Past Differentiation...

The differentiation rules we've learned so far:

- Power Rule: _____
- Sum Rule: _____
- Difference Rule: _____
- Product Rule: _____
- Quotient Rule: _____

Preliminaries: What We Know So Far

Question With the tools we have, how do we find $\frac{d}{dx}(x^3+1)^3$?

Question Does the same idea practically let us find $\frac{d}{dx}(x^3+1)^{100}$?

Question Let $L(E)$ be the life expectancy (years) of a person with E years of education. Let $E(i)$ be the average number of years of education obtained by people whose parents have an average annual income of i dollars.

- Describe the function $L(E(i))$ in practical terms. Include units in your answer.
- What are the units of $\frac{dL}{dE}$, $\frac{dE}{di}$, and $\frac{dL}{di}$?

The Chain Rule

More general question: How do we differentiate the composition of two functions (functions inside of functions)?

The Chain Rule
If $f(x)$ and $g(x)$ are differentiable, then

$$\frac{d}{dx}\left[f(g(x))\right] = f'(g(x)) \cdot g'(x)$$

If we let $z = g(x)$ and $y = f(z)$, then $y = f(g(x))$ and we can write the chain rule in the following way:

$$\frac{dy}{dx} = \frac{dy}{dz} \cdot \frac{dz}{dx}$$

Examples

- Find the derivative of $(x^3 + 1)^3$ using the chain rule. Check that you get the same as the answer above.

- Find $\frac{d}{dx}(x^3 + 1)^{100}$.

Proving the Quotient Rule Using the Chain Rule

The quotient rule tells us how to differentiate $\frac{f(x)}{g(x)}$. We previously proved it using the product rule. But in a sense, it's more natural to do it using the chain rule.

First, we note that $\frac{f(x)}{g(x)} = f(x)[g(x)]^{-1}$. Then:

- The derivative of $[g(x)]^{-1}$ can be computed using the chain rule by taking $f(x) = x^{-1}$. So $f'(x) =$ _____. So:

$$\frac{d}{dx}[g(x)]^{-1} = \underline{\hspace{6cm}}$$

- We can now compute $\frac{d}{dx}f(x)[g(x)]^{-1}$ using the product rule:

The Triple Chain Rule (and more)

The chain rule tells us how to differentiate the composition of two function. How about three (or more)? That is, what is the derivative of $f(g(h(x))$? The idea is to take the inner function to be $g(h(x))$ and the outer to be $f(x)$:

The Triple Chain Rule (and higher)
If $f(x)$, $g(x)$, and $h(x)$ are differentiable, then

$$\frac{d}{dx}\left[f(g(h(x)))\right] = f'(g(h(x)))g'(h(x))h'(x)$$

Similarly,

$$\frac{d}{dx}f(g(h(i(x)))) = f'(g(h(i(x))))g'(h(i(x)))h'(i(x))i'(x)$$

Etc

Questions

1. Differentiate the functions below:

□Regrade

(a) $(8x^2+2)^7$.

(b) $\left(\frac{1}{x^2}+x+1\right)^3$.

(c) $\left(\frac{(2x+3)^2}{(x+1)^3}\right)^{\frac{3}{2}}$.

2. Let $f(x)$ be a differentiable function. Find the following. You'll need the triple chain rule for the second and third.

□Regrade

(a) $\frac{d}{dx}f(x)^7$

(b) $\frac{d}{dx}f((x^5+1)^7)$

(c) $\frac{d}{dx}(f(x)^5+1)^7$

3. Find the equation of the tangent line to $f(x) = \left(\frac{2}{x} - 1\right)^3$ at the point where $x = 3$.

□Regrade

The Chain Rule in Applied Context

4. You are driving north from Durham. The temperature gets colder and colder farther and farther north of Durham. The function $F(x)$ gives the temperature, in degrees Fahrenheit, x miles north of Durham. The function $s(t)$ gives your distance north of Durham, in miles, after you have been driving for t hours.

□Regrade

(a) Explain the meaning of the function $f(t) = F(s(t))$.

(b) What are the units of $f'(t)$?

(c) Explain the meaning of each of the following (give units):

i. $s(3) = 160$

ii. $s'(3) = 80$

iii. $F(160) = 75$.

iv. $F'(160) = -0.05$

(d) Which of the following is the correct formula for $f'(3)$? Circle your answer:

- $f'(3) = F'(3) \cdot s'(3)$
- $f'(3) = F'(s'(3))$
- $f'(3) = F'(s(3)) \cdot s'(3)$

(e) After 3 hours of driving, how fast is the temperature changing outside your car? Give units. Use your answers from above.