Differentiating Power Functions and Polynomials

Purpose

In this lab we will learn how to differentiate functions like $x^3 + 6x^2 + 2x + 1$, called *polynomials*.

Part I: Review

Let's review some basic facts about derivatives. Earlier this semester, we computed the derivatives of constant functions and linear functions:

1. $\frac{d}{dx}(c) =$ _______ $\quad \frac{d}{dx}(mx+b) =$ _______

2. You may also have shown previously that $\frac{d}{dx}\left(\frac{1}{x}\right) = -\frac{1}{x^2}$, $\frac{d}{dx}\left(\sqrt{x}\right) = \frac{1}{2\sqrt{x}}$ and $\frac{d}{dx}\left(\frac{1}{x^2}\right) = -\frac{2}{x^3}$. Pick one of these and write out the full calculation involving limits.

3. We know that if $\frac{d}{dx}(f(x))$, we can easily find $\frac{d}{dx}(cf(x))$ because:

$$\begin{aligned}\frac{d}{dx}(cf(x)) &= \lim_{h\to 0} \frac{cf(x+h) - cf(x)}{h} \\ &= \lim_{h\to 0} \frac{c(f(x+h) - f(x))}{h} \\ &= c\lim_{h\to 0} \frac{f(x+h) - f(x)}{h} \\ &= \underline{\qquad\qquad}\end{aligned}$$

   This is called the *Constant Multiple Rule*.

4. Similarly, if we know $\frac{d}{dx}(f(x))$ and $\frac{d}{dx}(g(x))$, we can easily find $\frac{d}{dx}(f(x)+g(x))$ because (fill in the calculation):

$$\frac{d}{dx}(f(x)+g(x)) = \lim_{h\to 0}$$

   This is called the *Sum Rule*.

Examples

5. Compute the following derivatives:

   (a) $\frac{d}{dx}(\pi x)$

   (b) $\frac{d}{dx}\left(\left(r^2+1\right)\frac{1}{x}\right)$

   (c) $\frac{d}{dw}\left(3^{2y-x}\right)$ (Hint: What is the independent variable?)

Part II: Differentiating Monomials - The Power Rule

A function of the form $f(x) = ax^r$ is called a *monomial.* In this part of the lab, we will learn how to find its derivative.

6. Multiply out each of the following:
   (a) $(z-x)(z+x)$
   (b) $(z-x)(z^2+zx+x^2)$
   (c) $(z-x)(z^3+z^2x+zx^2+x^3)$
   (d) $(z-x)(z^4+z^3x+z^2x^2+zx^3+x^4)$

7. Write out each of the following a product of $z-x$ and another polynomial:
   (a) $z^2-x^2 = (z-x)$__________________________
   (b) $z^3-x^3 = (z-x)$__________________________
   (c) $z^4-x^4 = (z-x)$__________________________
   (d) $z^5-x^5 = (z-x)$__________________________
   (e) $z^n-x^n = (z-x)$__________________________

8. In the factorization of z^3-x^3, how many terms are in the second factor? In general, in the factorization of z^n-x^n, how many terms are in the second factor?

9. By letting $z = x+h$ and carefully thinking about what z approaches as h approaches 0, show that
$$f'(x) = \lim_{h\to 0}\frac{f(x+h)-f(x)}{h} = \lim_{z\to x}\frac{f(z)-f(x)}{z-x}.$$

10. Now use your factorization of z^n-x^n above to find $\frac{d}{dx}(x^n) = \lim_{z\to x}\dfrac{z^n-x^n}{z-x}$.

11. Note that we have previously shown that $\frac{d}{dx}\left(\frac{1}{x}\right) = -\frac{1}{x^2}$, $\frac{d}{dx}\left(\sqrt{x}\right) = \frac{1}{2\sqrt{x}}$ and $\frac{d}{dx}\left(\frac{1}{x^2}\right) = -\frac{2}{x^3}$.

    Rewrite each of these differentiation formulas using exponential notation (the first is done for you as an example):
   (a) $\frac{d}{dx}\left(\frac{1}{x}\right) = -\frac{1}{x^2} \Leftrightarrow \frac{d}{dx}(x^{-1}) = -1\cdot x^{-2}$
   (b) $\frac{d}{dx}\left(\sqrt{x}\right) = \frac{1}{2\sqrt{x}}$
   (c) $\frac{d}{dx}\left(\frac{1}{x^2}\right) = -\frac{2}{x^3}$

Notice that in each of these cases, we have that $\frac{d}{dx}(x^r) = rx^{r-1}$. This is called the *Power Rule.*

In Question 10 above, we showed that we have $\frac{d}{dx}(x^r) = rx^{r-1}$ whenever r is _______. Later in the semester we will show that this is true for any value of r, including fractions and negative numbers.

Differentiation Rules:

1. Power Rule: $\dfrac{d}{dx}(x^r) = rx^{r-1}$
2. Sum Rule: $\dfrac{d}{dx}(f(x)+g(x)) = \dfrac{d}{dx}(f(x)) + \dfrac{d}{dx}(g(x))$
3. Constant Multiple Rule: $\dfrac{d}{dx}(cf(x)) = c\dfrac{d}{dx}(f(x))$

Part III: Differentiating Polynomials and Applications

12. Use these rules (and only these rules, no others!) to find the derivatives of the following functions:

    (a) $y = -3x^4 - 4x^3 - 6x + 2$
    (b) $y = 3t^5 - 5\sqrt{t} + \frac{7}{t}$
    (c) $y = z^2 + \frac{1}{2z}$
    (d) $y = \frac{x^2+1}{x}$
    (e) $y = (x+1)^2$

13. Find the equation of the tangent line to the graph of $y = 2x^3 - 2x^2 + 1$ at $(1, 1)$.
14. Suppose that $f(x) = x^{10}$. What is the smallest number n such that $f^{(n)}(x) = 0$? (Recall that $f^{(n)}$ means the n^{th} derivative.)
15. Find values of a and b such that the line $3x + y = a$ is tangent to the graph of $f(x) = bx^2$ at $x = 1$.
16. If n is even then the derivative of x^n is (choose one):

    (a) an even function (b) an odd function (c) neither even nor odd (d) not enough information

    In a few complete sentences, and using a typical graph of a function of the form $f(x) = x^n$ for even n, explain why this should be true.
17. If n is odd then the derivative of x^n is (choose one):

    (a) an even function (b) an odd function (c) neither even nor odd (d) not enough information

    In a few complete sentences, and using a typical graph of a function of the form $f(x) = x^n$ for odd n, explain why this should be true.
18. Consider the function $y = x^5 + 2x$. What is the smallest slope that a line tangent to this curve can have?

Report

For the report, each group should hand in answers to the following questions: 7, 10, 13, 15, 17.