t - Y_t|) dt
\]

Let \( f(r) = -r^{-a}, a > 0 \). Apply Ito’s formula to obtain

\[
df(|X_t - Y_t|^2) = dM_t + dV_t
\]

For some \( a, b > 0 \), \( dV_t \leq -b dt \)

Planar convex domains – arbitrary couplings

Idea from “differential games” theory:

\( X_t \) can chase \( Y_t \)

Graphs with no endpoints

Skew Brownian motion: \( X_t = B_t + \beta L^X_t \)

Combinatorial example

\[
\begin{array}{c}
(1,4) \leftrightarrow (2,3) \leftrightarrow (3,5) \leftrightarrow (4,6) \\
(1,2) \leftrightarrow (5,6) \\
(2,1) \leftrightarrow (6,5) \\
(4,1) \leftrightarrow (3,2) \leftrightarrow (5,3) \leftrightarrow (6,4)
\end{array}
\]
Open problem

For given Markov transition probabilities, and an integer n, can one find n Markov processes $X^i_t$ with the given transition probabilities, such that for every pair,

$$dist(X^i_t, X^j_t) > c(\omega) > 0 \quad \forall t \geq 0$$