

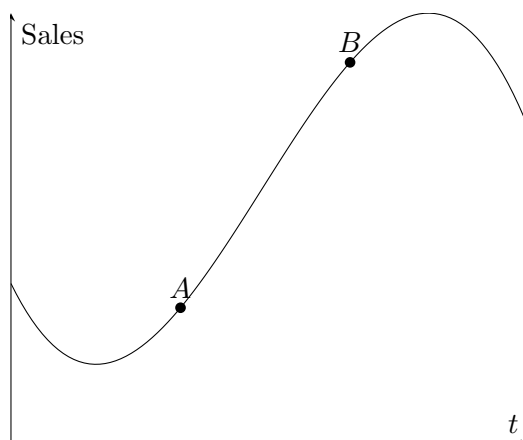
Derivatives and Roots

Purpose

The purpose of this lab is for you to acquire an understanding of the relationships among a function, its first and second derivatives, and the roots of these functions. You will apply this understanding to critical analysis of non-mathematical statements.

Preview

Suppose the curve below gives the daily sales totals for a new item in your department store. It would certainly be of value to be able to identify the high and low points. But consider the points A and B on the curve. The rates of increase of sales at those two points are about the same, but it is obvious that at point A you would want to restock the item, and at B it might be preferable to let your supply on hand dwindle.



This type of marketing decision would be easy if we had a crystal ball that could show us the entire curve—in particular, the future curve. But, of course, we only have a “local” view. We shall see in this lab how we can use the local behavior of the graph to analyze a situation such as that above.


Part I: Zooming, Roots, and Extrema

Open the link to Geogebra. This lab will largely take place using this app. Every member of the group should have the page open and be working to follow the lab on their own laptop. Each group will be responsible for all members being able to use the app and get the lab done.

1. Graph the function $f(x) = -2x^4 + 11x^3 + 15x^2 - 70x - 10$ on Geogebra (just type it into an input box¹). Use the default graphing ranges. Zoom out a couple of times and then zoom in on interesting parts of the curve until you are confident that you have both the “big picture” and a clear idea of the details of the graph.

¹If typing on your keyboard doesn't do anything, click the keyboard icon at the bottom left and use it to input.

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2. Now have Geogebra redraw the graph with a horizontal range of $[-3, 6]$ and a vertical range of $[-100, 200]$ (click on the  icon, then the 'Settings' option).
3. Geogebra can find the roots and extrema of the function for you! Click on the formula and select 'Special Point'. This will add the roots, extrema, and y-intercept to the graph. You can delete the y-intercept, as we will not be using that. You can see the coordinates in the input column.

Part II: The First Derivative

4. Our next step is to superimpose the graph of f' on the graph of f . Compute the derivative of $f(x)$ by hand, then add it to your graph. When you're done, your screen should show the graphs of both f and f' .
5. Just as you did with f , add in the roots and extrema of f' (delete the y-intercept).

The rest of the questions in this part of the lab should be done on paper using your graphs, not on Geogebra.

6. Describe the relationship between the roots of f' and the graph of f ? Use the concept of tangent lines to explain your observation.
7. Describe the relationship between the sign of f' and the graph of f . Explain your observation in the context of rates of change and values of f .
8. What is the relationship between the locations of the roots of f and the locations of the roots of f' ?
9. Consider the function $g(x) = f(x) + 70$. How many roots does this function have? What is the derivative of $g(x)$?
10. Using $g(x)$ instead of $f(x)$, answer the same questions you did in question 8.
11. Use your conclusions from questions 8 and 10 to make general statements about the relationship between the number of roots of a function h and the number of roots of h' :
 - If a differentiable function h has roots located at r_1 and r_2 , what can you conclude about the behavior of h' between r_1 and r_2 ?
 - If h' has exactly k roots, how many roots could h have?
 - If h has exactly k roots, how many roots could h' have?
12. Complete the statement of each of the following theorems.




Maximum/Minimum Value Theorem: If the function f has a local minimum or a local maximum at $x = c$, and if $f'(c)$ exists, then $f'(c) =$ _____.

Rolle's Theorem: If $f(r_1) = 0$ and $f(r_2) = 0$ and if f is differentiable on the interval $[r_1, r_2]$, then f' has _____ on that interval.

Part III: Concavity and Inflection Points

13. Refer again to the graph shown in the preview. If you look closely at a small region on the curve around each of the points A and B , you will see that although the rates of increase of sales at these two points are the same, there is a difference in the shape of the curve at the two points. Can you describe the difference?
14. Hopefully, you used the “concave up” and “concave down” in answering the previous question. State a definition or a characterization of the phrases “concave up” and “concave down” sufficient to enable someone who knows no calculus to use them correctly in describing features of a graph.
15. You will notice that this graph changes concavity between points A and B . As you may know, the point at which this change occurs is called an inflection point, which we will designate by “IP.” On the diagram in the preview indicate the location of the IPs and note that the curve “flexes” at that point.
16. Where on the sales curve in the preview is the tangent line the steepest? Describe the trend of the slopes of tangent lines just before and just after the point that you’ve chosen.

Part IV: Derivatives and Inflection Points

17. We will now turn our attention back to the function f which you graphed, along with its derivative function, in Part II. On the curve representing f locate the IPs as best you can just by looking (no need to write down any coordinates. Just look to see if you can see approximately where the IPs are).
18. Describe the relationship between the location of the IPs of f and the graph of f' ? Explain why this relationship should be true.
19. Your graph already includes the locations of the extrema of f' . Explain how to use this information to find the IPs of f . Using this, calculate the coordinates of the IPs of f .
20. Calculate f'' by hand and add it to your graph. Add its roots to the graph as well (delete its extrema and y-intercept).
21. Click on the Tools icon . Using the Line tool , draw vertical lines connecting the extrema of f' to the roots of f'' . How could you use these lines to find the IPs of f ?
22. Use the Intersect tool  icon to locate the IPs of f exactly (hint: your answer to the previous question should help). Compare your answers to the ones from question 19.

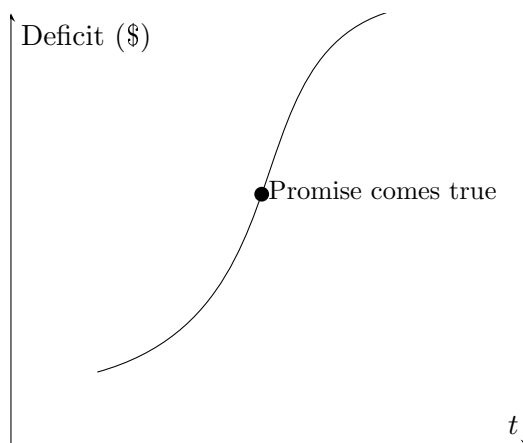
Part V: Critical Analysis

Consider the following statement.

If I am elected to office, I promise to decrease the annual national deficit.

We will use calculus to analyze what is being said here. The statement is actually about the national debt over time because the annual deficit is simply the rate at which the debt is growing, where time is measured in years. Let $D(t)$ represent the size of the national debt at year t . The fact that there is an annual deficit implies that $D'(t) > 0$ for the current year. If the politician speaking these words can keep this promise, then there will soon be an economic situation where $D''(t) < 0$.

This curve could represent what is happening: the debt has been rising at a (presumably) increasing rate. Our politician promises to bring about the inflection in the curve below. It is clear, of course, that there was no promise to make $D'(t) < 0$.



Part VI: Exercises

In each of the following exercises there is a statement and a definition of a function. Based upon what is given in the statement, make a sketch of the function and its derivatives. Explain why your graphs look the way they do.

23. The Dow Jones Industrial Average is continuing to increase, but at a decreasing rate. Let $A(t)$ represent the DJIA on day t .
24. There is new data that suggests that the annual rate of growth of the world population peaked last year. Let $P(t)$ be the number of people alive in year t .
25. Finally, look back at the preview graph of sales over time. Explain in terms of the first and second derivatives how you could know when to restock and when to cut back on orders. Explain how you could use sales data from electronic cash registers to get the information that you would need to make the restocking decision.

Report: Derivatives and Roots

Your lab report should consist of answers to the questions below, written neatly on a separate piece of paper. Each lab group will submit one report.

1. If possible, draw graphs of differentiable functions whose domains are $(-\infty, \infty)$ and that have the following characteristics. If it is not possible, explain why. Make sure that it is clear what the behavior of the function is as $x \rightarrow \infty$ and $x \rightarrow -\infty$.
 - (a) f' has four distinct zeros and f has one zero.
 - (b) f' has one zero and f has four distinct zeros.
 - (c) $f(x) < 10$ for all x , f' has one zero and f has one zero.
 - (d) f' has two distinct zeros and f has no zeros.
2. If possible, draw graphs of continuous functions whose domains are $(-\infty, \infty)$ and that have the following characteristics. If it is not possible, explain why. Make sure that it is clear what the behavior of the function is as $x \rightarrow \infty$ and $x \rightarrow -\infty$.
 - (a) $f(a) = 0$ and $f(b) = 0$ but $f'(x) \neq 0$ for any x between a and b .
 - (b) $f(a) = 0$ and $f(b) = 0$ (these are the only two zeros of f) and $f'(x) = 0$ for exactly two values between a and b .
3. Give formulas for functions whose domains are $(-\infty, \infty)$ with the following characteristics:
 - (a) f is continuous at 2, has a local minimum at 2, but $f'(2) \neq 0$.
 - (b) f is continuous at 0, $f'(0) = 0$, but f has neither a maximum nor a minimum at 0.
 - (c) f is continuous at 0, $f'(0) = 0$, $f''(0) \neq 0$, but f does not have an inflection point at 0.
 - (d) f is differentiable, f' has two distinct zeros and f has two distinct zeros.
4. State Rolle's theorem and explain (in a couple of sentences) why it must be true.