Solving Differential Equations

This entire worksheet concerns the differential equation
\[ \frac{dy}{dx} = xy. \]

You will investigate the solutions to this differential equation three ways: sketching approximate graphs of the solutions using 1 slopefields, approximating values of a solution using 2 Euler’s method, and finding an exact solution using 3 separation of variables. Finally, you will check and compare your answers in 4.

Slopefields

1. Sketch the slopefield for the differential equation using the grid below. Be sure to make your slopefield neat and accurate, and show your calculations.

```
  -2  -2  -1  -1  1  1  2  2
-2  -2  -1  -1  1  1  2  2
  .  .  2  .  .  .  .  .
  .  .  1  .  .  .  .  .
  .  .  0  .  .  .  .  .
  .  .  1  .  .  .  .  .
  .  .  2  .  .  .  .  .
```

2. On the slopefield you drew above, use a different color to sketch and label solution curves for the following initial values: \( y(0) = -1 \) and \( y(-1) = 1 \).

3. Consider the solution curve that you drew for the initial value \( y(-1) = 1 \). Try to approximate the value of \( y(0) \) for this curve using the graph that you drew (your approximation will not be very accurate).

Euler’s Method

For this problem we will only consider the solution that has initial value \( y(-1) = 1 \). In other words we will consider the initial value problem:
\[ \frac{dy}{dx} = xy, \quad y(-1) = 1. \]
4. Use Euler’s method with $\Delta x = .5$ to approximate the value of $y(0)$. Be sure to show your work and all your calculations clearly.

5. Use Euler’s method with $\Delta x = .25$ to approximate the value of $y(0)$. Be sure to show your work and all your calculations clearly.

6. In each of your approximations above, you actually produced a list of points that would be on an approximate sketch of the solution curve $y(x)$. Make an approximate sketch of the solution $y(x)$ on the interval $[-1,0]$ using the points you obtained in question 4. Then (in a different color) sketch the corresponding approximation from question 5. Be sure to label which is which.

**Separation of Variables**

7. Use separation of variables to find the family of solutions for the differential equation

$$\frac{dy}{dx} = xy.$$

Be sure to show all your work clearly and in order.

8. Find the solutions $y(x)$ that correspond to the initial values $y(0) = -1$, $y(-1) = 0$ and $y(-1) = 1$.

9. Graph the three solutions you found above (use Geogebra, or any other tool you wish). Make a careful sketch of your solutions.

**Checking**

In this problem you will do various checks and comparisons of your answers to the questions above. First let’s consider the problems that concerned the IVP

$$\frac{dy}{dx} = xy, \quad y(-1) = 1.$$

10. Which approximation for $y(0)$ should be better, the one found in question 4 or the one found in question 5? Why?

11. Compare your approximation for $y(0)$ from the slopefield in question 3 with your approximation from Euler’s Method in question 5. Are they similar? Why or why not? Which do you think is more accurate?

12. Compare the sketches of $y(x)$ that you obtained in question 6 from Euler’s method with the sketch of $y(x)$ that you drew using the slopefield in question 3. Do they match?

13. Find the actual value of $y(0)$ using the formula for the solution that you found in question 8. Is it close to the approximations of $y(0)$ that you made in questions 3, 4 and 5?
Now we’ll consider the family of solutions to the differential equation

\[ \frac{dy}{dx} = xy. \]

14. Compare the graphs of the three exact solution curves from question 9 to the curves you drew using the slopefield in question 3. Do they correspond?

15. Check that the family of solutions that you found in question 7 is correct. In other words, check that any function in that family of solutions will satisfy the differential equation.