Introduction

Let $f(x) = 2x$. Consider the following function:

$$F(x) = \int_0^x 2t \, dt.$$

1. Draw a graph of $f(x)$. On it, illustrate $F(1)$, $F(2)$ and $F(3)$.

2. What does $F(x)$ measure?

3. Without further computation, how do you know $F(x)$ is increasing? Concave up?

4. Calculate $F(1)$, $F(2)$, $F(3)$ and $F(-1)$ and draw a graph of $F(x)$.
5. What function do you think \( F(x) \) is?

6. Use the FTC to prove your hypothesis from the previous question, then fill in the blanks below:

\[
\frac{d}{dx} \int_0^x 2t \, dt = \underline{\phantom{0}}, \text{ so } \int_0^x 2t \, dt \text{ is an } \underline{\phantom{0}} \text{ of } f(x) = 2x.
\]

7. Is \( G(x) = \int_0^x 2t \, dt \) also an antiderivative of \( f(x) = 2x \)? If so, what constant do \( F(x) \) and \( G(x) \) differ by?

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**Stuff from the Past that will be Important Today!**

* Extreme Value Theorem: If \( f(t) \) is continuous on the closed interval \([a, b]\), then it has a \underline{\phantom{0}} \underline{\phantom{0}} and a \underline{\phantom{0}} \underline{\phantom{0}} on that interval.

** Bounding Integrals: If \( m \leq f(t) \leq M \) for \( a \leq t \leq b \), then

\[
\leq \int_a^b f(t) \, dt \leq
\]
FTC II

The Second Fundamental Theorem of Calculus
Let \( f \) be continuous on an interval. Then for \( x \) and \( a \) in that interval
\[
\frac{d}{dx} \int_a^x f(t) \, dt = f(x)
\]

Proof: Suppose \( f(t) \) is a continuous function and let \( g(x) = \int_a^x f(t) \, dt \).

Then
\[
\frac{g(x+h) - g(x)}{h} = \frac{h}{h} = \frac{\int_a^x f(t) \, dt}{h}
\]

By (⋆), since \( f(t) \) is a continuous on \([x, x+h]\), then it attains a smallest value \( m \) and a largest value \( M \). Then, by (⋆⋆),
\[
\leq \int_a^{x+h} f(t) \, dt \leq
\]

Dividing everything by \( h \), we get
\[
\leq \frac{\int_a^{x+h} f(t) \, dt}{h} \leq
\]

As \( h \to 0 \), what happens to the interval \([x, x+h]\)?

As \( h \to 0 \), what happens to \( m \) and \( M \)?

Therefore,
\[
\frac{d}{dx}[g(x)] = \lim_{h \to 0} \frac{\int_a^{x+h} f(t) \, dt}{h} = \frac{\int_a^{x+h} f(t) \, dt}{h}
\]
Questions

1. Why should you care (other than the test)? (Hint: can you write down an antiderivative for $e^{-x^2}$?)

2. Find the following derivative in two different ways: $\frac{d}{dx} \int_2^x \cos t \, dt$
   (a) Using FTC I:
   (b) Using FTC II:

3. Let $g(x) = \int_1^x \sqrt{1 + t^2} \, dt$.
   (a) What is $g'(x)$?
   (b) What is $g(x^3)$?
   (c) What is $\frac{d}{dx} g(x^3)$? (Hint: you need the chain rule here. This is a composite function.)
4.  (a) Find a function $g(x)$ such that $g'(x) = \sqrt{1 + x^2}$ and $g(2) = 0$.

(b) Find a function $g(x)$ such that $g'(x) = \sqrt{1 + x^2}$ and $g(2) = 10$.

5. Let $g(x) = \int_{0}^{x} f(t) \, dt$, with $f(x)$ is continuous. Circle out the right answer for each of the following:

- If $f(x) > 0$ and $x > 0$, then $g(x)$ is positive/negative and increasing/decreasing.
- If $f(x) > 0$ and $x < 0$, then $g(x)$ is positive/negative and increasing/decreasing.
- If $f(x) < 0$ and $x > 0$, then $g(x)$ is positive/negative and increasing/decreasing.
- If $f(x) < 0$ and $x < 0$, then $g(x)$ is positive/negative and increasing/decreasing.

6. What constant do the following two antiderivatives of $f(x)$ differ by?

$$F(x) = \int_{-1}^{x} f(t) \, dt \quad G(x) = \int_{1}^{x} f(t) \, dt$$

(Hint: draw pictures!)