Integrating to Infinity

Up to now, we’ve only dealt with integrals over a finite domain $[a, b]$. In fact, if you look back to when we proved FTC I, you’ll see that it only deals with finite domains...

**Definition**  Integrating over infinite domains (Part 1):

If $f(x)$ is continuous on $(-\infty, \infty)$ then

$$\int_a^\infty f(x) \, dx = \lim_{b \to \infty} \int_a^b f(x) \, dx \quad \text{and} \quad \int_{-\infty}^b f(x) \, dx = \lim_{a \to -\infty} \int_a^b f(x) \, dx.$$  

These integrals are said to converge if the limits exist and are finite.

**Examples**

Compute the following integrals:

1. $\int_0^\infty e^{-2x} \, dx$

2. $\int_1^\infty \frac{1}{x^2} \, dx$

3. $\int_{-\infty}^1 \frac{1}{x} \, dx$
**Definition** Integrating over infinite domains (Part 2):

If \( f(x) \) is continuous on \((-\infty, \infty)\) then

\[
\int_{-\infty}^{\infty} f(x) \, dx = \int_{-\infty}^{a} f(x) \, dx + \int_{a}^{\infty} f(x) \, dx,
\]

where \( a \) is any number. This integral only converges (i.e. only exists) when both integrals in the sum exist and are finite.

**Example**

4. Compute the integral \( \int_{-\infty}^{\infty} \frac{1}{1 + x^2} \, dx \).

**Exercises**

Calculate the following integrals, if they converge.

5. \( \int_{0}^{\infty} \frac{e^x}{(e^x + 1)^2} \, dx \) (This is a substitution. Don’t forget to change bounds!)
6. $\int_{0}^{\infty} x e^{-x} \, dx$ (You may need L’Hopital’s rule somewhere along the way...)

7. $\int_{0}^{\infty} \frac{1}{(x + 4)^2} \, dx$

8. $\int_{\frac{\pi}{4}}^{\frac{\pi}{2}} \csc^2(x) \, dx$ (Hint: $\csc^2(x) = \frac{\sec^2(x)}{\tan^2(x)}$)
9. The mass of pollutants over city up to height \( u \) meters is given by 
\[ \int_{0}^{u} 25,600 \pi e^{-0.0025h} \, dh \]
kilograms. Compute the total mass of pollutants over the city. (Compare Varying Density Lab, Question 15.)

10. Compare the following integrals to other integrals to see if they converge.

(a) \[ \int_{2}^{\infty} \frac{1}{\sqrt{x^2 - 1}} \, dx \]

(Hint: \( \sqrt{x^2 - 1} \) is a little smaller than \( \sqrt{x^2} = x \). What does that tell you about \( \frac{1}{\sqrt{x^2 - 1}} \) compared to \( \frac{1}{x} \)?)

(b) \[ \int_{0}^{\infty} \frac{1}{e^x + 2^x} \, dx \]
(c) \( \int_1^\infty \frac{1 + \sin x}{x^2} \, dx \) (Hint: ______ \( \leq \) \( \sin(x) \leq _____ \))

11. Find \( c \) such that \( \int_{-\infty}^{\infty} f(t) \, dt = 1 \)

\( f(t) = \begin{cases} 
cte^{-\frac{t}{2}} & t > 0 \\
0 & \text{otherwise}
\end{cases} \)