

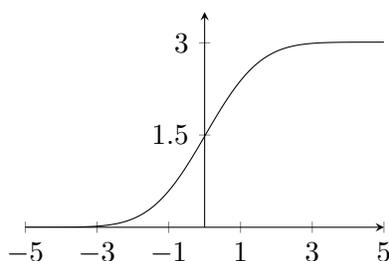
# Introduction to Euler's Method

## Purpose:

In this lab we will learn how to draw the graph of a function and how to approximate values of a function when we know only its rate of change and a starting point for the graph.

## Preview:

From the information that  $f'(t) = 2^{-t^2/2}$  and  $f(-5) = 0$  the following approximate graph of  $f$  can be produced, even though we do not have a formula for the function  $f$ .



## Background:

The idea of tracing a path on a chart when you know the direction of travel (but not a position function) is very old. Ancient mariners estimated the position of their ships by keeping track of the directions in which they sailed day-by-day. This process is called dead reckoning, because a navigator deduced where a ship was by using careful notes on its headings and the elapsed time. The mariners did know, of course, the point from which the ship started. In much the same way Leonhard Euler (1707-1783) developed a method that uses the starting point of a function and its direction (i.e., derivative) to approximate the graph of the function.

## Part I: Sketching a Solution Curve Geometrically

Let's suppose we know that the graph of an unknown function starts at the point  $(0, 1)$ , i.e.,  $y(0) = 1$ . Suppose, also, that its rate of change at any time is given by the function  $\frac{dy}{dt} = 2t$ . By anti-differentiation, we can deduce that  $y(t) = t^2 + 1$ . However, we pretend we cannot find such a formula, so that we can learn how to approximate the graph directly from the derivative. After we have constructed the approximate graph, we will compare it to the exact solution. The significance of Euler's method lies in the fact that we can construct the approximate graph even when we do not have an explicit expression for  $y(t)$ .

1. We start by plotting on the attached graphing page the known point  $(0, 1)$ . Next, choose a time step  $\Delta t$  which will remain fixed in the construction of the graph. The choice of  $\Delta t$

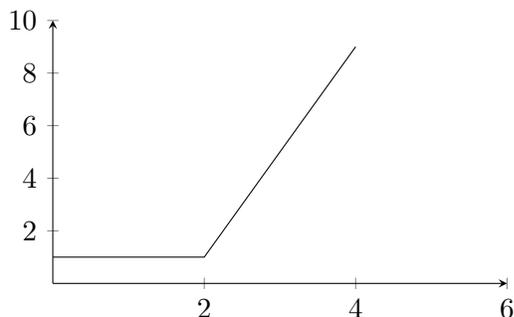
## 106L Labs: Introduction to Euler's Method

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will determine how many points we will plot over a given interval: the smaller the value of  $\Delta t$ , the more numerous the points and the more accurate (we hope) the graph. For this first approximation we will start with a value for  $\Delta t$  of 2. From the initial point,  $(0, 1)$ , of the graph we move to the right by  $\Delta t = 2$  units and estimate the new vertical position of the graph. Because  $\frac{dy}{dt} = 2t$ , this graph has a slope of  $2 \cdot 0 = 0$  when  $t = 0$ .

We then know that the “heading” of the curve at time 0 has slope 0. Thus, we draw a horizontal line segment from the point  $(0, 1)$  toward the right over a time interval of length  $\Delta t = 2$ . At that location,  $(2, 1)$ , we plot the second point. We have now approximated the first part of the curve.

2. We know that the curve has a slope of  $2 \cdot 2 = 4$  at time 2, so from the last point,  $(2, 1)$ , we draw a line segment toward the right with a slope of 4. This segment should extend over the time interval  $[2, 4]$  only. Plot the point at the end of this segment, and at this point you should have a plot similar to the one pictured below.



Note that this graphical approximation of  $f$  has the exact value and slope at time 0. At time 2 it has the correct slope, but the location of the point is an approximation of the actual location.

3. Now continue the process above until you reach the edge of the graphing page. You now have a crude approximation of the graph of the function  $y(t)$ .

## Part II: Stepping Formulas for a Solution Curve

In the last part we plotted points by locating them geometrically. In this part we will develop some stepping formulas (also called recursive formulas) that we will use to compute the coordinates of the points we just plotted.

- Let  $(t_0, y_0)$  denote the initial point. We will construct points  $(t_k, y_k)$  for  $k = 1, 2, 3, \dots$  which, after they're plotted and connected with line segments, will approximate the graph of  $y$ . After

## 106L Labs: Introduction to Euler's Method

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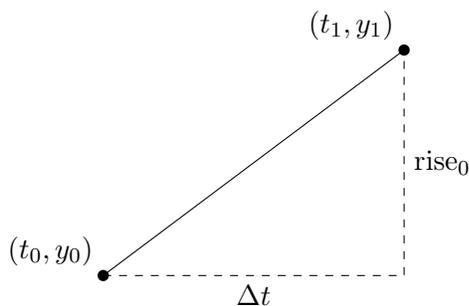
$t_0$  is given and  $\Delta t$  is chosen, all the other times,  $t_1, t_2, t_3, \dots$  are determined by

$$t_k = t_{k-1} + \Delta t \text{ for } k = 1, 2, 3, \dots$$

Or, because we started with  $t_0$  and added  $\Delta t$  at each step, we can write

$$t_k = t_0 + k \cdot \Delta t \text{ for } k = 1, 2, 3, \dots$$

- To develop the formula for  $y_k$  we will first compute  $y_1$ , and then we will generalize the result. Geometrically we found  $y_1$  by drawing a tangent at the initial point and extrapolating along the tangent for a horizontal distance of  $\Delta t$ . Let  $\text{rise}_0$  denote the rise along this tangent from  $(t_0, y_0)$  to  $(t_1, y_1)$  as pictured below.



It follows that  $y_1 = y_0 + \text{rise}_0$ . But for any line,  $\text{slope} = \frac{\text{rise}}{\text{run}}$ . Thus,

$$\text{rise} = \text{slope} \cdot \text{run}.$$

Therefore,

$$\text{rise}_0 = \text{slope}_0 \cdot \Delta t$$

and

$$y_1 = y_0 + \text{slope}_0 \cdot \Delta t.$$

We know  $y_0$ , we have chosen a value to use for  $\Delta t$ , and we can determine  $\text{slope}_0$  by evaluating the derivative at  $t_0$ , namely  $\text{slope}_0 = 2 \cdot t_0$ . Therefore, in this case,

$$y_1 = y_0 + (2 \cdot t_0) \cdot \Delta t = 1 + (2 \cdot 0) \cdot 2 = 1.$$

- We repeat the process to find the next point. And, in general, the process for extrapolating from the point  $(t_{k-1}, y_{k-1})$  to next point  $(t_k, y_k)$  is exactly the same. Thus, we have Euler's stepping formula for  $y_k$ , namely

$$y_k = y_{k-1} + \text{slope}_{k-1} \cdot \Delta t \text{ for } k = 1, 2, 3, \dots$$

Note that  $\text{slope}_{k-1}$  is given by the value of the derivative at time  $t_{k-1}$ .

If we use  $y'_{k-1}$  to denote the value of the derivative at time  $t_{k-1}$ , then Euler's formula becomes

$$y_k = y_{k-1} + y'_{k-1} \cdot \Delta t \text{ for } k = 1, 2, 3, \dots$$

## 106L Labs: Introduction to Euler's Method

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4. Use these results to complete the following table, given  $\frac{dy}{dt} = 2t$ ,  $y(0) = 1$  with  $\Delta t = 2$ .

$k$	$t_k$	$y_k$	slope $_k$	rise $_k$
0	0	1	0	0
1	2	1		

5. In this table we have computed approximations for the values of  $y(2)$ ,  $y(4)$ ,  $y(6)$ ,  $y(8)$  and  $y(10)$ ; i.e., the actual values  $y(t_k)$  are being approximated by the values  $y_k$  produced by Euler's method. On the attached graphing page plot the points  $(t_k, y_k)$  from your table. These points should coincide with the ones you plotted in Part I, and they provide a crude approximation of the graph of  $y(t)$  over the interval  $[0, 10]$ .

### Part III: Euler's Method

The method we have been using to approximate a graph using only the derivative and a starting point is called Euler's Method. To see the effect of the choice of  $\Delta t$  in Euler's method we will repeat the process above, but with a smaller value for  $\Delta t$ . To do this, we'll move our work to a spreadsheet.

6. Open the spreadsheet for this lab, make a copy, and rename it with your group names.
7. We'll start by reproducing the table from question 4 in our spreadsheet:
  - Enter  $2t$  in cell H2. (Note: this won't do anything, it's just there as a reminder.)
  - Enter 2 in cell H3 for  $\Delta x$ .
  - In column A, enter  $k$  values from 0 to 5.
  - In cells B2 and C2, enter values for  $t_0$  and  $y_0$  respectively.
  - In cell D2, enter a formula calculating slope $_0$ .
  - In cell E2, enter a formula to calculate rise $_0$ . Make sure you use a reference to cell H3, and make sure that reference will be fixed when you copy cells down!
  - In cell B3, enter a formula to calculate  $t_1$ . Again, be sure to use a fixed reference to H3.
  - In cell C3, enter a formula to calculate  $y_1$  using the previous row.
  - Copy cells D2 and E2 to D3 and E3 and verify the values are correct.
  - Select cells B3 to E3 and copy them down to fill the table. If you did everything correctly, your table should be identical to the one above.

## 106L Labs: Introduction to Euler's Method

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- In the second tab of your spreadsheet, repeat the above with  $\frac{dy}{dt} = 2t$ ,  $y(0) = 1$  use  $\Delta t = 1$  to compute a new set of values of  $y_k$ .
- Plot the points  $(t_k, y_k)$  from the table above on the same graph where you plotted your previous approximation.
- We will reduce the value of  $\Delta t$  one more time so we can see clearly what is happening as  $\Delta t$  is taken to be smaller and smaller. Given  $\frac{dy}{dt} = 2t$ ,  $y(0) = 1$  with  $\Delta t = 0.5$ , fill in the third tab of your spreadsheet. Then plot the points on the same graphing sheet as before.

### Part IV: Investigations

In this last part we will superimpose on our previous approximating plots the graph of the actual function  $y(t)$ . *But keep in mind that we are doing this operation to enhance our understanding of what we have been doing.* Euler's method is normally used when we cannot find an explicit formula for  $y(t)$ .

- We know that  $\frac{dy}{dt} = 2t$ . We've also previously learned enough about differentiation to determine what function the original  $y$  must have been. There are an infinite number of possibilities, but all the solutions are quite similar. Write down several functions  $y(t)$  which have the property that  $\frac{dy}{dt} = 2t$  and explain what these functions have in common.
- Among all the functions which satisfy the requirement that  $\frac{dy}{dt} = 2t$  there is only one that also has the property that  $y(0) = 1$ . Find that one now and then graph it on the same graph where you have made the previous plots from Euler's method.
- The attached graph paper should now have four plots. There are three approximations to the graph of  $y(t)$ , created by using Euler's method with values of  $\Delta t = 2$ , 1, and 0.5. There is also the "true" graph of the function  $y(t)$ . Describe the positions of the approximate graphs relative to the true graph. Explain why they are positioned relative to each other the way that they are. Would these relationships be any different if the curve had been concave down rather than up?

### Part V: Exercises

The following exercises should be done on a spreadsheet.

- The rate of change (velocity) of a certain share of stock, measured in \$/week, can be modeled by the equation  $\frac{dP}{dt} = \frac{5}{\sqrt{t+1}}$  where  $t$  is measured in weeks. If the initial value of a share is \$25 ( $P(0) = \$25$ ), estimate the value of a share of this stock after 10 weeks. Assuming that the stock price follows this model is your estimate of  $P(10)$  an over or under approximation. Explain.

## 106L Labs: Introduction to Euler's Method

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15. The rate of change of the temperature of a chemical solution, measured in  $^{\circ}\text{C}/\text{sec}$ , can be modeled by the equation  $\frac{dT}{dt} = \sqrt{t^2 + 1}$  where  $t$  is measured in seconds. Assume that  $T(0) = 0^{\circ}$ . Approximate the graph of  $T(t)$  over the interval  $[0, 1]$ . Use  $\Delta t = 0.2$ . Assuming that the temperature follows this model is this an over or under approximation? Explain.
16. In the preview you saw a graph of a function  $f$  given only that  $f'(t) = 2^{-t^2/2}$  and  $f(-5) = 0$ . Use Euler's method to make your own approximation of this graph. Plot your answer and compare it to the plot in the preview.
17. Consider the differential equation

$$\frac{dz}{du} = z - u.$$

- (a) For each of the following initial conditions, use Euler's method with  $\Delta u = 0.5$  to estimate the solution. Continue until  $u = 5$ , and plot your answers.
- $z(0) = 0.5$ .
  - $z(0) = 1$ .
  - $z(0) = 2$ .
- (b) If  $z(0) = 1$ , what do you notice about your solution? Is it an underestimate, overestimate, or neither?
- (c) Use your answers to the previous questions to solve the initial value problem

$$\frac{dz}{du} = z - u, z(0) = 1.$$

## Graphing Page

In this lab you are asked to make three plots using Euler's method and one graph using the formula for a function. All of these graphs should be superimposed on the grid below. You may want to plot each with a different color so that you can easily distinguish them.

