This worksheet will give you practice in setting up and solving initial value problems using the technique of separation of variables. Carefully read the example below before starting on the problems.

**Example:** The population of a certain tropical island is currently 1000 and, in the absence of immigration or emigration, increases at a rate proportional to the population itself (with proportionality constant \( k = .02 \) when time is measured in years). Each year, however, 12 people emigrate to the mainland. Let \( N(t) \) denote the population in year \( t \) where the present is year \( t = 0 \). The assumptions made in this problem imply that

\[
\text{rate of change of } N = \left( \text{"natural" growth of } N \right) - \text{ (rate of emigration)}.\]

Writing this as an initial value problem we have

\[
\frac{d}{dt} N(t) = .02 N(t) - 12, \quad N(0) = 1000.
\]

Factoring out the .02 then separating the variables yields

\[
\frac{1}{(N - 600)} dN = .02 \, dt.
\]

We now antidifferentiate both sides and solve for \( N \), and conclude that

\[
N(t) = Ce^{.02t} + 600
\]

for some arbitrary constant \( C \). Finally, using the fact that \( N(0) = 1000 \), we can solve for \( C \) yielding the solution

\[
N(t) = 400e^{.02t} + 600.
\]

Since we now have a formula for \( N(t) \) we can predict the population at any year in the future.

1. Newton formulated the principle that the rate of change of the temperature of an object is proportional to the difference between the object’s temperature and the temperature of the surroundings. Suppose that a cup of coffee, which has temperature 110\(^\circ\)F, is placed in a room which is at temperature 72\(^\circ\)F. Suppose that we measure time in minutes after the placement of the coffee in the room and that the constant of proportionality is .7. Let \( T(t) \) denote the temperature of the coffee at time \( t \). According to Newton’s Law of Cooling,

\[
T'(t) = -.7(T(t) - 72), \quad T(0) = 110.
\]

How long will it take for the temperature of the cup of coffee to reach 90\(^\circ\)F?
2. Suppose that the cup of coffee in problem 1 has temperature 40\(^\circ\)F at \(t = 0\) since it was mistakenly placed in the refrigerator. Find a formula for the temperature as a function of time.

3. An ingot of iron ore at 1000\(^\circ\)F is plunged into a water bath whose temperature is kept at 60\(^\circ\)F. The temperature of the ingot decreases according to Newton’s Law of Cooling with constant of proportionality \(k = 3\) when time is measured in minutes. Find a formula for the temperature of the ingot as a function of time. How long will it take until the temperature reaches 70\(^\circ\)F?

4. The temperature in a certain Duke dorm is a stuffy 76\(^\circ\)F at midnight when the power goes out. Outside the temperature holds steady at 35\(^\circ\)F. The students measure the temperature in the dorm at 1 am and find that it has decreased to 65\(^\circ\)F. Worried, they compute how low the temperature will fall by 7 am when Duke Energy Corporation has pledged that the heat will come back on. Explain what they did.

5. The population of a certain tropical island is currently 800 and, in the absence of immigration or emigration, increases at a rate proportional to the population itself (with proportionality constant \(k = .03\) when time is measured in years). However, each year 36 people emigrate to the mainland. Find and solve an initial-value problem for the population \(N(t)\). Graph \(N(t)\) and find the time when the population becomes zero.

6. An 800 gallon tank is filled with brine which contains 300 pounds of salt. Every minute two gallons of dilute brine containing .1 pound of salt per gallon are pumped in and two gallons of the (well-mixed) brine are pumped out. Find a differential equation and initial condition satisfied by \(S(t)\), the salt in the tank at time \(t\). Solve the initial value problem to find \(S(t)\). Draw a graph of \(S(t)\) and describe its behavior as \(t \to \infty\).

7. A tank of salt water has 100 pounds of salt dissolved in 1000 gallons of water. Every minute four gallons of salt water having 0.5 pounds of salt per gallon are pumped in and four gallons of the (well-mixed) salt water are pumped out. Find a differential equation and initial condition satisfied by \(S(t)\), the salt in the tank at time \(t\). Solve the initial value problem to find \(S(t)\). Draw a graph of \(S(t)\) and describe its behavior as \(t \to \infty\).

8. Suppose that we are in the situation described in problem 7 except that 2 pounds of salt but no water is added per minute. As in the example, 4 gallons of brine are pumped out each minute. Write down (but do not try to solve) a differential equation for \(S(t)\). Use Euler’s method to describe the behavior of \(S(t)\). Sketch a graph of \(S(t)\).

9. Let \(B(t)\) denote the number of infectious bacteria in a patient’s body. Suppose that, left unchecked, these bacteria would multiply at a rate proportional to the number there (with constant of proportionality equal to 1.6 if time is measured in hours). Suppose that the patient’s immune system kills off 800 of these bacteria per hour. Suppose that at \(t = 0\) the patient has \(B_0\) bacteria in his system. For which values of \(B_0\) will the immune system eventually conquer the infection?
10. In a certain polluted lake, the fish population is decreasing at a rate proportional to its size (with constant of proportionality equal to .04 when \( t \) is measured in months). However, environmentalists add 2000 fish per month to restock the lake. If the environmentalists continue this restocking for a long time, approximately how many fish will be in the lake?

11. (a) Suppose that the cup of coffee in problem 1 has temperature 72°F at \( t = 0 \). Show that the constant function \( T(t) = 72 \) satisfies the differential equation and the initial condition. Why does this make sense? \( T(t) = 72 \) is called an equilibrium solution because its value remains constant.

(b) Show that, no matter what the temperature of the coffee is at \( t = 0 \), the temperature will approach 72°F as \( t \to \infty \). An equilibrium solution is called stable if solutions which start close to it converge to it as \( t \to \infty \). Thus, in problem 1, the equilibrium solution is stable.