Introduction

1. Let \( f(x) = 2x \). Consider the following function:

\[
F(x) = \int_0^x 2t \, dt.
\]

(a) Draw a graph of \( f(x) \). On it, illustrate \( F(1), F(2) \) and \( F(3) \).

(b) What does \( F(x) \) measure?

(c) Without further computation, how do you know \( F(x) \) is increasing? Concave up?

(d) Calculate \( F(1), F(2), F(3) \) and \( F(-1) \).

(e) What function do you think \( F(x) \) is?

(f) Use the FTC to prove your hypothesis from the previous question, then fill in the blanks below:

\[
\frac{d}{dx} \int_0^x 2t \, dt = \quad \text{so} \quad \int_0^x 2t \, dt \text{ is an} \quad \text{of} \quad f(x) = 2x.
\]

(g) Is \( G(x) = \int_2^x 2t \, dx \) also an antiderivative of \( f(x) = 2x \)? If so, what constant do \( F(x) \) and \( G(x) \) differ by?
Stuff from the Past that will be Important Today!

2. ★ Extreme Value Theorem (105L Worksheet 11-3): If $f(t)$ is continuous on the closed interval $[a, b]$, then it has a __________ __________ and a __________ __________ on that interval.

★★ Bounding Integrals (106L Worksheet 6-3): If $m \leq f(t) \leq M$ for $a \leq t \leq b$, then

\[ \leq \int_a^b f(t)dt \leq \]

FTC II

**The Second Fundamental Theorem of Calculus**

Let $f$ be continuous on an interval. Then for $x$ and $a$ in that interval

\[ \frac{d}{dx} \int_a^x f(t)dt = f(x) \]

3. **Proof:** Suppose $f(t)$ is a continuous function and let $g(x) = \int_a^x f(t)dt$.

(a) Then

\[ \frac{g(x + h) - g(x)}{h} = \frac{h}{h} = \]

(b) By (★), since $f(t)$ is a continuous on $[x, x + h]$, then it attains a smallest value $m$ and a largest value $M$. Then, by (★★),

\[ \leq \int_x^{x+h} f(t)dt \leq \]

(c) Dividing everything by $h$, we get

\[ m \leq \frac{\int_x^{x+h} f(t)dt}{h} \leq M \]

(d) As $h \to 0$, what happens to the interval $[x, x + h]$?

(e) As $h \to 0$, what happens to $m$ and $M$?

(f) Therefore,

\[ \frac{d}{dx} g(x) = \lim_{h \to 0} \frac{\int_x^{x+h} f(t)dt}{h} = \]
4. Recall that we were never able to antidifferentiate $e^{-x^2}$. Can you now write down an antiderivative for it?

5. Find the following derivative in two different ways: $\frac{d}{dx} \int_2^x \cos t \, dt$

   (a) Using FTC I:

   (b) Using FTC II:

6. Let $g(x) = \int_1^x \sqrt{1 + t^2} \, dt$.

   (a) What is $g'(x)$?

   (b) What is $g(x^3)$?

   (c) What is $\frac{d}{dx} g(x^3)$? (Hint: you need the chain rule here. This is a composite function.)
7. (a) Find a function \( g(x) \) such that \( g'(x) = \sqrt{1 + x^2} \) and \( g(2) = 0 \).

(b) Find a function \( g(x) \) such that \( g'(x) = \sqrt{1 + x^2} \) and \( g(2) = 10 \).

8. Let \( g(x) = \int_0^x f(t) \, dt \), with \( f(x) \) is continuous. Cross out the wrong answer for each of the following:

- If \( f(x) > 0 \) and \( x > 0 \), then \( g(x) \) is positive/negative and increasing/decreasing.
- If \( f(x) > 0 \) and \( x < 0 \), then \( g(x) \) is positive/negative and increasing/decreasing.
- If \( f(x) < 0 \) and \( x > 0 \), then \( g(x) \) is positive/negative and increasing/decreasing.
- If \( f(x) < 0 \) and \( x < 0 \), then \( g(x) \) is positive/negative and increasing/decreasing.

9. What constant do the following two antiderivatives of \( f(x) \) differ by?

\[
F(x) = \int_{-1}^{x} f(t) \, dt \quad G(x) = \int_{1}^{x} f(t) \, dt
\]

(Hint: draw pictures, and see Worksheet 6-3, property 4!)