A Key Tool!

The chain rule:
\[ \frac{d}{dx}(f(g(x))) = \\text{__________}. \]

Implicit Differentiation

Questions

1. Consider the equation \(xy + y = e^x\). This defines \(y\) as function of \(x\). Find \(y(x)\) and use it to find \(\frac{dy}{dx}\).

2. Now, consider the equation \(x^2 + y^2 = 4\).

   (a) What curve does this equation describe? Draw it!

   (b) The equation \(x^2 + y^2 = 4\) does not define \(y\) as a function of \(x\). Why not?

   (c) However, “near the point \((0, 2)\)”, \(y\) is function of a \(x\). What is this function? What is the derivative of this function at \(x = 1\)?

   (d) Do the same for the point \((0, -2)\) (and \(x = -1\)).

   (e) Notice that in both the above cases, we can write \(\frac{dy}{dx} = \text{______}\).
(f) Even though $x^2 + y^2 = 4$ does not define $y$ as a function of $x$, it is still close to a function that the derivative makes sense, and the usual rules apply. Let’s pretend $y$ is a function of $x$, differentiate both sides, and solve for $\frac{dy}{dx}$:

\[ \frac{d}{dx}(x^2 + y^2) = \frac{d}{dx}(4) \]

(g) Notice that $\frac{dy}{dx}$ depends on both $x$ and $y$. For “normal” functions, the derivative should only depend on the independent variable. Why isn’t this true for our equation?

(h) Are there any points where $\frac{dy}{dx}$ is not defined? What happens to the tangent line to the curve at these points?

3. Let’s revisit $xy + y = e^x$. What is $\frac{dy}{dx}$? Check that it’s the same as what you got previously!
4. Find the tangent line to $x^2 + xy - y^3 = xy^2$ at the point $(1, 1)$.

Here’s a picture of this curve. Odd, eh? What happens to $\frac{dy}{dx}$ at $(0, 0)$ and $(1, -1)$? More on this in lab...

5. (a) Find $\frac{dy}{dx}$ if $e^x + e^{y^2} = 3$.

That curve looks like this:

(b) As $x$ approaches $-\infty$, what does $\frac{dy}{dx}$ approach? Make sure you see how this matches up with the graph!

(c) At the point $(\ln(2), 0)$, what is $\frac{dy}{dx}$? What direction is the tangent line at that point?
6. Where does the curve $y^3 - xy = 1$ have
   
   - a vertical tangent? (Hint: see question 5c.)

   - a horizontal tangent? (Hint: if the tangent line is horizontal, what must $\frac{dy}{dx}$ be?)

Here’s a picture of this curve. Check that your two answers above make sense in the context of the picture!