HODGE THEORY: A REPRESENTATION THEORETIC PERSPECTIVE

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Abstract. Three lectures prepared for the Universitè de Paris summer school on *Representation theory and flag or quiver varieties*, June 2022. Lecture 1: an introduction to formal Hodge theory covering polarized Hodge structures; Hodge groups, domains and representations; and period maps. Lecture 2: Gross's geometric realization problem for Hodge representations of CY type. Lecture 3: characteristic cohomology of the infinitesimal period relation. Lectures 2 and 3 include several open questions. Many exercises are given.

Errata. If you find any errors or typos, please send them to `robles@math.duke.edu`. Updates will be posted at `www.math.duke.edu/∼robles/papers.html`.

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1. Hodge theory

Fix a lattice $V_\mathbb{Z} \simeq \mathbb{Z}^d$ of rank d, and let $V = V_\mathbb{Z} \otimes_\mathbb{Z} \mathbb{Q}$ be the associated rational vector space of dimension d. Fix an integer $n \in \mathbb{Z}$ and a non-degenerate bilinear form

$$Q : V \times V \;\to\; \mathbb{Q}$$

that is (skew-)symmetric

$$Q(u,v) \;=\; (-1)^n \, Q(v,u)\,, \qquad \text{for all} \quad u,v \in V\,.$$

Let $\mathrm{Aut}(V) \simeq \mathrm{GL}_d$ be the group of invertible linear maps $V \to V$, and let $\mathrm{End}(V) \simeq \mathfrak{gl}_d$ be the Lie algebra of linear maps $V \to V$. Let

$$\mathrm{Aut}(V,Q) \;=\; \{\alpha \in \mathrm{Aut}(V) \mid Q(\alpha u, \alpha v) = Q(u,v)\,,\ \forall\ u,v \in V\}$$

be the subgroup of automorphisms preserving Q, and let

$$\mathrm{End}(V,Q) \;=\; \{X \in \mathrm{End}(V) \mid Q(Xu,v) = Q(u,Xv) = 0\,,\ \forall\ u,v \in V\}$$

be its Lie algebra.

A brief review of Hodge theory follows: references include [CMSP17, CEZGT14, GGK12].

1.1. **Hodge structures.** A (pure, rational) *Hodge structure of weight $n \ge 0$* on the vector space V is given by either of the following two equivalent objects: A *Hodge decomposition*

$$V_\mathbb{C} \;=\; \bigoplus_{p+q=n} V^{p,q}\,, \qquad \text{such that} \quad \overline{V^{p,q}} = V^{q,p}\,. \tag{1.1}$$

A (finite, decreasing) *Hodge filtration*

$$0 \,\subset\, F^n \,\subset\, F^{n-1} \,\subset \cdots \subset\, F^1 \,\subset\, F^0 \;=\; V_\mathbb{C} \tag{1.2}$$

such that

$$V_\mathbb{C} \;=\; F^k \,\oplus\, \overline{F^{n+1-k}}\,.$$

The equivalence of the two definitions is given by

$$F^k \;=\; \bigoplus_{p\geq k} V^{p,n-p} \quad \text{and} \quad V^{p,q} \;=\; F^p \cap \overline{F^q}\,.$$

1.1.1. The *Hodge numbers* $\mathbf{h} = (h^{p,q})$ and $\mathbf{f} = (f^p)$ are

$$h^{p,q} \;=\; \dim_{\mathbb{C}} V^{p,q} \quad \text{and} \quad f^p \;=\; \dim_{\mathbb{C}} F^p\,.$$

1.1.2. *Example: weight one.* A weight $n = 1$ Hodge structure is given by a subspace $H^{1,0} = F^1 \subset V_{\mathbb{C}}$ such that $V_{\mathbb{C}} = H^{1,0} \oplus \overline{H^{1,0}}$. We will denote the Hodge numbers $\mathbf{h} = (h^{1,0}, h^{0,1}) = (g, g)$. The Hodge filtration is $F^1 = H^{1,0}$.

1.1.3. *Example: weight two.* A weight $n = 2$ Hodge structure is given by subspaces $H^{2,0} \oplus H^{1,1} \subset V_{\mathbb{C}}$ so that $\overline{H1,1} = H^{1,1}$ and $V_{\mathbb{C}} = H^{2,0} \oplus H^{1,1} \oplus \overline{H^{2,0}}$. We will denote the Hodge numbers $\mathbf{h} = (h^{2,0}, h^{1,1}, h^{0,2}) = (a, b, a)$. The Hodge filtration is $F^2 = H^{2,0}$ and $F^1 = H^{2,0} \oplus H^{1,1}$.

1.1.4. *Example: weight three.* A weight $n = 3$ Hodge structure is given by subspaces $H^{3,0} \oplus H^{2,1} \subset V_{\mathbb{C}}$ so that $V_{\mathbb{C}} = H^{3,0} \oplus H^{2,1} \oplus \overline{H^{2,1} \oplus H^{3,0}}$. We will denote the Hodge numbers $\mathbf{h} = (h^{3,0}, h^{2,1}, h^{1,2}, h^{0,3}) = (a, b, b, a)$. The Hodge filtration is $F^3 = H^{3,0}$, $F^2 = H^{3,0} \oplus H^{2,1}$, and $F^1 = H^{3,0} \oplus H^{2,1} \oplus H^{1,2}$.

1.1.5. Note that (1.1) implies that $\dim V = 2g$ is even when n is odd.

1.1.6. It is implicit in the definition above that we are assuming that the Hodge structure on V is *effective*: $V^{p,q} = 0$ if either $p < 0$ or $q < 0$. Neither this nor the assumption that the weight n is non-negative is necessary (or even desirable, cf. §1.7.2). We restrict to this case for expository convenience.

1.1.7. *Example: compact Kähler manifolds.* The Hodge Theorem asserts that the n-th cohomology group $V = H^n(X, \mathbb{Q})$ of a compact Kähler manifold admits a Hodge structure of weight n. Here $V^{p,q} = H^{p,q}(X) \subset H^n(X, \mathbb{C})$ are the de Rham cohomology classes that can be represented by (p, q)–forms.

1.1.8. The Hodge structure is *Q-polarized* if the *Hodge–Riemann bilinear relations* hold:

$$Q(V^{p,q}, V^{r,s}) \;=\; 0 \qquad \text{if} \qquad (p,q) \neq (r,s)\,, \tag{1.3}$$

$$\mathbf{i}^{p-q}\, Q(v,\bar v) \; > 0 \quad \text{for all} \quad 0 \neq v \in V^{p,q}\,. \tag{1.4}$$

1.1.9. *Example: weight one.* The first Hodge–Riemann bilinear relation is $Q(F^1, F^1) = 0$. Note that F^1 is maximal with this property: $(F^1)^\perp = F^1$. The second Hodge–Riemann bilinear relation is $\mathbf{i}Q(v,\bar v) > 0$ for all $0 \neq v \in H^{1,0}$.

1.1.10. *Example: weight two.* The first Hodge–Riemann bilinear relation is $Q(F^2, F^1) = 0$. In this case we have $(F^2)^\perp = F^1$. The second Hodge–Riemann bilinear relation asserts that $-Q(u,u) > 0$ for all $0 \neq u \in H^{2,0}$ and $Q(v,v) > 0$ for all $0 \neq v \in H^{1,1}$.

1.1.11. *Example: weight three.* The first Hodge–Riemann bilinear relation is $Q(F^2, F^2) = 0$. Again, F^2 is maximal with this property: $(F^2)^\perp = F^2$. The second Hodge–Riemann bilinear relation is $-\mathbf{i}Q(u,\bar u) > 0$ for all $0 \neq u \in H^{3,0}$, and $\mathbf{i}Q(v,\bar v) > 0$ for all $0 \neq v \in H^{2,1}$.

1.1.12. *Example: smooth projective varieties.* Let $X \subset \mathbb{P}^N$ be a projective manifold of dimension d with hyperplane class $\omega \in H^2(X,\mathbb{Z})$. Given $n \leq d$, the primitive cohomology

$$V \;=\; \{\alpha \in H^n(X,\mathbb{Q}) \mid \omega^{d-n+1} \wedge \alpha = 0\}$$

inherits the weight n Hodge decomposition

$$V_{\mathbb{C}} \;=\; \bigoplus_{p+q=n} H^{p,q}(X) \cap V_{\mathbb{C}}$$

from $H^n(X,\mathbb{Q})$. The Hodge–Riemann bilinear relations for X assert that this Hodge structure is polarized by

$$Q(\alpha,\beta) \;=\; (-1)^{(d-n)(d-n-1)/2} \int_X \alpha \wedge \beta \wedge \omega^{d-n}\,.$$

1.1.13. *Exercise.* The real automorphism group $\mathrm{Aut}(V_\mathbb{R}, Q)$ is isomorphic to:

- $\mathrm{Sp}(2g, \mathbb{R})$, where $2g = \dim V$, when n is odd;
- $\mathrm{O}(b, 2a)$, where

$$b \;=\; \sum_k h^{m+2k,m-2k} \quad \text{and} \quad 2a \;=\; \sum_k h^{m+1+2k,m-1-2k}\,,$$

  when $n = 2m$ is even.

Verify this, at least for $n \le 3$.

1.1.14. *Exercise.* The assignment

$$\mathcal{H}(u,v) \;=\; \mathbf{i}^n\, Q(u,\bar v)$$

defines a nondegenerate Hermitian form on $V_\mathbb{C}$ of signature

- (g,g), where $2g = \dim V$, when n is odd;
- $(b, 2a)$, when $n = 2m$ is even.

Verify this, at least for $n \le 3$.

1.2. **Period domains and compact duals.** The first Hodge–Riemann bilinear relation (1.3) asserts that the Hodge filtration (1.2) is *Q–isotropic*

$$Q(F^p, F^q) \;=\; 0\,, \qquad \text{for all} \quad p+q = n+1\,.$$

This is precisely the statement that the Hodge filtration is an element of the complex flag manifold

$$\check{\mathcal{D}} \;=\; \mathrm{Flag}^Q(\mathbf{f}, V_\mathbb{C}) \tag{1.5}$$

of Q–isotropic filtrations $F = (F^p)$ of $V_\mathbb{C}$. The variety $\check{\mathcal{D}}$ is the *compact dual* of the period domain $\mathcal{D}$ (which will be defined next).

1.2.1. *Example: weight one.* The compact dual is $\mathbb{P}^1$, when $g = 1$.

For $g \geq 1$, the compact dual is the Lagrangian grassmannian $\mathrm{LG}(g, \mathbb{C}^{2g})$ of g-dimensional subspaces $F^1 \subset V_{\mathbb{C}} =\simeq \mathbb{C}^{2g}$ that are isotropic with respect to a nondegenerate skew-symmetric bilinear form.

1.2.2. *Example: weight two.* The compact dual is the Grassmannian $\mathrm{Gr}^Q(a, \mathbb{C}^{2a+b})$ of a–dimensional subspaces $F^2 \subset V_{\mathbb{C}} \simeq \mathbb{C}^{2a+b}$ that are isotropic with respect to the nondegenerate, symmetric bilinear form Q.

1.2.3. *Example: weight three.* The compact dual is the isotropic flag manifold $\mathrm{Flag}^Q(a, g; \mathbb{C}^{2g})$, consisting of pairs $F^3 \subset F^2$ with $F^2 \in \mathrm{LG}(g, \mathbb{C}^{2g})$, $\dim_{\mathbb{C}} F^3 = a$, and where $g = a + b$.

1.2.4. *Exercise.* The complex automorphism group $\mathrm{Aut}(V_{\mathbb{C}}, Q)$ acts transitively on $\check{D}$. Verify this, at least for weight $n \leq 3$.

1.2.5. The *period domain* $\mathcal{D} = \mathcal{D}_{\mathbf{h},Q}$ is the set of all Q-polarized Hodge structures on V with Hodge numbers $\mathbf{h}$.

1.2.6. *Example: weight one.* When $g = 1$ the period domain is the upper-half plane, and $\mathrm{Aut}(V_{\mathbb{R}}, Q) = \mathrm{Sp}(2, \mathbb{R}) = \mathrm{SL}(2, \mathbb{R})$ acts transitively.

For $g \geq 1$, the period domain $\mathcal{D} = \mathrm{Sp}(2g, \mathbb{R})/U(g)$ is the Siegel upper-half space of symmetric $g \times g$ matrices with complex entries and positive definite imaginary part. Alternatively $\mathcal{D}$ is the set of $E \in \mathrm{LG}(g, \mathbb{C}^{2g})$ with the property that the Hermitian form $\mathbf{i}Q(u, \bar{u})$ restricts to be positive definite on E.

We recover the Hodge decomposition from E by setting $H^{1,0} = E$ and $H^{0,1} = \overline{E}$.

1.2.7. *Example: weight two.* The period domain $\mathcal{D} = \mathrm{O}(b, 2a)/\mathrm{U}(a) \times \mathrm{O}(b)$ is the subset of elements $E \in \mathrm{Gr}^Q(a, \mathbb{C}^{2a+b})$ on which the Hermitian bilinear form $-Q(u, \bar{v})$ restricts to be positive definite.

We recover the Hodge decomposition from E by setting $H^{2,0} = E$ and $H^{0,2} = \overline{E}$, and $H^{1,1} = (E \oplus \overline{E})^{\perp}$.

1.2.8. *Example: weight three.* The period domain $\mathcal{D} = \mathrm{Sp}(2g,\mathbb{R})/\mathrm{U}(a)\times\mathrm{U}(b)$ is the subset of filtrations $(F^3 \subset F^2) \in \mathrm{Flag}^Q(a,g;\mathbb{C}^{2g})$ with the property that the Hermitian form $-\mathbf{i}Q(u,\bar{v})$ restricts to be positive definite on F^3, and nondegenerate on F^2 with signature (a,b).

1.2.9. *Exercise.* The real automorphism group $\mathrm{Aut}(V_\mathbb{R},Q)$ of §1.1.13 acts transitively on $\mathcal{D}$ with compact isotropy H isomorphic to:

- $\mathrm{U}(h^{n,0})\times\cdots\times\mathrm{U}(h^{m+1,m})$, if $n=2m+1$ is odd;
- $\mathrm{U}(h^{n,0})\times\cdots\times\mathrm{U}(h^{m+1,m-1})\times\mathrm{O}(h^{m,m})$, if n is even.

Verify this, at least for $n\le 3$.

1.2.10. *Exercise.*

(a) Show that $\mathcal{D}\subset\check{\mathcal{D}}$ is open (in the analytic topology). In particular, $\mathcal{D}$ inherits the structure of a complex manifold from $\check{D}$, and is a "flag domain" in the sense of [Wol69, FHW06].

(b) The stabilizer of $F\in\check{\mathcal{D}}$ in $\mathrm{Aut}(V_\mathbb{C},Q)$ is a parabolic subgroup P. Let $U\subset P$ denote the unipotent racial. Show that the compact isotropy $H=\mathrm{Aut}(V_\mathbb{R},Q)\cap P$ is a real form of the reductive Levi quotient P/U.

1.2.11. *Remark.* The compact dual $\check{\mathcal{D}}$ parameterizes filtrations satisfying the first Hodge–Riemann bilinear relation, and the period domain $\mathcal{D}\subset\check{\mathcal{D}}$ parameterizes filtrations satisfying both Hodge–Riemann bilinear relations.

1.3. **Hodge structures: a third definition.** We have seen that a Hodge structure may be defined by either a Hodge decomposition (1.1), or by a Hodge filtration (1.2). There is a third definition by group homomorphisms. Let $\mathbb{C}^\times=\mathbb{C}\backslash\{0\}$ be the group of nonzero complex numbers. Define a homomorphism

$$\tilde{\varphi}:\mathbb{C}^\times \to \mathrm{Aut}(V_\mathbb{R}) \tag{1.6}$$

by specifying

$$\tilde\varphi(z) \;=\; z^p \bar z^q \, v\,, \qquad \text{for all} \quad v \in V^{p,q}\,;$$

that is, we specify that the Hodge decomposition (1.1) is an eigenspace decomposition for $\tilde\varphi$.

1.3.1. Observe that (1.6) satisfies $\varphi(x) = x^n \, \mathrm{Id}$, for all nonzero real numbers $x \in \mathbb{R}^\times$.

1.3.2. *Exercise.*

(a) Verify that $\tilde\varphi$ does indeed take value in $\mathrm{Aut}(V_\mathbb{R})$.

(b) Verify that the restriction $\tilde\varphi|_{S^1}$ takes value in $\mathrm{Aut}(V_\mathbb{R}, Q)$ if and only if the Hodge structure is Q–polarized.

1.3.3. We wish to view $\mathbb{C}^\times$ as real group. At the very least you should think of it a real Lie group. If you are familiar with algebraic groups, then you should think of $\mathbb{C}^\times$ as the real points

$$\mathbf{S}(\mathbb{R}) \;=\; \left\{ \begin{bmatrix} x & -y \\ y & x \end{bmatrix} \;\middle|\; \begin{array}{l} x, y \in \mathbb{R} \\ x^2 + y^2 \neq 0 \end{array} \right\}$$

of the *Deligne torus*, the $\mathbb{R}$–algebraic group $\mathbf{S} = \mathrm{Res}_{\mathbb{C}/\mathbb{R}}\, G_{m,\mathbb{C}}$. Likewise, we identify $S^1 \subset \mathbb{C}^\times$ with the maximal compact subgroup

$$\mathbf{U}(\mathbb{R}) \;=\; \left\{ \begin{bmatrix} x & -y \\ y & x \end{bmatrix} \;\middle|\; \begin{array}{l} x, y \in \mathbb{R} \\ x^2 + y^2 = 1 \end{array} \right\}.$$

1.3.4. *Exercise.* Conversely suppose that you are given a homomorphism (1.6), with the property that $\tilde\varphi|_{\mathbb{R}^\times}$ is defined over $\mathbb{Q}$.

(a) Show that $V \;=\; \oplus_{n\in\mathbb{Z}} V_n$ where

$$V_n \;=\; \{ v \in V \mid \tilde\varphi(x)(v) = x^n \, \mathrm{Id}\,,\ x \in \mathbb{R}^\times \}\,.$$

(b) Set $V^{p,q} = \{ v \in V_\mathbb{C} \mid \tilde\varphi(z)(v) = z^p \bar z^q \, v\,,\ \forall\, z \in \mathbb{C}^\times \}$. Show that $V_{n,\mathbb{C}} = \oplus_{p+q=n} V^{p,q}$ is a Hodge decomposition of weight n.

1.3.5. The upshot of the discussion above is that we may define a (real) *Hodge structure* as a homomorphism (1.6) of $\mathbb{R}$–algebraic groups. The Hodge structure is *rational* if $\tilde{\varphi}|_{\mathbb{R}^\times}$ is defined over $\mathbb{Q}$; it is *pure* of weight $n \in \mathbb{Z}$ if $\tilde{\varphi}(r) = r^n \,\mathrm{Id}$ for all $r \in \mathbb{R}^\times$; and if the Hodge structure is Q–polarized, then $\varphi = \tilde{\varphi}|_{S^1}$ takes value in $\mathrm{Aut}(V_{\mathbb{R}}, Q)$. We may identify the period domain $\mathcal{D}$ with the $\mathrm{Aut}(V_{\mathbb{R}}, Q)$ conjugacy classes of φ, and the isotropy group H is clearly seen to be the centralizer of the circle $\varphi : S^1 \to \mathrm{Aut}(V_{\mathbb{R}}, Q)$.

1.4. **Hodge groups and domains.** The image $\varphi(S^1) \subset \mathrm{Aut}(V_{\mathbb{R}}, Q)$ is an $\mathbb{R}$–algebraic subgroup. The *Hodge group* $\mathbf{G}_\varphi \subset \mathrm{Aut}(V, Q)$ is the $\mathbb{Q}$–Zariski closure of $\varphi(S^1)$; that is, $\mathbf{G}_\varphi$ is the smallest $\mathbb{Q}$–algebraic subgroup of $\mathrm{Aut}(V, Q)$ that contains $\varphi(S^1)$.

The *Hodge (sub)domain* is the real orbit $D_\varphi = \mathbf{G}_\varphi(\mathbb{R}) \cdot \varphi \subset \mathcal{D}$. The *compact dual* is the complex orbit $\check{D}_\varphi = \mathbf{G}_\varphi(\mathbb{C}) \cdot \varphi \subset \check{D}$. Again, the complex stabilizer $P_\varphi \subset \mathbf{G}_\varphi(\mathbb{C})$ of φ is a parabolic subgroup (see §1.7.6 for a description of the Lie algebra), and $\check{D}_\varphi = \mathbf{G}_\varphi(\mathbb{C})/P_\varphi$ is a generalized flag manifold (a.k.a. a rational homogeneous variety). Likewise, the real stabilizer $H_\varphi \subset G_\varphi$ of φ is (isomorphic to) a real form of the Levi quotient P_φ/U_φ, where U_φ is the unipotent radical of P_φ.

To ease notation, from this point forward, we will often drop the subscript φ from these objects.

1.5. **Hodge tensors.** Hodge groups are the symmetry groups of Hodge theory: $\mathbf{G} = \mathbf{G}_\varphi$ is precisely the subgroup of $\mathrm{Aut}(V, Q)$ that acts trivially on all the Hodge tensors of φ. A brief discussion follows; for details see [GGK12, Pat16].

1.5.1. Fix a Hodge structure $\varphi : S^1 \to \mathrm{Aut}(V_{\mathbb{R}}, Q)$. If the weight $n = 2m$ is even, then the *Hodge classes* of φ are the rational (m, m) classes $V \cap V^{m,m}$. These are precisely the elements of V upon which the circle φ acts trivially. (A priori the intersection may be trivial.) There are no Hodge classes in odd weight. But there may be Hodge tensors.

1.5.2. *Exercise.* Let

$$\mathcal{T}(V) \;=\; \bigoplus_{k,\ell} V^{\otimes k} \otimes (V^*)^{\otimes \ell}$$

denote the tensor algebra. The action of $\mathbb{C}^\times$ on $V_\mathbb{R}$ (via $\tilde\varphi$) naturally induces an action on $\mathcal{T}(V_\mathbb{R})$. We also denote this induced action by $\tilde\varphi : \mathbb{C}^\times \to \mathrm{Aut}(\mathcal{T}(V_\mathbb{R}))$. It naturally defines a Hodge structure on $\mathcal{T}(V)$ by

$$\mathcal{T}(V_\mathbb{C}) \;=\; \bigoplus_{p,q} \mathcal{T}(V)^{p,q} \,,$$

where

$$\mathcal{T}(V)^{p,q} \;=\; \{\tau \in \mathcal{T}(V_\mathbb{C}) \mid \tilde\varphi(z)\tau \;=\; z^p \bar z^q \tau\}\,.$$

1.5.3. The algebra of *Hodge tensors* is

$$\mathrm{Hg}(\varphi) \;=\; \mathcal{T}(V) \cap \bigoplus_p \mathcal{T}(V)^{p,p}\,.$$

These are precisely the (rational) elements of $\mathcal{T}(V)$ upon which the circle S^1 acts trivially. The Hodge group is the stabilizer

$$\mathbf{G}_\varphi \;=\; \{g \in \mathrm{Aut}(V,Q) \mid g\cdot\tau = \tau\,,\ \forall\, \tau \in \mathrm{Hg}(\varphi)\}$$

of the Hodge tensors.

1.5.4. *Exercise.* Verify that $Q \in \mathcal{T}(V)$ is a Hodge tensor.

1.5.5. *How to think of Hodge groups and domains.* For a generic choice of $\varphi \in \mathcal{D}$, we will have $\mathbf{G}_\varphi = \mathrm{Aut}(V,Q)$. Essentially, the only Hodge tensor is the polarization Q. At the other extreme, the Hodge group can be a torus. An example is the Hodge structure of an elliptic curve with complex multiplication, where $\mathbf{G}_\varphi(\mathbb{R}) = \varphi(S^1)$.

In general, the Hodge group is reductive, and has the property that $\mathbf{G}_\varphi(\mathbb{R})$ contains a compact maximal torus. Conversely, any such group may be realized as a Hodge group. So for example, $\mathrm{SU}(a,b)$ may be realized as some $\mathbf{G}_\varphi(\mathbb{R})$ so long as $a,b>0$; but $\mathrm{SL}_k\mathbb{R}$ can *not*, if $k \geq 3$.

For generic choice of Hodge structure $\varphi \in \mathcal{D}$, the Hodge domain D_φ is the entire period domain. However, at the other extreme, when $\mathbf{G}_\varphi$ is a torus, we have $D_\varphi = \{\varphi\}$.

In general, given another Hodge structure $\varphi' \in D_\varphi$, the Hodge groups and domains will satisfy $\mathbf{G}_{\varphi'} \subset \mathbf{G}_\varphi$, and $D_{\varphi'} \subset D_\varphi$. In both cases, equality will hold for generic choice of $\varphi' \in D_\varphi$.

When the containment $D \subset \mathcal{D}$ is strict, the Hodge domain is parameterizing Hodge structures with nongeneric Hodge tensors.

1.6. **Hodge representations.** A (polarized) *Hodge representation* (of weight n) consists of:

(i) a faithful morphism

$$\rho : \mathbf{G} \hookrightarrow \mathrm{Aut}(V, Q)$$

of $\mathbb{Q}$–algebraic groups, and

(ii) a nontrivial morphism $\varphi : S^1 \to \mathbf{G}(\mathbb{R})$ of $\mathbb{R}$–algebraic groups such that the eigenspaces

$$V_\varphi^{p,q} \;=\; \{v \in V_\mathbb{C} \mid \varphi(z)\, v = z^{p-q}\, v\}\,,$$

are the summands in a weight n, Q–polarized Hodge decomposition

$$V_\mathbb{C} \;=\; \bigoplus_{p+q=n} V_\varphi^{p,q}\,.$$

A priori $\mathbf{G}$ contains the Hodge group $\mathbf{G}_\varphi$ of φ. Replacing $\mathbf{G}$ with the Hodge group if necessary we may assume without loss of generality that equality holds. ss

Additional discussion of Hodge representations may be found in [GGK12, HR20].

Set

$$G \;=\; \mathbf{G}(\mathbb{R}) \quad \text{and} \quad \check{G} \;=\; \mathbf{G}(\mathbb{C})\,.$$

1.6.1. Given a Hodge representation (φ,ρ), the circle φ may be viewed as an element of the period domain $\mathcal{D}$ parameterizing Q–polarized Hodge structures on V with Hodge numbers $h^{p,q} = \dim_{\mathbb{C}} V^{p,q}_\varphi$. Let $D = G\cdot\varphi \subset \mathcal{D}$ be the Hodge domain. Note that the inclusion $\iota_\rho : D \hookrightarrow \mathcal{D}$ is a G–equivariant embedding. In fact, this map is the restriction to D of a $\check{G}$–equivariant embedding $\iota_\rho : \check{D} \hookrightarrow \check{\mathcal{D}}$.

1.6.2. *Exercise.* As an abstract homogeneous space D does not depend on the choice of ρ. Show that $D = G/H$ is determined by $\varphi : S^1 \to G$ alone.

1.6.3. *Exercise.* Show that $\rho : \mathbf{G} \hookrightarrow \mathrm{Aut}(V,Q)$ is a Hodge representation for every choice of $\varphi' \in D$.

1.7. **Exercises on the induced Hodge structure on the Lie algebra.** Fix a Hodge structure φ with Hodge group $\mathbf{G} \subset \mathrm{Aut}(V,Q)$, and let $\mathfrak{g}$ be the Lie algebra.

1.7.1. *Exercise.* Show that

$$\mathfrak{g}_{\mathbb{C}} \;=\; \bigoplus_{p\in\mathbb{Z}} \mathfrak{g}^{p,-p}_\varphi\,, \tag{1.7}$$

where

$$\begin{aligned}
\mathfrak{g}^{p,-p}_\varphi &= \{X \in \mathfrak{g}_{\mathbb{C}} \mid \mathrm{Ad}_{\varphi(z)}(X) = z^{2p}\,X\}\\
&= \{X \in \mathfrak{g}_{\mathbb{C}} \mid X(V^{r,s}_\varphi) \subset V^{r+p,s-p}_\varphi\}\,.
\end{aligned}$$

1.7.2. *Exercise.* Show that (1.7) is a weight zero Hodge structure on $\mathfrak{g}$.

1.7.3. *Exercise.* Show that the Hodge structure is polarized by $-\kappa$, where κ is the Killing form. In particular, given a Hodge structure as in §1.3.5, the pair (Ad,φ) is always a Hodge representation (§1.6).

From this we may deduce that $\mathfrak{k}_{\mathbb{C}} = \oplus_p\,\mathfrak{g}^{2p,-2p}$ is the complexification of a maximal compact subalgebra $\mathfrak{k} \subset \mathfrak{g}_{\mathbb{R}}$. This is the unique maximal compact subalgebra containing $\mathfrak{h}$.

1.7.4. *Remark.* The pair $\mathfrak{k}_{\mathbb{C}} \subset \mathfrak{g}_{\mathbb{C}}$ is enough to determine the real form $\mathfrak{g}_{\mathbb{R}}$.

1.7.5. *Exercise.* For convenience we set

$$\mathfrak{g}^p \;=\; \mathfrak{g}_\varphi^{p,-p}\,.$$

Show that $[\mathfrak{g}^p\,,\,\mathfrak{g}^q] \subset \mathfrak{g}^{p+q}$.

1.7.6. *Exercise.* Let $P \subset \check{G}$ be the complex stabilizer of φ. (In particular, $\check{D} = \check{G}/P$.) Show that the Lie algebra is

$$\mathfrak{p} \;=\; F^0(\mathfrak{g}_\mathbb{C}) \;=\; \bigoplus_{p\geq 0} \mathfrak{g}^p\,.$$

1.7.7. *Exercise.* Let $H = G \cap P$ be the real stabilizer of φ. (In particular, $D = G/H$.) Show that the Lie algebra satisfies

$$\mathfrak{h}_\mathbb{C} \;=\; \mathfrak{g}^0\,.$$

1.7.8. *Exercise.* Given a Hodge representation, define $\mathtt{T}_\varphi \in \mathrm{End}(V_\mathbb{C})$ by specifying that $\mathtt{T}_\varphi(v) = \frac{1}{2}(p-q)\,v$ for all $v \in V_\varphi^{p,q}$. In particular the Hodge decomposition is the eigenspace decomposition of $\mathtt{T}_\varphi$.

Show that $\mathtt{T}_\varphi \in \mathbf{i}\mathfrak{g}_\mathbb{R}$.

1.7.9. *Exercise.* Fix a Cartan subalgebra $\mathfrak{t} \subset \mathfrak{g}_\mathbb{R}$ containing $\mathbf{i}\mathtt{T}_\varphi$. Note that $\mathfrak{t}$ is contained in the Lie algebra $\mathfrak{h}$ of the *compact* isotropy $H \subset G$. If $\Lambda_{\mathrm{rt}} \subset \Lambda \subset \Lambda_{\mathrm{wt}} \subset \mathfrak{t}_\mathbb{C}^*$ is the weight lattice of $\check{G}$, set $\Lambda^* = \mathrm{Hom}(\Lambda, 2\pi\mathbf{i}\,\mathbb{Z}) \subset \mathfrak{t}$, so that the exponential map identifies $\mathfrak{t}/\Lambda^*$ with a compact torus $T \subset G$. Show that $4\pi\mathbf{i}\mathtt{T}_\varphi \in \Lambda^*$, and $\varphi(e^{2\pi\mathbf{i}t}) = 4\pi\mathbf{i}\,t\,\mathtt{T}_\varphi$ mod Λ^*.

In particular $\frac{\partial\varphi}{\partial t}\big|_{t=1} = 4\pi\mathbf{i}\,\mathtt{T}_\varphi$, and suggests that we think of $\mathtt{T}_\varphi$ as a (rescaled) "infinitesimal Hodge structure".

1.7.10. *Exercise.* Let $\Delta \subset \mathfrak{t}_\mathbb{C}^*$ denote the roots of $\mathfrak{g}_\mathbb{C}$, and show that $\alpha(\mathtt{T}_\varphi) \in \mathbb{Z}$ for all $\alpha \in \Delta$. Show that

$$\mathfrak{g}^p \;=\; \{X \in \mathfrak{g}_\mathbb{C} \mid [\mathtt{T}_\varphi, X] = p\,X\}\,.$$

1.7.11. *Exercise.* Fix a choice of positive roots $\Delta^+ \subset \Delta$ with the property that $\alpha(\mathtt{T}_\varphi) \geq 0$ for all $\alpha > 0$. Let $\Sigma \subset \Delta^+$ be the simple roots. Show that the subspace $\mathfrak{g}^1$ generates the subalgebra $F^1(\mathfrak{g}_\mathbb{C}) = \oplus_{p>0}\, \mathfrak{g}^p$ if and only if

$$\alpha(\mathtt{T}_\varphi) \ \in\ \{0,1\} \qquad \forall \quad \alpha \in \Sigma\,. \tag{1.8}$$

1.8. **Period maps and the infinitesimal period relation.** The *infinitesimal period relation* (IPR) is differential constraint on period maps (§1.8.4). It is often expressed as

$$\mathrm{d}F^p \ \subset\ F^{p-1}\,.$$

Let $\mathcal{U} = \{z \in \mathbb{C} \text{ s.t. } |z| < 1\}$ denote the unit disc in the complex plane.

Exercise 1.9. Suppose that $\mathcal{D}$ is a period domain parameterizing weight $n = 1$ polarized Hodge structures. Prove that the IPR is trivial: that is, given any holomorphic curve $\alpha : \mathcal{U} \to \check{\mathcal{D}}$, we have $(\mathrm{d}F^p)(\dot\alpha(t)) \subset F^{p-1}_{\alpha(t)}$ for all p.

Exercise 1.10. Suppose that $\mathcal{D}$ is a period domain parameterizing weight $n = 2$ polarized Hodge structures with Hodge numbers $\mathbf{h} = (1, h, 1)$. Prove that the IPR is trivial: that is, given any holomorohic curve $\alpha : \mathcal{U} \to \check{\mathcal{D}}$, we have $(\mathrm{d}F^p)(\dot\alpha(t)) \subset F^{p-1}_{\alpha(t)}$ for all p.

Exercise 1.11. Suppose that $\mathcal{D}$ is a period domain parameterizing weight $n = 2$ polarized Hodge structures with Hodge numbers $\mathbf{h} = (2, h, 2)$. Prove that there exists a corank 1 subbundle $T^{\mathrm{ipr}}\check{\mathcal{D}}$ of the holomorphic tangent bundle $T\check{\mathcal{D}}$ such that $(\mathrm{d}F^p)(\dot\alpha(t)) \subset F^{p-1}_{\alpha(t)}$ for all p if and only if $\dot\alpha(t) \in T^{\mathrm{ipr}}_{\alpha(t)}\check{\mathcal{D}}$ for all t.

1.8.1. The IPR admits the following description as a homogeneous vector bundle. The holomorphic tangent bundle is

$$T\check{D} \ =\ \check{G} \times_P (\mathfrak{g}_\mathbb{C}/\mathfrak{p})\,.$$

Set

$$F^{-1}(\mathfrak{g}_{\mathbb{C}}) \;=\; \bigoplus_{p\geq -1} \mathfrak{g}^p\,.$$

Then

$$T^{\mathrm{ipr}}\check D \;=\; \check G \;\times_P\; F^{-1}(\mathfrak{g}_{\mathbb{C}})/\mathfrak{p}$$

Restricting the tangent bundle to the domain, we may express this as

$$T^{\mathrm{ipr}} D \;=\; G \;\times_H\; \mathfrak{g}^{-1}\,.$$

1.8.2. *Exercise.* The *horizontal bundle* of $T\check D$ contains a unique, minimal, bracket-generating, homogeneous subbundle $T^{\mathrm{h}}\check D$. Show that

$$T^{\mathrm{ipr}}\check D \;\subset\; T^{\mathrm{h}}\check D\,, \tag{1.12}$$

and that equality holds if and only if (1.8) holds.

1.8.3. We may reduce to the case that equality holds in (1.12), but possibly at the expense of the rational structure. The subalgebra $\tilde{\mathfrak{g}}_{\mathbb{C}} \subset \mathfrak{g}_{\mathbb{C}}$ generated by $\mathfrak{g}^1 \oplus \mathfrak{g}^{-1}$ is semisimple and defined over $\mathbb{R}$. (If the containment $\tilde{\mathfrak{g}}_{\mathbb{C}} \subset \mathfrak{g}_{\mathbb{C}}$ is strict, then $\tilde{\mathfrak{g}}_{\mathbb{C}}$ need not be defined over $\mathbb{Q}$.) Let $\tilde G_{\mathbb{C}} \subset \check G$ be the corresponding Lie subgroup. The IPR and horizontal subbundle coincide on the subvariety $\tilde D = \tilde G_{\mathbb{C}} \cdot \varphi \subset \check D$. Moreover, any horizontal submanifold of $\check D$ passing though φ is necessarily contained in $\tilde D$.

We will assume that equality holds in (1.12)*.*

1.8.4. Let B be a complex manifold with fundamental group $\pi_1(B)$. A *period map* is a holomorphic map

$$\Phi : B \;\to\; \Gamma\backslash D\,,$$

where $\Gamma \subset \mathbf{G}(\mathbb{Q})$ is discrete (for example, Γ is a subgroup of $\mathbf{G}(\mathbb{Z})$), and with the properties that Φ is locally liftable and satisfies the infinitesimal period relation $\mathrm{d}\Phi(T_bB) \subset T^{\mathrm{h}}_{\Phi(b)}D$.

1.8.5. *Example.* Period maps arise when considering families of smooth projective varieties. Specifically consider a proper, holomorphic submersion $f : \mathcal{X} \to B$. Then $X_b = f^{-1}(b)$ is a family of compact complex manifolds. It is a smooth fibre bundle: the fibres are all diffeomorphic, but not necessarily biholomorphic. (For example, you might consider the elliptic curves $E_\tau = \mathbb{C}/(\mathbb{Z} + \tau\mathbb{Z})$ parameterized by points τ in the upper half-plane.) If $\mathcal{X} \subset \mathbb{P}^N$, then the Chern class of the hyperplane bundle induces integral Kähler classes $\omega_b \in H^{1,1}(X_b) \cap H^2(X_b, \mathbb{Z})$. As discussed in §1.1.12 this gives us a polarized Hodge structure on the primitive cohomology $V_b \subset H^n(X_b, \mathbb{Q})$. The fibres V_b patch together to define a vector bundle over B. The vector bundle is equipped with a natural connection – the Gauss-Manin connection – which is flat. So parallel transport may be used to identify each V_b with a fixed $V = V_o$, $o \in B$. The identification is well-defined up to the action of the monodromy representation $\pi_1(B, o) \to \mathrm{Aut}(V, Q)$. (The vector bundle is equipped with an underlying local system $B \times_{\pi_1(B)} V_{\mathbb{Z}}$.) Under this identification, each $b \in B$ defines a Hodge structure on V; that is, an element of $\Gamma \backslash \mathcal{D}$, where $\Gamma \subset \mathrm{Aut}(V, Q)$ is the image of the monodromy representation. This is the period map $\Phi_f : B \to \Gamma \backslash \mathcal{D}$.

1.8.6. *Example.* The identity map $\mathrm{Id} : D \to D$ will be a period map (§1.8.4) if and only if the IPR is trivial; that is, if and only if $T^{\mathrm{h}}\check{D} = T\check{D}$. This is the case if and only if $\mathfrak{g}_{\mathbb{C}} = \mathfrak{g}^1 \oplus \mathfrak{g}^0 \oplus \mathfrak{g}^{-1}$; equivalently, D is Hermitian (equivalently, $\check{D}$ is cominuscule).

1.8.7. *Exercise.* Assume the setting of §1.8.6, and fix a Hodge representation as in §1.6.1. Show that the inclusion $\iota_\rho : D \hookrightarrow \mathcal{D}$ is a period map. (The substance of the exercise is to show that the IPR is satisfied.)

2. The geometric realization problem

2.1. **Open question: the geometric realization problem.** Exercise 1.8.7 raises a very interesting question: *Can the image of the embedding $\iota_\rho : D \hookrightarrow \mathcal{D}$ be realized geometrically?* That is, does there exist a family $f : \mathcal{X} \to B$, as in §1.8.5 (or more generally, a motivic variation of Hodge structure, as in §2.3.2), so that the image $\Phi(B) \subset \Gamma\backslash\mathcal{D}$ is an open subset of $\Gamma_\rho\backslash D = \mathrm{Image}\{D \hookrightarrow \mathcal{D} \twoheadrightarrow \Gamma\backslash\mathcal{D}\}$?

Following [Gro94], we will consider on the case that the Hodge representation is of Calabi–Yau type. In this situation, there are invariants (characteristic forms) that can determine whether or not a given period map $\Phi_f : B \to \Gamma\backslash\mathcal{D}$ is a geometric realization (Theorem 2.2). The Hodge representation is of *Calabi-Yau type* (CY) if the first Hodge number $h^{n,0} = \dim_{\mathbb{C}} V^{n,0} = 1$. As the following two exercises indicate, this restriction involves no real loss of information.

2.1.1. *Exercise.* Fix a Hodge representation (ρ, φ), as in §1.6, not necessarily of CY type. Let $d = h^{n,0}$ be the first Hodge number and set $V' = \bigwedge^d V$. Let $\rho' : \mathbf{G} \to \mathrm{Aut}(V')$ be the representation induced by ρ. Verify that φ defines a CY Hodge structure on V'.

This Hodge structure admits a natural polarization Q' with the property that ρ' takes value in $\mathrm{Aut}(V', Q')$, and in this way we obtain a Hodge representation (ρ', φ) of CY type.

2.1.2. *Exercise.* Verify that the Hodge domains $D \subset \mathcal{D}$ and $D' \subset \mathcal{D}'$ are isomorphic as G–homogeneous manifolds (cf. §1.6.2).

2.2. **Canonical CY Hodge representations over tube domains.** The irreducible Hermitian Hodge domains D admit canonical CY Hodge representations [Gro94, SZ10]. The case that D is a tube domain is the simplest, in the sense that the canonical representation is irreducible over $\mathbb{C}$.[1] The irreducible tube domains, and their canonical (real[2]) CY Hodge representations are given in [Gro94]:

(a) For $G/H = \mathrm{O}(2,b)/(\mathrm{U}(1) \times \mathrm{O}(b))$ we have $V_{\mathbb{C}} = \mathbb{C}^{2+b}$ (the irreducible representation of highest weight ω_1), and $D = G/H$ is the period domain for weight $n = 2$ Hodge structures with $h^{2,0} = 1$, cf. §1.2.7.

[1] In the other cases, $V_{\mathbb{C}}$ factors as $U_{\mathbb{C}} \oplus U_{\mathbb{C}}^*$ with $U_{\mathbb{C}}$ and irreducible $\check{G}$–module.

[2] The constructions of [Gro94, SZ10] are over $\mathbb{R}$.

(b) For $G/H = \mathrm{Sp}(2g,\mathbb{R})/\mathrm{U}(g)$ we have $V_\mathbb{C} = \bigwedge^g \mathbb{C}^{2g}$ (the irreducible representation of highest weight ω_g), and $D = G/H$ is the period domain for weight $n = 1$ Hodge structures, cf. §1.2.6. The points $\varphi \in D$ parameterize Hodge decompositions $\mathbb{C}^{2g} = E_\varphi \oplus \overline{E_\varphi}$, and these induce the Hodge decompositions $V^{p,q}_\varphi = (\bigwedge^p E_\varphi) \otimes (\bigwedge^q \overline{E_\varphi})$, as in §2.1.1.

(c) For $G/H = \mathrm{U}(n,n)/(\mathrm{U}(n)\times\mathrm{U}(n))$ we have $V_\mathbb{C} = \bigwedge^n \mathbb{C}^{2n}$ (irreducible representation of highest weight ω_n). The points $\varphi \in D = G/H$ parameterize subspaces $E \in \mathrm{Gr}(n, \mathbb{C}^{2n})$ upon which the Hermitian form restricts to be positive definite. The decomposition $\mathbb{C}^{2n} = E_\varphi \oplus \overline{E_\varphi}$ is a weight one Hodge decomposition. The induced Hodge decomposition is $V^{p,q}_\varphi = (\bigwedge^p E_\varphi) \otimes (\bigwedge^q \overline{E_\varphi})$.

(d) For $G/H = \mathrm{SO}^*(4r)/U(2r)$, with $r \geq 2$, $V_\mathbb{C}$ is a Spinor representation (irreducible representation of highest weight ω_{2r}), the weight of the Hodge structure is $n = r$, and the Hodge decomposition is $V^{p,q}_\varphi \simeq \bigwedge^{2p} \mathbb{C}^{2r}$.

(e) For $D = G/H$, with G the exceptional simple real Lie group of rank 7 having the maximal compact subgroup $H = U(1) \times_{\mu_3} E_6$, then $V_\mathbb{C}$ is the irreducible representation of highest weight ω_7 and the Hodge structure is weight 3 with Hodge numbers $(1, 27, 27, 1)$.

2.3. **Geometric realizations.** Some of the examples in §2.2 admit geometric realizations:

2.3.1. *K3 surfaces.* The tube domain in §2.2(a) is a period domain. For certain values of b, it is geometrically realized by families of K3 surfaces. And these Hodge structures are of CY-type: the dimension of $H^{2,0}(K3)$ is 1.

2.3.2. *Principally polarized abelian varieties.* The tube domain in §2.2(b) is a period domain. As such its is realized geometrically by principally polarized abelian varieties (ppav) of genus g. In fact, the moduli space $\mathcal{A}_g = \Gamma_g \backslash D$, and the period map is essentially the identity. Given $A \in \mathcal{A}_g$, the Hodge decomposition $H^1(A, \mathbb{C}) =$

$H^{1,0}(A) \oplus H^{0,1}(A)$ induces a CY-type Hodge structure on $\bigwedge^g H^1(A, \mathbb{C})$. In this way we obtain a motivic realization of the canonical CY Hodge representation over D by mapping $A \in \mathcal{A}_g$ to the Hodge structure on $\bigwedge^g H^1(A, \mathbb{C})$ as in §2.1.1. (This is an example of a "motivic" variation of Hodge structure: it is constructed from a geometric family and certain linear/tensor operations.)

2.3.3. *Calabi–Yau varieties.* One may obtain a family $f : \mathcal{X} \to B$ of n-folds by resolution of double covers of $\mathbb{P}^n$ branched over $2n+2$ hyperplanes in general position. The resulting period map $\Phi_f : B \to \Gamma\backslash\mathcal{D}$ has the same Hodge numbers as the CY Hodge representation of §2.2(c). So it is natural to ask if this family is a geometric realization of the corresponding embedding $\iota_\rho : D \hookrightarrow \mathcal{D}$?

When $n \leq 2$, the answer is "yes". For $n = 1$ this is the classical case of elliptic curves branched over fours points in $\mathbb{P}^1$. In the case $n = 2$ this was proved by Matsumoto, Sasaki and Yoshida [MSY92].

However, when $n \geq 3$ the family of Calabi–Yau's does *not* realize the canonical CY Hodge representation over $D = \mathrm{U}(n,n)/(\mathrm{U}(n) \times \mathrm{U}(n))$. This was proved by Gerkmann, Sheng, van Straten and Zuo [GSvSZ13] in the $n = 3$ case, and their argument was extended to $n \geq 3$ by Sheng, Xu and Zuo [SXZ15]. *The crux of the argument is to show that the second characteristic forms do not agree.*[3]

2.4. **Characteristic forms.** The compact dual is a flag manifold $\check{\mathcal{D}} = \mathrm{Flag}^Q(\mathbf{f}, V_\mathbb{C})$ and so admits a tautological filtration

$$\mathcal{F}^n \subset \mathcal{F}^{n-1} \subset \cdots \subset \mathcal{F}^1 \subset \mathcal{F}^0$$

[3] A similar argument was used by Sasaki, Yamaguchi and Yoshida [SYY97] to disprove a related conjecture on the projective solution of the system of hypergeometric equations associated with the hyperplane configurations.

of the trivial bundle $\mathcal{F}^0 = \check D \times V_{\mathbb{C}}$ over $\check D$. Given a holomorphic map $\psi : M \to \check{\mathcal{D}}$,[4] let

$$\mathcal{F}^p_\psi \ = \ \psi^* \mathcal{F}^p$$

denote the pull-back to M. The map ψ satisfies the IPR (§1.8) if and only if

$$\mathrm{d}\mathcal{F}^p_\psi \ \subset \ \mathcal{F}^{p-1}_\psi \otimes \Omega^1_M \, .$$

In this case we obtain a vector bundle map

$$\gamma_\psi : TM \ \to \ \mathrm{Hom}(\mathcal{F}^n_\psi, \mathcal{F}^{n-1}_\psi/\mathcal{F}^n_\psi) \, ;$$

sending $\xi \in T_x M$ to the linear map $\gamma_{\psi,x}(\xi) \in \mathrm{Hom}(\mathcal{F}^n_{\psi,x}, \mathcal{F}^{n-1}_{\psi,x}/\mathcal{F}^n_{\psi,x})$ defined as follows. Fix a locally defined holomorphic vector field X on M extending $\xi = X_x$. Given any $v_0 \in \mathcal{F}^n_{\psi,x}$, let v be a local section of $\mathcal{F}^n_\psi$ defined in a neighborhood of x and with $v(x) = v_0$. Then

$$\gamma_\psi(\xi)(v_0) \ := \ X(v)|_x \ \mathrm{mod}\ \mathcal{F}^n_{\psi,x}$$

yields a well-defined map $\gamma_\psi(\xi) \in \mathrm{Hom}(\mathcal{F}^n_\psi \, , \, \mathcal{F}^{n-1}_\psi/\mathcal{F}^n_\psi)$. More generally there is a vector bundle map

$$\gamma^k_\psi : \mathrm{Sym}^k TM \ \to \ \mathrm{Hom}(\mathcal{F}^n_\psi \, , \, \mathcal{F}^{n-k}_\psi/\mathcal{F}^{n-k+1}_\psi)$$

defined as follows. Given $\xi_1, \ldots, \xi_k \in T_x M$, let $X_1, \ldots, X_k$ be locally defined holomorphic vector fields extending the $\xi_j = X_{j,x}$. Given v_0 and v as above, define

$$\gamma^k_\psi(\xi_1, \ldots, \xi_k)(v_0) \ := \ X_1 \cdots X_k(v)|_x \ \mathrm{mod}\ \mathcal{F}^{n-k+1}_{\psi,x} \, . \tag{2.1}$$

It is straightforward to confirm that γ^k_ψ is well-defined. This bundle map is the *k-th characteristic form* of $\psi : M \to \check D$. Let $\mathbf{C}^k_\psi \subset \mathrm{Sym}^k T^*M$ denote the image of the dual map. In a mild abuse of terminology we will also call $\mathbf{C}^k_\psi$ the *k–th characteristic forms* of $\psi : M \to \check D$.

Recall the embedding $\iota_\rho : \check D \hookrightarrow \check{\mathcal{D}}$ of §1.6.1.

[4] We are interested in the case that ψ is either a local lift of the period map $\Phi_f : B \to \Gamma\backslash\mathcal{D}$ associated with a geometric family, as in §1.8.5, or is the embedding $\iota_\rho : D \hookrightarrow \mathcal{D}$ of §1.6.1.

Theorem 2.2. *The characteristic forms of ψ and ι_ρ are isomorphic if and only if there exists $g \in \mathrm{Aut}(V_\mathbb{C}, Q)$ so that $g \circ \psi(M)$ is an open subset of $\iota_\rho(\check{D})$.*

See [Rob18] for a precise statement of the theorem.

The upshot is that if we have a period map $\Phi_f : B \to \Gamma\backslash\mathcal{D}$, then it will be a geometric realization of $D \hookrightarrow \mathcal{D}$ if and only if the characteristic forms of a local lift of Φ_f are isomorphic to those of ι_ρ.

3. Characteristic cohomology of the infinitesimal period relation

Throughout this section we assume $T^{\mathrm{ipr}}\check{D} = T^{\mathrm{h}}\check{D}$ (§1.8.3).

A complex submanifold $M \subset \check{D}$ is *horizontal* if it satisfies the IPR (§1.8); that is, $T_xM \subset T^{\mathrm{h}}_x\check{D}$. More generally, we say that an irreducible subvariety $X \subset \check{D}$ is horizontal if its smooth locus is. The image of a period map is horizontal by definition (§1.8.4). Associated to the IPR is a “characteristic cohomology”, which we may think of as being “universal” in the sense that it is the cohomology that induces ordinary cohomology on all horizontal submanifolds $M \subset \check{D}$ by virtue of their being solutions of the system of differential equations.

The characteristic cohomology is relatively well understood over the compact dual (§3.2), but several interesting questions are open over the Hodge domain (§3.3).

3.1. **Definition.** Like de Rham cohomology, the characteristic cohomology is defined in terms of 1-forms. Let $T_\mathbb{R}\check{D}$ denote the real tangent bundle, and let

$$T^{\mathrm{h}}_\mathbb{R}\check{D} \;=\; (T^{\mathrm{h}}\check{D} \oplus \overline{T^{\mathrm{h}}\check{D}}) \cap T_\mathbb{R}\check{D}$$

denote the real horizontal subbundle. Let $\mathrm{Ann}(T^{\mathrm{h}}_\mathbb{R}\check{D}) \subset T^*_\mathbb{R}\check{D}$ denote the annihilator of the horizontal subbundle. Let $\mathcal{A}$ denote the ring of smooth, complex-valued differential forms, and let $\mathcal{I} \subset \mathcal{A}$ be the differential ideal generated by the sections of $\mathrm{Ann}(T^{\mathrm{h}}_\mathbb{R}\check{D})$. That is, $\mathcal{I}$ is generated by smooth 1-forms $\alpha \in \mathcal{A}^1$ that vanish on $T^{\mathrm{h}}_\mathbb{R}\check{D}$.

3.1.1. *Exercise.* Show that M is horizontal if and only if M is an integral submanifold of $\mathcal{I}$; that is, $\vartheta|_M = 0$ for all $\vartheta \in \mathcal{I}$.

In a mild abuse of notation, we will also call $\mathcal{I}$ the IPR.

By construction $\mathcal{I}$ is differentially closed $\mathrm{d}\mathcal{I} \subset \mathcal{I}$. So that de Rham complex $(\mathcal{A}, \mathrm{d})$ induces a quotient complex $(\mathcal{A}/\mathcal{I}, \mathrm{d})$. The *characteristic cohomology of the IPR* is

$$H^\bullet_{\mathcal{I}} \ = \ H^\bullet(\mathcal{A}/\mathcal{I}, \mathrm{d})\,.$$

3.1.2. *Exercise.* Suppose that $M \subset \check D$ is horizontal, and show that the characteristic cohomology pulls back to the ordinary cohomology. That is, there is a natural map $H^\bullet_{\mathcal{I}}(\check D) \to H^\bullet(M, \mathbb{C})$.

3.2. **Characteristic cohomology and Schubert classes.** The compact dual $\check D$ decomposes as a disjoint union $\cup_w C_w$ of affine, quasi-projective varieties. Each of the *Schubert cells* C_w is the orbit of a fixed Borel subgroup $B \subset \check G$. Their Zariski closures $X_w = \overline{C_w}$ are *Schubert varieties*, and their homology classes $\mathbf{x}_w$ generate $H_\bullet(\check D, \mathbb{Z})$. So it is natural to ask: *which Schubert varieties are horizontal?*

It turns out that there is a simple representation theoretic characterization of the horizontal Schubert varieties [Rob14]. Briefly, to every $\mathbf{x}_w$ there is associated a collection of positive roots $\Delta(w)$ with the property that $|\Delta(w)| = \dim_{\mathbb{C}} X_w$. This implies that $\dim_{\mathbb{C}} X_w \le \sum_{\alpha\in\Delta(w)} \alpha(\mathtt{T})$, cf. §1.7.10 and §1.7.11. Equality holds if and only if X_w is horizontal.

And one may show that the a homology class $\mathbf{y} \in H_\bullet(\check D, \mathbb{Z})$ may be represented by a horizontal subvariety only if it is a linear combination of horizontal Schubert classes. Reciprocally, the characteristic cohomology is spanned by the classes dual to the Schubert classes. More precisely, let $\mathbf{x}^w \in H^\bullet(\check D, \mathbb{Z})$ the cohomology classes dual to the Schubert classes $\mathbf{x}_w$. Then the kernel of the natural projection $H^\bullet(\check D) \to H^\bullet_{\mathcal{I}}(\check D)$ is spanned by the $\{\mathbf{x}^w \mid X_w \text{ is not horizontal}\}$. Together these results imply that there is a nondegenerate Poincaré pairing between the characteristic cohomology and the

$\mathcal{I}$–homology $H_{\bullet,\mathcal{I}} = \mathrm{span}\{[Y] \in H_\bullet(\check{D}) \mid Y$ *is horizontal*$\}$. For details and precise statements see [Rob16, §4].

3.3. **Characteristic cohomology over Hodge domains.** The characteristic cohomology over D is less well-understood, and there are a number of interesting open questions. What we can say is the following. The cohomology $H^\bullet_{\mathcal{I}}(D)$ may be realized as the cohomology of the total complex of a double complex of G–invariant differential operators on homogeneous vector bundles [Rob16, §5]. Using this complex, one may show that there is an integer $\nu = \nu(D) > 0$ so that: $H^k_{\mathcal{I}}(D) \simeq H^k(D)$; in particular, $H^k_{\mathcal{I}}(D)$ is finite dimensional when $k < \nu$, and zero when $k < \nu$ is odd. One may also show that the characteristic cohomology in degree $k < \nu$ satisfies a local Poincarè Lemma [Rob16, §6].

The integer is defined as follows. Let $\mathfrak{g}_- = \oplus_{p>0}\, \mathfrak{g}^{-p}$. The Lie algebra cohomology $H^\ell(\mathfrak{g}_-, \mathbb{C})$ decomposes into a direct sum of $\mathtt{T}$–eigenspaces with integer eigenvalues $\ell, \ell+1, \ldots, m(\ell)$. The integer ν is the largest value of ℓ for which $m(\ell) = \ell$.

3.3.1. *Example.* In the case that D is Hermitian the IPR is trivial (§1.8.6); equivalently, $\mathcal{I} = 0$. In this case D is bounded domain in $\mathbb{C}^d$, where $d = \dim_{\mathbb{C}} D$ and $\nu = d$.

3.3.2. *Example.* The simplest nontrivial case is that $T^{\mathrm{h}}D \subset TD$ has corank one. When $\check{G}$ is simple, these compact duals are adjoint varieties. Examples include the flag variety $\mathrm{Flag}(1, k, \mathbb{C}^{k+1})$ of lines in hyperplanes; and the Grassmannian $\mathrm{Gr}^Q(2, V_{\mathbb{C}})$ of 2-planes that are isotropic with respect to a non-degenerate symmetric bilinear form. In this case $\dim_{\mathbb{C}} \check{D} = 2\nu + 1$.

3.3.3. *Open question: computation of cohomology.* What can be said about $H^\bullet_{\mathcal{I}}(D)$ as a G–module?

3.3.4. *Open question: invariant cohomology.* Since the period map takes value in $\Gamma\backslash D$, it is natural to ask: what can be said about the Γ–invariant cohomology $H^\bullet_{\mathcal{I}}(D)^\Gamma$? (The G–invariant cohomology $H^\bullet_{\mathcal{I}}(D)^G$ is computed in [Rob14].)

3.3.5. *Open question: mixed Hodge structure on the characteristic cohomology?* Consider a period map $\Phi : B \to \Gamma\backslash D$ as in §1.8.4. In many cases of interest the complex manifold B is a smooth quasi-projective variety. We may assume without loss of generality that Φ is proper [Gri70, p. 158]. Then the proper mapping theorem implies that the image $\wp = \Phi(B)$ is a complex analytic space.[5] It is expected that mixed Hodge structures (§3.3.6) on $\wp$ arise universally, in the sense that they are induced from objects on $\Gamma\backslash D$. *Is there a mixed Hodge structure on $H^\bullet_{\mathcal{I}}(D)^\Gamma$?*

3.3.6. *Mixed Hodge structures.* Because, with the exception of §3.3.5, these lectures concern (pure) Hodge structures, rather than mixed Hodge structures, I will be very brief here.

A *mixed Hodge structure* on V is given by two filtrations: a decreasing filtration $F^\bullet$ of $V_\mathbb{C}$, as in (1.2), and a rational increasing filtration $W_0 \subset W_1 \subset \cdots \subset W_{2n} = V$ with the property that $F^\bullet$ induces a weight ℓ Hodge structure on $\mathrm{Gr}^W_\ell = W_\ell/W_{\ell-1}$.

Mixed Hodge structures arise in (at least) three contexts. Deligne [Del74] has shown that the cohomology $H^n(X,\mathbb{Q})$ of an algebraic variety X admits a (functorial) mixed Hodge structure. Here X need not be smooth or closed. However, when X is smooth and closed, Delignes MHS is the (usual) Hodge structure of §1.1.7. For an expository introduction to mixed Hodge structures on algebraic varieties see [Dur83]; for a thorough treatment see [PS08].

[5]The image $\wp = \Phi(B) \subset \Gamma\backslash D$ will be quasi-projective when Γ is arithmetic [BBT18]. This is striking because in "most" cases, the complex manifold $\Gamma\backslash D$ admits no compatible algebraic structure [GRT14]. (If Γ is not arithmetic, one may take an arithmetic group $\Gamma' \supset \Gamma$ and compose the period map with the surjection $\Gamma\backslash D \twoheadrightarrow \Gamma'\backslash D$. Then $\wp$ will be a finite cover of the quasi-projective image $\wp'$.)

Asymptotically the pure Hodge structures parameterized by a period map degenerate to mixed Hodge structures. (There the weight filtration W is the Jacobson–Morosov filtration of a nilpotent operator known as a logarithm of (local) monodromy.) For an expository account, and further references, see [Rob17].

The Hard Lefschetz Theorem and Hodge–Riemann bilinear relations of "polarized mixed Hodge structures" have found applications in combinatorics. See [Bak18] for an expository overview of recent applications; and [Rob21] for an introduction to the linear structures underlying the Hard Lefschetz Theorem and Hodge–Riemann bilinear relations.

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