

Implicit Differentiation

Purpose

In this lab, we will explore curves that are defined *implicitly*. That is, curves that are not defined as $y = f(x)$, but rather as a relationship between x and y .

Part I: A Curve We Know

Consider the equation $x^2 + y^2 = 9$.

- What shape does this equation describe? Why?
 - Why can we not solve for y in this case? That is, why can't we just write this in the form $y = f(x)$?
 - Plot the curve described by this equation. Be sure to label the axes.
 - Is the point $(1, 2)$ on this curve? Why or why not? What about the point $\left(-\frac{3}{\sqrt{2}}, \frac{3}{\sqrt{2}}\right)$?
 - Without differentiating, find the equations of the tangent lines to the curve at the following points. (Hint: draw the lines!)
 - $(0, 3)$
 - $(-3, 0)$
- Use the technique of *implicit differentiation* you learned in class to find $\frac{dy}{dx}$ as a function of x and y .
- Use your previous answer to find the equation of the tangent to the curve at the point $\left(-\frac{3}{\sqrt{2}}, \frac{3}{\sqrt{2}}\right)$.
- What happens to $\frac{dy}{dx}$ at the point $(-3, 0)$? How could you have told this just by looking at the graph of the curve you plotted above?
- Suppose that the point (a, b) is on the curve.
 - What is the slope of the tangent line to the curve at this point (in terms of a and b)?
 - What is the slope of the line connecting this point to the origin $(0, 0)$?

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- (c) What is the relationship between the slopes of these two lines?
- (d) What does this show about circles?

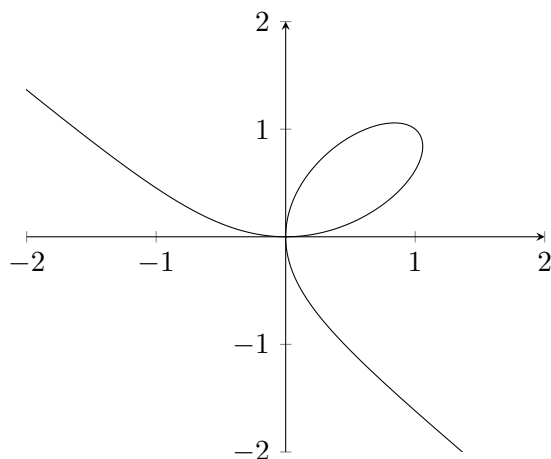
Part II: The Folium of Descartes

In 1638, the famous mathematician and philosopher René Descartes challenged his colleague Pierre de Fermat to find a general formula for the slope of the implicitly defined curve

$$x^3 + y^3 = 2xy.$$

We will start this part of the lab by doing the same.

Here is a plot of this curve, often called the *Folium of Descartes*, from the Latin word ‘folium’, meaning ‘leaf’:



- 6. Show that the points $(0,0)$ and $(1,1)$ are on this curve.
- 7. Does this curve have a tangent line at the point $(0,0)$? Explain.
- 8. Find $\frac{dy}{dx}$ in terms of x and y using implicit differentiation.
- 9. What happens to $\frac{dy}{dx}$ at the point $(0,0)$? How does that fit in with your answer to question 7?
- 10. To find where a curve has a horizontal tangent line, we set $\frac{dy}{dx}$ to 0 and solve. When we use implicit differentiation, we often need another step. This question explores the issue.

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- (a) Set your $\frac{dy}{dx}$ found in the previous question to 0 and solve for y .
- (b) Keeping in mind that any point on the curve has to satisfy the relationship $x^3 + y^3 = 2xy$, find the point(s) where the curve has a horizontal tangent line. Find these point(s) on the curve above.
11. Use a similar technique to the one employed in the previous question to find all points on the curve where it has a vertical tangent line.

Part III: A Disconnected Curve

We've seen that curves defined implicitly can have some strange properties:

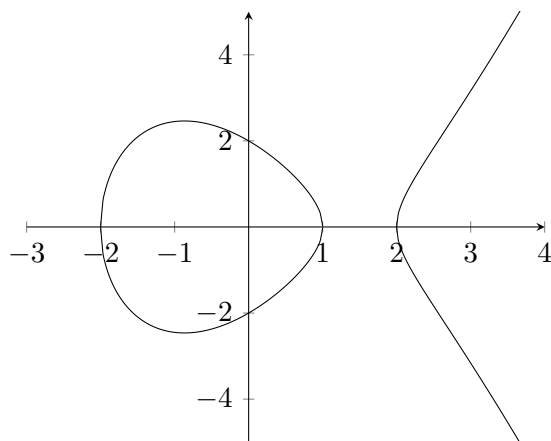
- They sometimes do not define y as a function of x . That is, they fail to pass the _____ test.
- They can cross themselves. At such points, they fail to have a well-defined _____ line.

In this part, we will investigate yet another such property that distinguishes implicitly defined curves from explicitly defined curves: they can be disconnected without being discontinuous and with no vertical asymptotes.

Consider the equation

$$y^2 = (x + 2)(x - 1)(x - 2).$$

This is an example of an *elliptic curve*. The study of elliptic curves has important applications in cryptography (code making and breaking) as well as in the prime factorization of large whole numbers (these two applications are in fact closely related). The graph of this curve is shown below:



12. By looking at the graph above, find all points on the curve where it has a vertical tangent line.

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13. Use implicit differentiation to find $\frac{dy}{dx}$ in terms of x and y .

14. Check your answer to question 12 above using the expression you just found for $\frac{dy}{dx}$ and the technique you learned in the previous part (Question 10).

15. Find all points on the curve where it has a horizontal tangent line. Verify your answer using the graph above.

16. Explain why the curve is not defined for $x < -2$ or for $1 < x < 2$. (Hint: look at the graph of $(x + 2)(x - 1)(x - 2)$.)

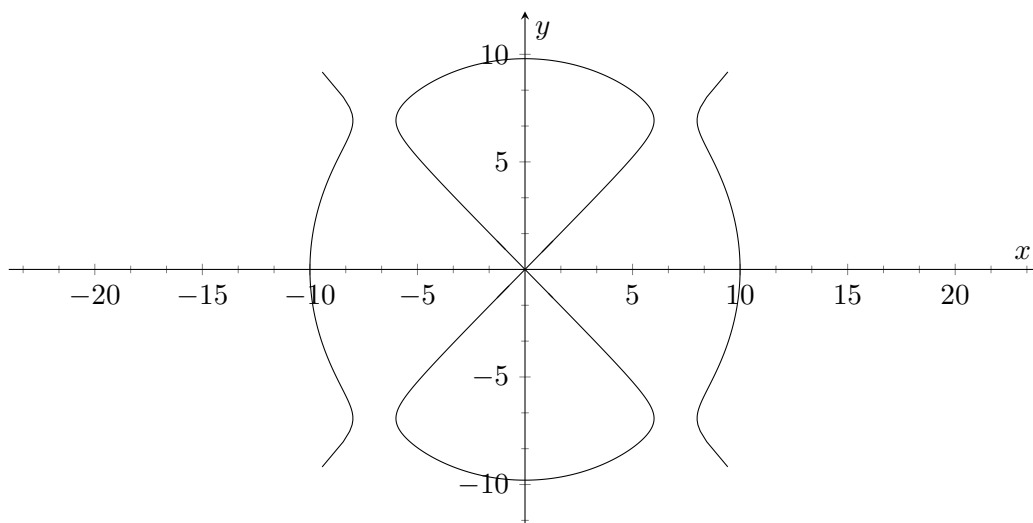
17. Find the equations of the tangent lines to the curve when $x = 0$.

Report – Pick a Curve

For this report, pick one of the following two curves and answer the questions on the back side of this page.

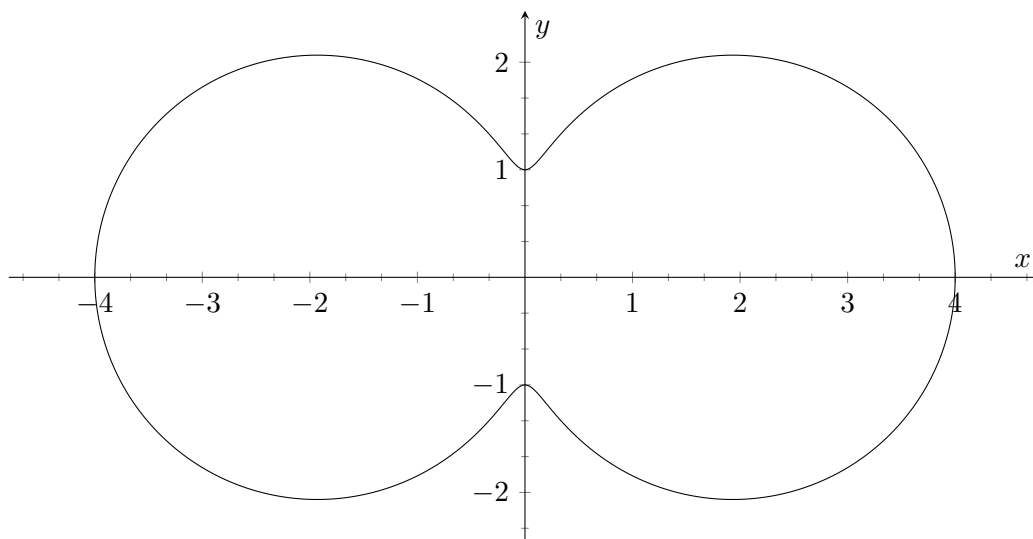
The Devil's Curve

$$y^2(y^2 - 96) = x^2(x^2 - 100)$$



The Hippopede

$$(x^2 + y^2)^2 = 16x^2 + y^2$$



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Questions

1. Research the curve online and write a short paragraph about its history, the etymology of its name, and so on.
2. Why was it necessary to define your curve implicitly? In other words, why can't it be written in the form $y = f(x)$?
3. On the plot of your chosen graph, show all vertical and horizontal tangent lines.
4. By using implicit differentiation, find $\frac{dy}{dx}$ for your chosen curve.
5. Using your derivative from the previous question, compute exactly the locations of all the vertical and horizontal tangent lines.
6. Each curve has exactly one point where $\frac{dy}{dx}$ is an indeterminate value. Compute that point. What happens on your chosen curve at that point? (Be sure to check that your point is indeed on the curve!)