Introduction

For many years now, you have been solving algebraic equations, such as

\[ 2x + 1 = 0, \, x^2 + 3x = 1, \, e^x = 7 \]

and so on. Today, we will begin to examine differential equations. First, let’s think about what we already know:

1. (a) What does it mean to solve an algebraic equation? For example, what does it mean to solve \( x^2 + 3x + 2 = 0 \)?

(b) How do you check a proposed solution to an algebraic equation? For example, how do you check if \( x = -2 \) is a solution to \( x^2 + 4 = -x^3 \)?

(c) Is it easier to solve an algebraic equation, or to check a proposed solution? Explain.

2. (a) Is the point \( x = 1, \, y = 3 \) (i.e. the point \( (1,3) \)) a solution to the equation \( 2y + x = y + 4 \)? Why?

(b) Is the point \( (2, 3) \) a solution to the equation \( y^2 + x = y + 8 \)? Why?

(c) Is the point \( (1, 1) \) a solution to the equation \( x^2 = 1 - y^2 \)? Why?

(d) What does it mean to say that a point \( (a, b) \) is a solution to an equation?

Differential Equations

Definition  Equations of the form \( \frac{dy}{dx} = y^2, \, \frac{dy}{dx} = 2x \) or \( \frac{dy}{dx} = y + x \) are called differential equations. In fact, any equation of the form \( \frac{dy}{dt} = g(y, t) \) is a differential equation. \( y = f(t) \) is a solution if when \( f(t) \) is substituted for \( y \) in the expression \( g(y, t) \), the result is \( \frac{dy}{dt} \). In other words, like any other equation, when you substitute your answer into both sides of the equation, you get a true statement.
Differential Equations and Antidifferentiation

3. (a) Check if the function \( y = x^2 \) a solution to the equation \( \frac{dy}{dx} = 2x \). Why?

(b) Consider the differential equation \( \frac{dy}{dx} = \cos x \). Use antidifferentiation to find a solution of this equation. Can you find more than one solution? How many can you find? Graph a few of them on the set of axes below. How are they related to each other?

In general, differential equations do not have unique solutions. In fact, a differential equation often has an infinite number of solutions, as you saw above. The reason is that we antidifferentiate, we introduce an arbitrary constant: if \( F(x) \) is an antiderivative of \( f(x) \), then so is \( F(x) + C \) for any value of \( C \). When you are asked to solve a differential equation, you are required to find the general solution. That is, the solution with a constant in it.

**Definition** An initial value problem is an a differential equation with a specified value of the solution provided. Such a value is called an initial condition. Initial value problems most commonly have a unique solution.

4. (a) Solve the differential equation \( \frac{dy}{dx} = 2e^x \).

(b) Solve the initial value problem \( \frac{dy}{dx} = 2e^x, \quad y(0) = 3 \).
Above, we saw differential equations of the form \( \frac{dy}{dx} = f(x) \). We found that solving such an equation is just antidifferentiating \( f(x) \). That is, the general solution is

\[
y(x) = \int f(x) \, dx.
\]

Solving differential equations is not always as straightforward as that, though!

**More Complex Differential Equations**

5. Consider the differential equation \( \frac{dy}{dx} = x + y \).

   (a) Why is the solution to this not \( y(x) = \int x + y \, dx \)? In fact, why does that integral not make any sense?

   (b) i. Check if \( y(x) = \frac{x^2}{2} \) is a solution to this equation.

      ii. Check if \( y(x) = e^x - x - 1 \) is a solution.

      iii. Check if \( y(x) = e^x - x + 1 \) is a solution.

      iv. Check if \( y(x) = 2e^x - x - 1 \) is a solution.

   v. Is \( y(x) = e^x - x + C \) a solution for any value of \( C \)? What about \( y(x) = Ce^x - x - 1 \)?

   vi. Solve the initial value problem \( \frac{dy}{dx} = x + y \), \( y(0) = 4 \).
6. Consider the differential equation $y'(t) = y(t)$. Complete the blank: The solution to this differential equation is a function $y(t)$ whose derivative is equal to __________.

(a) What function $y(t)$ satisfies the sentence you just wrote down? Plug in to both sides of the equation to check it.

(b) Is the function $y(t) = 2 + e^t$ also a solution?

(c) Can you figure out another function that solves the equation? Check it!

(d) Can you write down the general solution to this equation?

7. Consider the differential equation $y'(t) = 2y(t)$. Complete the blank: The solution to this differential equation is a function $y(t)$ whose derivative is equal to __________.

(a) Can you find one solution to this equation?

(b) Can you find another solution by adding a constant to your solution above?

(c) Write down the general solution to this equation.

(d) Solve the initial value problem $y'(t) = 2y, \ y(0) = 3$. 
8. By considering the previous two questions, find the general solution of the differential equation \( \frac{dy}{dx} = ky \), where \( k \) is a fixed constant. Then solve the following initial value problems:

(a) \( \frac{dy}{dt} = ky, \ y(0) = 2 \).

(b) \( \frac{dy}{dt} = ky, \ y(ln 2) = 2 \).

(c) \( \frac{dy}{dt} = ky, \ y(1) = 2 \).

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**Higher Order Differential Equations**

**Definition** The **order** of a differential equation is the highest derivative that appears in it. For example \( \frac{dy}{dx} = x + y \) is a first order equation; \( \frac{d^2y}{dx^2} + \frac{dy}{dx} = y \) is a second order equation.

9. Consider the differential equation \( \frac{d^2y}{dx^2} = -9y \).

(a) What is the order of this equation?

(b) Check that both \( \sin(3x) \) and \( \cos(3x) \) are solutions to this equation.

(c) Check that both \( 2\sin(3x) \) and \( 3\cos(3x) \) are solutions to this equation.
(d) Check that $2 \sin(3x) + 3 \cos(3x)$ is a solution to this equation.

(e) Can you write down the general solution of this equation?

(f) Solve the initial value problems:

i. $\frac{d^2 y}{dx^2} = -9y$, $y(0) = 1$, $y\left(\frac{\pi}{6}\right) = 0$.

ii. $\frac{d^2 y}{dx^2} = -9y$, $y(0) = 1$, $y'(0) = 2$.

(g) Graph the two solutions from the previous part below.

In general, an $n^{th}$ order differential equation has a general solution with $n$ arbitrary constants in it.

As you can see, the solutions to a differential equation may not related to each other as simply as the ones in Question 3b above!
What we can and can’t do (for now)

- The only differential equations we can currently solve analytically—that is, without (much) guessing—are those of the form

$$\frac{dy}{dt} = f(t),$$

such as \( \frac{dy}{dt} = t^2 \). We solve these by finding an \( \text{___________} \) of \( f(t) \).

- We cannot (for now) solve equations like

$$\frac{dy}{dt} = y^2.$$

We will do (some of) these later in the semester. For now, given a solution, we can check it.

- We cannot solve second (or higher) order differential equations other than \( \frac{d^2y}{dx^2} = f(t) \). We will leave other second order equations for a later course.

- We can also check whether a given function solves a particular differential equation.

Homework Questions

1. Is the function \( y = -\frac{1}{x} \) a solution to the equation \( \frac{dy}{dx} = y^2 \)? Check that \( y = \frac{1}{x} \) is \text{not} a solution. What about \( y = -\frac{1}{x+c} \)? Find a solution with \( y(1) = -\frac{1}{2} \).

2. Above, you found that the general solution to \( \frac{dy}{dx} = ky \) is \( y(x) = Ce^{kx} \). Graph solutions to the equation \( \frac{dy}{dx} = 3y \) with \( y(0) = 1 \), \( y(0) = 2 \), and \( y(0) = -1 \) on the same set of axes.

3. What is the order of the differential equation \( \frac{d^3y}{dx^3} = 24x \)? Find its the general solution by antidifferentiating three times. Check that your solution has three arbitrary constants in it! Solve the initial value problems:

   (a) \( \frac{d^3y}{dx^3} = 24x, \ y(0) = 1, \ y'(0) = 2, \ y''(0) = 3. \)

   (b) \( \frac{d^3y}{dx^3} = 24x, \ y(0) = 1, \ y(1) = 2, \ y(2) = 3. \)

4. Solve the following initial value problems by antidifferentiation:

   (a) \( \frac{dy}{dx} = x^2, \ y(1) = 1. \)

   (b) \( \frac{d^2h}{dt^2} = -32, \ h(0) = 100, \ h'(0) = 10. \) (See the Newton’s Law of Motion lab!).

5. Find a function of the form \( y = x^n \) that is a solution to the differential equation \( \frac{1}{2} x \frac{dy}{dx} = y. \)