Back to Calculus...

- Given a function \( f(x) \), its *derivative function* is given by

  \[ f'(x) = \quad \cdash \quad \]

- To approximate the derivative of a function at a point \( x = a \), we plug in small values of ___ into the *difference quotient*

  \[ f'(a) \approx \quad \cdash \quad \]

- The quantity \( A \) is directly proportional to the quantity \( B \) if \( A = \quad \cdash \quad \).

Playing with Derivatives

Let \( f(x) = 2^x \) and \( g(x) = 3^x \). Using a spreadsheet to approximate the derivatives, fill out the following table.

<table>
<thead>
<tr>
<th>( x )</th>
<th>( f(x) )</th>
<th>( f'(x) )</th>
<th>( \frac{f'(x)}{f(x)} )</th>
<th>( g(x) )</th>
<th>( g'(x) )</th>
<th>( \frac{g'(x)}{g(x)} )</th>
</tr>
</thead>
<tbody>
<tr>
<td>(-4)</td>
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</tr>
<tr>
<td>(-3)</td>
<td>\</td>
<td>\</td>
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<td></td>
</tr>
<tr>
<td>(-2)</td>
<td>\</td>
<td>\</td>
<td>\</td>
<td>\</td>
<td>\</td>
<td></td>
</tr>
<tr>
<td>(-1)</td>
<td>\</td>
<td>\</td>
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<td></td>
</tr>
<tr>
<td>(0)</td>
<td>\</td>
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<tr>
<td>(1)</td>
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<tr>
<td>(2)</td>
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<td>(3)</td>
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<tr>
<td>(4)</td>
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</tbody>
</table>

Sample calculation for \( x = -4 \):

Questions

1. It seems that \( f(x) \) is \quad \cdash \quad \quad \quad to its derivative. Mathematically, it seems that \( f'(x) = k \quad \cdash \quad \). Use the limit definition of the derivative to show that this is indeed the case. What can you say about \( k \)?
2. If \( h(x) = a^x \), show a similar relationship between \( h'(x) \) and \( h(x) \).

3. Fill in the following: if \( h(x) = a^x \), then
\[
h'(x) = k \text{____}, \quad \text{where} \quad k = \lim \text{______}.
\]

4. Suppose we could find a number \( a \) such that the constant of proportionality becomes 1? What would \( \frac{d}{dx} a^x \) be in that case? Using guess and check, estimate this \( a \) to two decimal places (take \( h = 0.001 \)).

Why \( e \) is Important in Calculus

Such an \( a \) satisfies \( \frac{a^h - 1}{h} \approx 1 \) for small \( h \). We can rewrite this as
\[
a \approx \frac{\ln(h + 1)}{\ln(h)}
\]
In the limit as \( h \to 0 \) we get a number we’ve encountered before: We define
\[
e = \lim_{h \to 0} (h + 1)^{\frac{1}{h}}.
\]
This is the same \( e \) we saw before, when we did compound interest and natural logs. Our previous results give us that

<table>
<thead>
<tr>
<th>Derivative of ( e^x )</th>
</tr>
</thead>
<tbody>
<tr>
<td>( \frac{d}{dx} e^x = e^x )</td>
</tr>
</tbody>
</table>

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Now, back to the general exponential function $b^x$. By the chain rule, $\frac{d}{dx}e^{kx} =$ 
Then, we note that

$$b^x =$$

so, using the chain rule result above:

$$\frac{d}{dx}b^x =$$

### Derivatives of Exponential Functions

$$\frac{d}{dx}b^x =$$

### Questions

5. Find the derivatives of the following functions:

   (a) $f(x) = 8x + 8^x$

   (b) $g(x) = e^2 - 2^x$

   (c) $h(x) = e^\pi + e^x + x^e$

6. Find the tangent line to $x^2e^x$ at $x = 1$. 

7. Find the tangent line to $e^{(x+1)^3}$ at $x = 1$ (Hint: see the triple chain rule from the last worksheet).