Back to Calculus...

- Given a function $f(x)$, its *derivative function* is given by

$$f'(x) = \underline{\hspace{6cm}}.$$

- To approximate the derivative of a function at a point $x = a$, we plug in small values of _____ into the *difference quotient*

$$f'(a) \approx \underline{\hspace{4cm}}.$$

- The quantity A is directly proportional to the quantity B if $A =$ _____.

Preliminaries: Playing with Derivatives

Let $f(x) = 2^x$ and $g(x) = 3^x$. Using a spreadsheet to approximate the derivatives using $h = 0.01$, fill out the following table.

x	$f(x)$	$f'(x)$	$\frac{f'(x)}{f(x)}$	$g(x)$	$g'(x)$	$\frac{g'(x)}{g(x)}$
-4						
-3						
-2						
-1						
0						
1						
2						
3						
4						

Sample calculation for $x = -4$:

- It seems that $\frac{f'(x)}{f(x)}$ is always _____, giving

$$\frac{d}{dx}2^x = \underline{\hspace{2cm}}$$

.

- It seems that $\frac{g'(x)}{g(x)}$ is always _____, giving

$$\frac{d}{dx}3^x = \underline{\hspace{2cm}}$$

.

In both cases, it seems that the derivative of each of these exponential functions is _____ to its derivative.

Examining the Evidence

Two questions arise:

- Why is the derivative of an exponential directly proportional to the exponential itself?
- What is the the constant of proportionality?

Let's compute $\frac{d}{dx}2^x$ using the definition of the derivative:

By plugging in a small value of h into that last limit, estimate it.

Do the same for $\frac{d}{dx}a^x$ using the definition of the derivative:

It would be great to find a formula for that limit! Why is it approximately 0.693 when $a = 2$ and approximately 1.0987 when $a = 3$? What would it be for other values of a?

A Pretty (and Important!) Derivative

We now know that

$$\frac{d}{dx}a^x = ka^x \text{ where } k = \lim_{h\to 0}\frac{a^h - 1}{h}$$

Suppose we could find a number a such that the constant of proportionality is 1. For that value of a, we'd get $\frac{d}{dx}a^x =$ _____!

- From our work above it seems that such an a should be between 2 and 3. Starting by taking $a = 2.5$ and estimate the above limit using $h = 0.001$:

- Keep trying values of a until you get the limit to be 1, at least to three decimal places.

- The number you got should be familiar. Fill in the following: it seems that when $a =$ _____, $\lim_{h\to 0}\frac{a^h - 1}{h}$. That is (fill in the blank):

$$\lim_{h\to 0}\frac{___^h - 1}{h} = 1.$$

- Let's see why this true: Such an a satisfies $\frac{a^h-1}{h} \approx 1$ for small h. We can rewrite this as

$$a \approx _____.$$

- We now want to take $h \to 0$ to make that $\approx$ an $=$. Let $n = \frac{1}{h}$. As $h \to 0$, $n \to \infty$, so

$$a = \lim_{h\to 0}(h+1)^{\frac{1}{h}} = \lim_{n\to\infty}\left(1 + \frac{1}{n}\right)^n = ___.$$

Therefore:

Derivative of e^x

$$\frac{d}{dx}e^x = e^x$$

Derivative of a^x

Question What is $\frac{d}{dx}e^{kx}$? Use the chain rule.

Recall from the exponential functions and logs worksheets that we can write any exponential as $a^x = e^{rx}$, where $r =$ _____. So, using the chain rule result above:

$$\frac{d}{dx}a^x = \underline{\hspace{6cm}} = \underline{\hspace{6cm}} = \underline{\hspace{6cm}}$$

Derivatives of Exponential Functions

$$\frac{d}{dx}a^x = \underline{\hspace{3cm}}$$

Questions

1. Find the derivatives of the following functions:

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(a) $f(x) = 8x + 8^x$

(b) $g(x) = e^2 - 2^x$

(c) $h(x) = e^\pi + e^x + x^e$

(d) $f(x) = 2^{x^2-1}$

2. (a) Find the tangent line to x^2e^x at $x = 1$.

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(b) Find the tangent line to $e^{(x+1)^3}$ at $x = 1$ (Hint: see the triple chain rule from the last worksheet).

3. Looking back at the first page of this worksheet, can you explain where the numbers 0.693 and 1.0987 came from? What would that number be if we carried out the same analysis for $h(x) = 10^x$?

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4. Find a number a such that $\frac{d}{dx}a^x = 2a^x$.

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