Integrating to Infinity

Up to now, we’ve only dealt with integrals over a finite domain \([a,b]\). In fact, if you look back to when we proved FTC I, you’ll see that it only deals with finite domains...

**Definition** Integrating over infinite domains (Part 1):

If \(f(x)\) is continuous on \((-\infty, \infty)\) then

\[
\int_{a}^{\infty} f(x) \, dx = \lim_{b \to \infty} \int_{a}^{b} f(x) \, dx \quad \text{and} \quad \int_{-\infty}^{b} f(x) \, dx = \lim_{a \to -\infty} \int_{a}^{b} f(x) \, dx.
\]

These integrals are said to converge if the limits exist and are finite.

**Examples**

Compute the following integrals:

1. \(\int_{0}^{\infty} e^{-2x} \, dx\)

2. \(\int_{1}^{\infty} \frac{1}{x^2} \, dx\)

3. \(\int_{-\infty}^{1} \frac{1}{x} \, dx\)
**Definition**  Integrating over infinite domains (Part 2):

If \( f(x) \) is continuous on \((-\infty, \infty)\) then

\[
\int_{-\infty}^{\infty} f(x) \, dx = \int_{-\infty}^{a} f(x) \, dx + \int_{a}^{\infty} f(x) \, dx,
\]

where \( a \) is any number. This integral only converges (i.e. only exists) when both integrals in the sum exist and are finite.

**Example**

4. Compute the integral \( \int_{-\infty}^{\infty} \frac{1}{1 + x^2} \, dx \).

**Exercises**

Calculate the following integrals, if they converge.

5. \( \int_{0}^{\infty} \frac{e^x}{(e^x + 1)^2} \, dx \) (This is a substitution. Don’t forget to change bounds!)
6. $\int_{0}^{\infty} xe^{-x} \, dx$ (You may need L’Hopital’s rule somewhere along the way...)

7. $\int_{0}^{\infty} \frac{1}{(x + 4)^2} \, dx$

8. $\int_{\frac{\pi}{4}}^{\frac{\pi}{2}} \csc^2(x) \, dx$ (Hint: $\csc^2(x) = \frac{\sec^2(x)}{\tan^2(x)}$)
9. The mass of pollutants over city up to height \( u \) meters is given by 
\[
\int_0^u 25,600 \pi^{-0.0025h} \, dh
\]
kilograms. Compute the total mass of pollutants over the city. (Compare Varying Density Lab, Question 15.)

10. Compare the following integrals to other integrals to see if they converge.

(a) \( \int_2^\infty \frac{1}{\sqrt{x^2 - 1}} \, dx \)

(Hint: \( \sqrt{x^2 - 1} \) is a little smaller than \( \sqrt{x^2} = x \). What does that tell you about \( \frac{1}{\sqrt{x^2 - 1}} \) compared to \( \frac{1}{x} \)?)

(b) \( \int_0^\infty \frac{1}{e^x + 2^x} \, dx \)

\[
\int_0^\infty \frac{1}{e^x + 2^x} \, dx \leq \int_0^\infty \frac{1}{2e^x} \, dx \quad (\text{Since } e > 2)
\]
\[
= \frac{1}{2} \lim_{b \to \infty} \left( -e^{-x} \bigg|_0^b \right)
\]
\[
= \frac{1}{2}
\]

Since the original integrand is positive, and we know it’s less than \( \frac{1}{2} \), it must also converge (though we don’t know to what!).
(c) \[ \int_{1}^{\infty} \frac{1 + \sin x}{x^2} \, dx \] (Hint: _____ \leq \sin(x) \leq _____)

11. Find c such that \[ \int_{-\infty}^{\infty} f(t) \, dt = 1 \]

\[ f(t) = \begin{cases} 
cte^{-\frac{t}{2}} & t > 0 \\
0 & \text{otherwise}
\end{cases} \]