Differentiating Power Functions and Polynomials

Purpose

In this lab we will learn how to differentiate functions like $x^3 + 6x^2 + 2x + 1$, called polynomials.

Part I: Review

Let’s review some basic facts about derivatives. Earlier this semester, we computed the derivatives of constant functions and linear functions:

1. $\frac{d}{dx}(c) = \underline{}$  $\frac{d}{dx}(mx + b) = \underline{}$

2. You may also have shown previously that $\frac{d}{dx}\left(\frac{1}{x}\right) = -\frac{1}{x^2}$, $\frac{d}{dx}(\sqrt{x}) = \frac{1}{2\sqrt{x}}$ and $\frac{d}{dx}\left(\frac{1}{x^2}\right) = -\frac{2}{x^3}$.

Pick one of these and write out the full calculation involving limits.

3. We know that if $\frac{d}{dx}(f(x))$, we can easily find $\frac{d}{dx}(cf(x))$ because:

$$
\frac{d}{dx}(cf(x)) = \lim_{h \to 0} \frac{cf(x + h) - cf(x)}{h} = \lim_{h \to 0} \frac{c(f(x + h) - f(x))}{h} = c \lim_{h \to 0} \frac{f(x + h) - f(x)}{h} = \underline{}
$$

This is called the **Constant Multiple Rule**.

4. Similarly, if we know $\frac{d}{dx}(f(x))$ and $\frac{d}{dx}(g(x))$, we can easily find $\frac{d}{dx}(f(x) + g(x))$ because (fill in the calculation):

$$
\frac{d}{dx}(f(x) + g(x)) = \lim_{h \to 0} \underline{}
$$

This is called the **Sum Rule**.

Examples

5. Compute the following derivatives:

(a) $\frac{d}{dx}(\pi x)$

(b) $\frac{d}{dx}\left((r^2 + 1)^\frac{1}{2}\right)$

(c) $\frac{d}{dxw}(3^{2y-x})$ (Hint: What is the independent variable?)
Part II: Differentiating Monomials - The Power Rule

A function of the form \( f(x) = ax^r \) is called a monomial. In this part of the lab, we will learn how to find its derivative.

6. Multiply out each of the following:
   (a) \((z - x)(z + x)\)
   (b) \((z - x)(z^2 + 2z + x^2)\)
   (c) \((z - x)(z^3 + z^2x + x^2 + x^3)\)
   (d) \((z - x)(z^4 + z^3x + z^2x^2 + zx^3 + x^4)\)

7. Write out each of the following a product of \( z - x \) and another polynomial:
   (a) \(z^2 - x^2 = (z - x)\)
   (b) \(z^3 - x^3 = (z - x)\)
   (c) \(z^4 - x^4 = (z - x)\)
   (d) \(z^5 - x^5 = (z - x)\)
   (e) \(z^n - x^n = (z - x)\)

8. In the factorization of \( z^3 - x^3 \), how many terms are in the second factor? In general, in the factorization of \( z^n - x^n \), how many terms are in the second factor?

9. By letting \( z = x + h \) and carefully thinking about what \( z \) approaches as \( h \) approaches 0, show that
   \[ f'(x) = \lim_{h \to 0} \frac{f(x + h) - f(x)}{h} = \lim_{z \to x} \frac{f(z) - f(x)}{z - x}. \]

10. Now use your factorization of \( z^n - x^n \) above to find \( \frac{d}{dx}(x^n) = \lim_{z \to x} \frac{z^n - x^n}{z - x} \).

11. Note that we have previously shown that \( \frac{d}{dx} \left( \frac{1}{x} \right) = -\frac{1}{x^2}, \frac{d}{dx} (\sqrt{x}) = \frac{1}{2\sqrt{x}} \) and \( \frac{d}{dx} \left( \frac{1}{x^2} \right) = -\frac{2}{x^3} \).

   Rewrite each of these differentiation formulas using exponential notation (the first is done for you as an example):
   (a) \( \frac{d}{dx} \left( \frac{1}{x} \right) = -\frac{1}{x^2} \leftrightarrow \frac{d}{dx}(x^{-1}) = -1 \cdot x^{-2} \)
   (b) \( \frac{d}{dx} (\sqrt{x}) = \frac{1}{2\sqrt{x}} \)
   (c) \( \frac{d}{dx} \left( \frac{1}{x^2} \right) = -\frac{2}{x^3} \)

   Notice that in each of these cases, we have that \( \frac{d}{dx}(x^r) = rx^{r-1} \). This is called the **Power Rule**.

   In Question 10 above, we showed that we have \( \frac{d}{dx}(x^r) = rx^{r-1} \) whenever \( r \) is _______. Later in the semester we will show that this is true for any value of \( r \), including fractions and negative numbers.
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Differentiation Rules:

1. Power Rule: \( \frac{d}{dx}(x^r) = rx^{r-1} \)

2. Sum Rule: \( \frac{d}{dx}(f(x) + g(x)) = \frac{d}{dx}(f(x)) + \frac{d}{dx}(g(x)) \)

3. Constant Multiple Rule: \( \frac{d}{dx}(cf(x)) = c\frac{d}{dx}(f(x)) \)

Part III: Differentiating Polynomials and Applications

12. Use these rules (and only these rules, no others!) to find the derivatives of the following functions:
   (a) \( y = -3x^4 - 4x^3 - 6x + 2 \)
   (b) \( y = 3t^5 - 5\sqrt{t} + \frac{7}{t} \)
   (c) \( y = z^2 + \frac{1}{z^2} \)
   (d) \( y = \frac{x^2+1}{x} \)
   (e) \( y = (x + 1)^2 \)

13. Find the equation of the tangent line to the graph of \( y = 2x^3 - 2x^2 + 1 \) at \((1, 1)\).

14. Suppose that \( f(x) = x^{10} \). What is the smallest number \( n \) such that \( f^{(n)}(x) = 0 \)? (Recall that \( f^{(n)} \) means the \( n^{th} \) derivative.)

15. Find values of \( a \) and \( b \) such that the line \( 3x + y = a \) is tangent to the graph of \( f(x) = bx^2 \) at \( x = 1 \).

16. If \( n \) is even then the derivative of \( x^n \) is (choose one):
   (a) an even function      (b) an odd function      (c) neither even nor odd      (d) not enough information

   In a few complete sentences, and using a typical graph of a function of the form \( f(x) = x^n \) for even \( n \), explain why this should be true.

17. If \( n \) is odd then the derivative of \( x^n \) is (choose one):
   (a) an even function      (b) an odd function      (c) neither even nor odd      (d) not enough information

   In a few complete sentences, and using a typical graph of a function of the form \( f(x) = x^n \) for odd \( n \), explain why this should be true.

18. Consider the function \( y = x^5 + 2x \). What is the smallest slope that a line tangent to this curve can have?

Report

For the report, each group should hand in answers to the following questions: 7, 10, 13, 15, 17.