Review - Exponential Functions

- An exponential function is any function of the form
  \[ f(x) = ax, \]
  where \( a > 0 \).
- \( P_0 \) is the \underline{initial value} of the function. In other words, \( P_0 \) is the ___ \underline{initial point} of the graph of the function.
- If \( a > 1 \), then the function is \underline{increasing}.
- If \( 0 < a < 1 \), then the function is \underline{decreasing}.
- A bank account that pays interest rate \( r \), compounded continuously, will yield \$\underline{A} \) if we leave \$\( A \) in it for \( t \) years.

Logarithms

Question  Is every exponential function invertible? How do you know?

On the last worksheet, you were asked to find the doubling time of a bank account with interest rate 5\%, compounded continuously. This required you to solve the equation

\[ 2A = Ae^{0.05t} \]

for \( t \) (where \( A \) is the initial deposit). To do so, you plugged in various numbers for \( t \) until you got pretty close to the answer. This is a little unsatisfactory.

In other terms, if we define \( P(t) = Ae^{0.05t} \), we wish to compute \underline{__________}.

If \( f(x) = 10^x \), then we define the \underline{logarithm base 10} of \( x \) to be \( f^{-1}(x) \). In other words

\[ \log_{10} x = c \iff 10^c = x. \]

Often, we leave off the 10 and just write \( \log x \).

If \( f(x) = e^x \), then we define the \underline{logarithm base e} of \( x \) to be \( f^{-1}(x) \). In other words

\[ \log_e x = c \iff e^c = x. \]

For reasons we’ll see next time, log base \( e \) is called the \underline{natural logarithm} and is most often written \( \log_e x = \ln x \).
Questions

1. (a) What are the range and domain of $\log x$?

   (b) On the same axes, draw the graphs of $10^x$ and $\log x$. (Hint: recall that the graph of the inverse of a function $f(x)$ is given by reflecting the graph of $f(x)$ in the line $y=x$).

   (c) Why does it make no sense to find $\log(0)$? What about $\log(x)$ when $x$ is negative?

   (d) Does $\log(x)$ have a vertical asymptote? Where?

   (e) Do all the previous questions for $\ln(x)$.
Properties of Logarithms

All the following properties can be deduced from the properties of exponents (eg \( x^{a+b} = x^a x^b \)):

| \( \log(AB) = \log A + \log B \) | \( \ln(AB) = \ln A + \ln B \) |
| \( \log \left( \frac{A}{B} \right) = \log A - \log B \) | \( \ln \left( \frac{A}{B} \right) = \ln A - \ln B \) |
| \( \log(A^p) = p \log(A) \) | \( \ln(A^p) = p \ln(A) \) |
| \( \log(10^x) = x \) | \( \ln(e^x) = x \) |
| \( 10^{\log x} = x \) | \( e^{\ln x} = x \) |

Applications of Logs

Questions

2. Find the doubling time of a bank account that has 5% interest rate compounded continuously.

3. Suppose we start with 10 moles of radon-222, a radioactive element. After two days, we find that there are \( 6.943 \times 10^6 \) moles of radon-222 remaining. Find the half-life of radon-222.

4. Find the inverses of the following functions:
   
   (a) \( f(t) = 10(7)^t \)

   (b) \( g(t) = 2 \ln(t) + 5 \)
5. Solve the following equations:

(a) \( 7^x = 2 \)

\[ (b) \quad 10^{2x+9} = 6 \cdot 7^{x-4} \]

(c) \( 9^{7-2x} + 6 = e \)

6. Let \( f(t) = 7(2^t) \). Write \( f(t) \) in the form \( P_0e^{rt} \). (Hint: \( e^{rt} = (e^r)^t \). If you can figure out \( P_0 \) and \( r \), you’re done...)

Note: As seen in the last question, it is possible to write any exponential function in the form \( P(t) = P_0e^{rt} \). This is far more common than \( P(t) = P_0a^t \), so we’ll be using it from now on.
Homework Problems

1. (a) Can the log of a number be negative? Explain.
   (b) Can we take the log of a negative number? Explain.

2. State the domain and range of the function $y = \log(x)$.

3. Evaluate the following expressions without using a calculator.
   (a) $\log\left(\frac{1}{100}\right)$
   (b) $\log(200^2) - \log(40)$
   (c) $\log\left(\frac{1}{10,000}\right) + \log\sqrt{1,000}$

4. Let $x = \log A$ and $y = \log B$. Rewrite the following expressions in terms of $x$ and $y$.
   (a) $\sqrt{\log(AB)}$
   (b) $\frac{\log B}{\log A}$
   (c) $\log\left(\sqrt{AB^2}\right)$
   (d) $\log\left(\frac{A}{B}\right)$

5. If possible, use the properties of logarithms to find exact solutions of the following equations for $x$.
   (a) $\log(1 - x) - \log(1 + x) = 2$
   (b) $\log(10x - 4) \cdot \log(16x^2) = 0$
   (c) $\frac{1}{5} \cdot 5^x - 25 = 100$
   (d) $\log(6x) - \log(2x - 1) = 2$