1. **Glucose in the Bloodstream**

A physician decides to give a patient an infusion of glucose at a rate of \( c \) grams per hour. The body of the patient simultaneously converts the glucose and removes it from the bloodstream at a rate proportional to the amount present in the bloodstream, say, at \( r \) grams per hour per gram of glucose present.

(a) Explain why the amount \( G = G(t) \) of glucose present at time \( t \) can be modeled by a differential equation of the form

\[
\frac{dG}{dt} = c - rG.
\]

(b) Given an initial amount \( G_0 \) of glucose in the bloodstream at time 0, find an explicit formula for the amount present at any time \( t \geq 0 \).

(c) What rate of infusion would keep the glucose level in the bloodstream constant?

(d) A physician orders an infusion of 10 grams of glucose per hour for a patient. Laboratory technicians determine that the patient has 2 grams of glucose in his bloodstream and that his body will remove glucose from the bloodstream at a rate of 3 grams per hour per gram of glucose. How much glucose will be in the bloodstream \( t \) hours after the infusion is started? In particular, how much after 2 hours? How long will it take for the glucose level in the bloodstream to reach 3 grams?

(e) Suppose a patient’s bloodstream has 2 grams of glucose, and her physician wants to raise this amount to 3.5 grams in 3 hours. It is determined that her system removes glucose from the bloodstream at a rate of 4 grams per hour per gram of glucose. How fast should the physician order the glucose to be infused into the patient’s body?
2. A Falling Body with Air Resistance

Suppose you drop a marble from a height of 900 meters. In this problem we will consider the effects of air resistance on such a marble. (You will also need to find formulas, without air resistance, for the position, velocity, and acceleration of such a marble, and compare them to your answers here.)

Taking air resistance into account, the total force $F$ acting on the marble has two components, the gravitational force $F_g$, and the retarding force (i.e., force due to air resistance) $F_r$; in other words $F = F_g + F_r$. We will assume here that the retarding force $F_r$ is proportional to the velocity of the marble, say $F_r = -kv$.

(a) Why is the coefficient of $v$ (in the equation for retarding force) negative?

(b) Use Newton’s Law of Motion to argue that $v$ must satisfy a differential equation of the form

$$\frac{dv}{dt} = g - cv$$

for some constant $c$. How is $c$ related to $k$ and the mass $m$ of the object?

(c) Under the assumptions stated above, set up an initial value problem for the velocity of the marble.

(d) Assume further that the numerical values of $k$ and of the mass are such that $c$ turns out to be 0.08. Solve the initial value problem to find an explicit formula for the velocity of the marble as a function of time.

(e) Your explicit formula for velocity also provides a differential equation for the position function $s = s(t)$. Set up and solve an initial value problem for the position function.

(f) Under the assumptions above, how long will it take the marble to reach the ground? How fast will it be going when it hits the ground?

(g) Compare your answers to the problem above with the answers you get when you do not take air resistance into account. Are they much different? How are they different? It may help to compare the position, velocity, and acceleration functions rather than just the values for when the marble hits the ground or how fast it is going then.

(h) Look back again at the velocity function you got when you took air resistance into account. Is there a limiting value? In other words, if the marble could fall for a very long time (imagine the ground was not there, or a big hole in the ground, or that the marble fell from a greater height), would it approach a “terminal velocity”? If so, find this terminal velocity two ways; using the differential equation, and using the velocity function itself. Is there a terminal velocity for the marble when we do not take air resistance into account? If so, what is it? If not, why not?
3. **RL Circuits**

Suppose you are given an electrical circuit with a resistance of $R$ ohms, an inductance of $L$ Henries, and a battery (i.e., a constant voltage source) of $V$ volts. If the current (in amperes) at time $t$ (in seconds after closing the switch and completing the circuit) is $i = i(t)$, then the voltage drop at the resistor is $Ri$, and the voltage drop at the inductor is $L\frac{di}{dt}$. According to Ohm’s Law, the voltage input $V$ must balance the voltage drops. That is,

$$L\frac{di}{dt} + Ri = V.$$

(a) Find the current as a function of time.

(b) If the switch is left closed for a long time, what is the limiting value of the current?

4. **Brine Mixing**

A tank contains 25 pounds of salt dissolved in 200 gallons of water. There is a pipe at the top of the tank to bring in additional solution and a pipe at the bottom of the tank to drain off solution. The tank itself contains a large stirrer that keeps the solution in the tank thoroughly mixed. Starting at time $t = 0$, water containing $\frac{1}{2}$ pound of salt per gallon enters the tank at the rate of 4 gallons per minute, and the well-stirred solution leaves the tank at the same rate.

(a) If this process continues for a long time, how much salt do you think will eventually be in the tank? Does your answer about the eventual salt content depend on how much salt there was in the tank at the start of the process? Why or why not? Does your answer about the eventual salt content depend on the rate at which water enters and leaves the tank? Why or why not?

(b) Find a differential equation that describes the rate of change of the amount of salt in the tank. Supply an initial condition for your differential equation, and solve for an explicit formula that gives the amount of salt as a function of time. (*Hints: Let $S = S(t)$ be the amount of salt in the tank at time $t$ and let $C = C(t)$ be the concentration of the salt at time $t$. How are $S$ and $C$ related? The flow of salt out of the tank depends on the concentration; use the relationship between $S$ and $C$ to express that flow in terms of $S$.*)

(c) How much salt is in the tank after 10 minutes? After 2 hours? What does your formula tell you about how much salt there will be in the tank after a long time? Does your answer agree with what you determined in part (a)? Why or why not?