Riemann Sums

Purpose:

The purpose of this lab is to review the ideas, discussed in class, concerning how to compute sums of the form \( \sum_{i=1}^{n} f(a + i \Delta x) \Delta x \) and to explore what happens as we compute those sums for large and larger \( n \).

Open the spreadsheet from this lab in Sakai. Then go to the File menu, make a copy, and rename the sheet with your group names.

Part I

1. Consider the area above the \( x \)-axis and under the graph of \( y = \sqrt{25 - x^2} \) between \( x = 0 \) and \( x = b \) as shown in the picture below. Note that this is \( \int_{0}^{b} \sqrt{25 - x^2} \, dx \) (we’ll compute the value of \( b \) in a second).

(a) Carefully copy the above picture and shade in the area under discussion. Make sure to check this with your TA before proceeding.

(b) What is the value of \( a \)? Store it in cell B1.

(c) Find the value of \( b \) and store it in cell B2.
(d) Find the exact area of the region using what you know about geometry and store it in cell B7 (be sure to enter the formula you used to compute it, so it’s as precise as possible).

(e) Let’s take \( n = 5 \) rectangles, enter 5 in cell B3, and enter a formula into cell B4 that computes \( \Delta x \).

(f) Draw a picture showing the Right Hand Sum (RHS) for \( n = 5 \).

2. Recall that the \( i^{th} \) interval in a Riemann sum is \([a, a + i\Delta x]\). Use the following steps to compute left-hand and right-hand sums for this integral with \( n = 5 \):

- In Column F, enter \( i \) values from 0 to 5.
- In cell G2, enter a formula that computes \( a + i\Delta x \) for the given \( i \) in column F. Recall that you previously stored \( a \) and \( \Delta x \) in cells B1 and B4 respectively. So that things don’t go wrong later, be sure to refer to B1 as B$1 and to B4 as B$4.
- Copy your formula down from cell G2 to fill in cells G3 through G7.
- In cell H2, compute the height of the first rectangle in a left hand sum (that is, \( f(a) \)). Note that you have \( a \) in cell G2!
- Copy your formula down to cells H3 through H7.

3. We will now find the areas of rectangles in the LHS and the RHS, then add them up. It turns out we don’t have to duplicate most of the work!

- In cell I2, compute the area of the first rectangle in a LHS. Then, in cell I3, compute the area of the second rectangle in a LHS. How does this relate to the area of the the first rectangle in a RHS?
- In the rest of Column I, find the areas of rectangles by multiplying your height from column H by \( \Delta x \). Once again, refer to \( \Delta x \) as B$4, not just B4.
- We want to compute the LHS in cell L1. Which five areas should we add to do this? Enter a formula in L1 that does this.
- We want to compute the RHS in cell L2. Which five areas should we add to do this? Enter a formula in L2 that does this.
- In cell L6, enter a formula for the average of the LHS and RHS. In Cell L7, enter a formula for the difference of this average from the true value.

4. Next, we will compute the errors. That is, how far off from the true area our sums are.

- Since our function is decreasing, should the LHS overestimate the true value of the integral or underestimate it? What about the RHS?
- In cell L3, enter a formula for the difference between the left hand sum in cell L1 and the true value (from cell B7). Do the same for the right hand sum in cell L4. Check that the first is positive and the second negative. Why must this be the case?
- Compute the error for the average in cell L7. It should be negative. Why is this the case?
5. Record your answers for LHS and RHS, as well as their respective errors, in the first row of the table in rows 13-19 in the spreadsheet. Do the same for the average of the LHS and RHS and its error. To make sure we have as accurate an answer as possible, use Ctrl-C (or Command-C or a Mac) to copy L2 (the LHS), then use Ctrl-Shift-V (or Command-Shift-V) to paste it in. This copies the value rather than the formula for it.

6. Next, modify your spreadsheet to complete the rest of the table. For each row, you’ll need to:
   - change the value of \( n \) (which will automatically change \( \Delta x \));
   - add more values of \( i \);
   - copy down heights and areas;
   - and modify your formulas for summing up for the LHS and RHS.

7. By looking at the LHS and RHS columns of your table, estimate \( \lim_{n \to \infty} \sum_{i=1}^{n} f(a + i\Delta x)\Delta x \) correct to one decimal place. Enter your estimate in cell L9. What do you think the exact value is?

**Midpoint Sums**

Note that we can also compute an estimate of the area using the following sum:

\[
\sum_{i=1}^{n} f \left( a + \left( i - \frac{1}{2} \right) \Delta x \right) \Delta x.
\]

This is the *Midpoint Sum* or MPS. Instead of using the left or the right endpoint of the interval, this estimate uses the midpoint of the interval.

8. Explain why, if \( 1 \leq i \leq n \), then \( a + (i - \frac{1}{2})\Delta x \) gives the midpoint of the \( i^{th} \) interval. (Hint: multiply out those parentheses, then think about how many steps of length \( \Delta x \) from \( a \) your formula shows you.)

9. On a copy of the following axes, draw a picture of this sum, for \( n = 5 \).
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10. Modify your spreadsheet to compute the mid-point sum with five intervals. Note that you will need to compute the new $x$ values (use the formula for the midpoints of the intervals above), then the height of the rectangles, then their areas. Also find the error from the exact answer. Fill your answers in the last two columns of the table that begins on row 23. Then do the same for the larger values of $n$. Again, be sure to use copy by value to record your answers accurately.

11. Compare the averages of the LHS and RHS to the MPSs. Are they the same? Answer in terms of over and underestimates.

Part II

12. Graph $y = e^{-x}$ between $x = 1$ and $x = 2$ and consider the area above the $x$-axis and under the graph between these two points. Draw a picture that shows the Left-Hand Sum for $n = 5$.

13. Use the second tab of the spreadsheet to compute Riemann sums for this integral. Complete the tables in that tab.

14. Estimate $\lim_{n \to \infty} \sum_{i=1}^{n} f(a + i\Delta x)\Delta x$ correct to three decimal places.

Part III

15. Graph $y = \sin(x)$ between $x = 0$ and $x = \pi$ and consider the area above the $x$-axis and under the graph between these two points. Draw a picture that shows the Left-Hand Sum for $n = 6$.

16. Use the third tab of the spreadsheet to compute Riemann sums for this integral. Complete the tables in that tab.

17. What do you think $\lim_{n \to \infty} \sum_{i=1}^{n} f(a + i\Delta x)\Delta x$ is?

Part IV: The Definite Integral

**Definition:** The *definite integral* of $f$ from $a$ to $b$ is

$$\int_{a}^{b} f(x)dx = \lim_{n \to \infty} \sum_{i=1}^{n} f(a + i\Delta x)\Delta x,$$

where $\Delta x = \frac{b-a}{n}$. If $f$ is continuous then this limit always exists.

Note also that $\int_{a}^{b} f(t)dt = \lim_{n \to \infty} \sum_{i=0}^{n-1} f(a + i\Delta x)\Delta x = \lim_{n \to \infty} \sum_{i=1}^{n} f\left(a + \left(i - \frac{1}{2}\right) \Delta x\right)\Delta x$.

18. Using only the work done in parts I-III of this lab, find (precisely if possible, otherwise to three decimal places)
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(a) \( \int_0^b \sqrt{25 - x^2} \, dx \), where \( b = \frac{5}{\sqrt{2}} \).

(b) \( \int_1^2 e^{-x} \, dx \).

(c) \( \int_0^\pi \sin x \, dx \).

19. Note that this definition allows for \( f(x) < 0 \) for some \( x \) in the interval \([a, b]\). Use some geometry and logic (i.e. no Fundamental Theorem) to find:

(a) \( \int_0^{2\pi} \sin x \, dx \).

(b) \( \int_1^4 2 - x \, dx \).

Part V: Overestimates and Underestimates

Let’s examine whether, given certain conditions, certain sums overestimate or underestimate \( \int_a^b f(x) \, dx \). Assume \( f(x) \) is continuous and positive for all \( x \) in \([a, b]\). If the conditions in a particular column hold for all \( x \) in \([a, b]\), decide whether the sum in a particular row overestimates \( \int_a^b f(x) \, dx \) or underestimates \( \int_a^b f(x) \, dx \), or whether there is not enough information to tell.

<table>
<thead>
<tr>
<th>( f'(x) &gt; 0 ), ( f''(x) &gt; 0 )</th>
<th>( f'(x) &lt; 0 ), ( f''(x) &gt; 0 )</th>
<th>( f'(x) &gt; 0 ), ( f''(x) &lt; 0 )</th>
<th>( f'(x) &lt; 0 ), ( f''(x) &lt; 0 )</th>
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</thead>
<tbody>
<tr>
<td>RHS</td>
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<tr>
<td>LHS</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>LHS + RHS</td>
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<td></td>
<td></td>
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<tr>
<td>( \frac{LHS}{2} )</td>
<td></td>
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<tr>
<td>MPS</td>
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</tbody>
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Explain your answers to the second column.