Stuff We Already Know

- Given a right-angled triangle with angle $x$, write down the definitions of the following:

\[
\sin x = \quad, \quad \cos x = \\
\tan x = \quad, \quad \sec x = \\
\csc x = \quad, \quad \cot x = \\
\]

- Write down the following limits:

\[
\lim_{x \to 0} \frac{\sin x}{x} = \quad, \quad \lim_{x \to 0} \frac{\cos x - 1}{x} = \\
\]

- Write down the definition of the derivative for a function $f(x)$:

\[
\]

Derivatives of sin and cos

Questions

1. (a) Use Geogebra to graph the function, $g(x) = \frac{\sin(x+0.001)-\sin(x)}{0.001}$ on the axes below with domain $[-2\pi, 2\pi]$. What does this approximate?

(b) What other function that we know does this graph look like? Graph it on top of your graph of $g(x)$ using Geogebra to verify this.

2. (a) Write down a formula (using a limit) that would give the derivative of $\sin x$:

\[
\frac{d}{dx} (\sin x) = \lim \\
\]

(b) What other function that we know does this graph look like? Graph it on top of your graph of $g(x)$ using Geogebra to verify this.
(b) Use the fact that \( \sin(A + B) = \sin A \cos B + \cos A \sin B \) to rewrite the derivative of \( \sin(x) \) as the sum of two different limits.

(c) Use your two limits to find what \( \frac{d}{dx} \sin x \) should be.

3. Use the same sorts of tricks to find \( \frac{d}{dx} \cos x \):
   (Note: \( \cos(A + B) = \cos A \cos B - \sin A \sin B \).)

Other Trig Derivatives

Questions

4. Show that \( \frac{d}{dx} \tan x = \sec^2 x \). (Hint: Quotient Rule)

5. Show that \( \frac{d}{dx} \sec x = \sec x \tan x \), \( \frac{d}{dx} \csc x = -\csc x \cot x \), and \( \frac{d}{dx} \cot x = -\csc^2 x \). (Hint: for example, write \( \sec x = \frac{1}{\cos x} = (\cos x)^{-1} \) and use the chain rule...)
6. Find the equation of the tangent line to the graph of $f(x) = \tan x$ at $x = \frac{\pi}{4}$.

7. A particle is moving along a straight line. Its position from its origin (in meters) at time $t$ seconds is given by $s(t) = \sin^2 t$. Find its velocity and acceleration at time $t = 2$ seconds. At what times is the particle at rest? (Hint: The identity $\sin 2x = 2 \sin x \cos x$ may be useful.)

8. (a) Find the linear approximation to the curve $g(x) = \sec x$ at $x = \frac{5\pi}{6}$.

(b) Use your answer above to estimate $\sec \frac{11\pi}{12}$.

The derivatives we found above are important. While all trig functions have multiple forms (e.g. we write $\sec^2(x)$ instead of $\frac{1}{1-\sin^2(x)}$), these derivatives have standard forms. For future reference, write these below:

$$\frac{d}{dx} \sin x = \cos x$$
$$\frac{d}{dx} \csc x = -\csc x \cot x$$
$$\frac{d}{dx} \tan x = \sec^2 x$$
$$\frac{d}{dx} \cos x = -\sin x$$
$$\frac{d}{dx} \sec x = \sec x \tan x$$
$$\frac{d}{dx} \cot x = -\csc^2 x$$
Homework Exercises

1. (a) Use Geogebra to graph the function \( y = \sin(2x) - 2\sin(x) \) over the horizontal range \([0, 2\pi]\). Insert special points on your graph to find decimal values of the points where the tangent line to the curve is horizontal.

(b) Use the derivative to find the exact \( x \)-coordinates of all points on the interval \([0, 2\pi]\) where the tangent line to the graph of the function \( y = \sin(2x) - 2\sin(x) \) is horizontal. What is the period of this function?

   (You will find the identity \( \cos(2x) = 2\cos^2(x) - 1 \) useful.)

2. Consider a particular point on a vibrating string as it moves vertically up and down. The position of this point (in mm) at time \( t \) (in seconds) is given by

   \[ s(t) = 10 + \frac{1}{4}\sin(10\pi t). \]

   (a) What is the period of oscillation of this point on the vibrating string?

   (b) Find a formula for the velocity of the point on the string after \( t \) seconds.

   (c) Describe the position and the motion (up or down) of this point on the string at \( t = 0 \) and \( t = 0.3 \) seconds. (Your answers should have units.)

3. If a projectile is fired from ground level with initial velocity \( v_0 \) and inclination angle \( \alpha \) and if air resistance can be ignored, the horizontal distance (in feet) it travels is

   \[ R = \frac{1}{16}v_0^2 \sin(\alpha) \cos(\alpha). \]

   (a) Assuming that a soccer player kicks the ball at a 40° angle of inclination, find the initial velocity needed for a kick of 120 feet. (No calculus needed.)

   (b) What value of \( \alpha \) maximizes \( R \)?

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\(^1\)If you don’t remember how to use Geogebra, look back to the Derivatives and Roots lab from Math 105L!