Review - Exponential Functions

- An exponential function is any function of the form
  \[ f(x) = \text{__________}, \]
  where \( a > 0 \).
- \( P_0 \) is the __________ of the function. In other words, \( P_0 \) is the __________ of the graph of the function.
- If \( a > 1 \), then the function is __________.
- If \( 0 < a < 1 \), then the function is __________.
- A bank account that pays interest rate \( r \), compounded continuously, will yield $\text{__________}$ if we leave $A$ in it for \( t \) years.

Logarithms

Question  Is every exponential function invertible? How do you know?

On the last worksheet, you were asked to find the doubling time of a bank account with interest rate 5%, compounded continuously. This required you to solve the equation

\[ 2A = Ae^{0.05t} \]

for \( t \) (where \( A \) is the initial deposit). To do so, you plugged in various numbers for \( t \) until you got pretty close to the answer. This is a little unsatisfactory.

In other terms, if we define \( P(t) = Ae^{0.05t} \), we wish to compute __________.

If \( f(x) = 10^x \), then we define the \textit{logarithm base 10} of \( x \) to be \( f^{-1}(x) \). In other words

\[ \log_{10} x = c \iff 10^c = x. \]

Often, we leave off the 10 and just write \( \log x \).

If \( f(x) = e^x \), then we define the \textit{logarithm base e} of \( x \) to be \( f^{-1}(x) \). In other words

\[ \log_e x = c \iff e^c = x. \]

For reasons we’ll see next time, log base \( e \) is called the \textit{natural logarithm} and is most often written \( \log_e x = \ln x \).
Questions

1. What are the range and domain of \( \log x \)?

2. On the same axes, draw the graphs of \( 10^x \) and \( \log x \). (Hint: recall that the graph of the inverse of a function \( f(x) \) is given by reflecting the graph of \( f(x) \) in the line ________).

3. Why does it make no sense to find \( \log(0) \)? What about \( \log(x) \) when \( x \) is negative?

4. Does \( \log(x) \) have a vertical asymptote? Where?

5. Do all the previous questions for \( \ln(x) \).
Properties of Logarithms

All the following properties can be deduced from the properties of exponents (eg $x^{a+b} = x^a x^b$):

<table>
<thead>
<tr>
<th>Logarithmic Property</th>
<th>Exponential Property</th>
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</thead>
<tbody>
<tr>
<td>$\log(AB) = \log A + \log B$</td>
<td>$\ln(AB) = \ln A + \ln B$</td>
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<tr>
<td>$\log \left( \frac{A}{B} \right) = \log A - \log B$</td>
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<tr>
<td>$\log(A^p) = p \log(A)$</td>
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</tr>
<tr>
<td>$\log(10^x) = x$</td>
<td>$\ln(e^x) = x$</td>
</tr>
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<td>$10^{\log x} = x$</td>
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</table>

Applications of Logs

Questions

1. Find the doubling time of a bank account that has 5% interest rate compounded continuously.

2. Suppose we start with 10 moles of radon-222, a radioactive element. After two days, we find that there are 6.943 moles of radon-222 remaining. Find the half-life of radon-222.

3. Find the inverses of the following functions:
   
   (a) $f(t) = 10(7)^t$

   (b) $g(t) = 2 \ln(t) + 5$
4. Solve the following equations:

(a) \(7^x = 2\)

(b) \(10^{2x+9} = 6 \cdot 7^{x-4}\)

(c) \(9^{7-2x} + 6 = e\)

5. Let \(f(t) = 7(2^t)\). Write \(f(t)\) in the form \(P_0e^{rt}\). (Hint: \(e^{rt} = (e^r)^t\). If you can figure out \(P_0\) and \(r\), you’re done...)

Note: As seen in the last question, it is possible to write any exponential function in the form \(P(t) = P_0e^{rt}\). This is far more common than \(P(t) = P_0a^t\), so we’ll be using it from now on.
Extra Homework Problems

1. (a) Can the log of a number be negative? Explain.
   
   (b) Can we take the log of a negative number? Explain.

2. State the domain and range of the function $y = \log(x)$.

3. Evaluate the following expressions without using a calculator.

   (a) $\log\left(\frac{1}{100}\right)$
   
   (b) $\log(200^2) - \log(40)$
   
   (c) $\log\left(\frac{1}{10,000}\right) + \log\sqrt{1,000}$

4. Let $x = \log A$ and $y = \log B$. Rewrite the following expressions in terms of $x$ and $y$.

   (a) $\sqrt{\log(AB)}$
   
   (b) $\frac{\log B}{\log A}$
   
   (c) $\log\left(\sqrt{AB^{-2}}\right)$
   
   (d) $\log\left(\frac{A}{B}\right)$

5. If possible, use the properties of logarithms to find exact solutions of the the following equations for $x$.

   (a) $\log(1 - x) - \log(1 + x) = 2$
   
   (b) $\log(10x - 4) \cdot \log(16x^2) = 0$
   
   (c) $\frac{1}{5} \cdot 5^x - 25 = 100$
   
   (d) $\log(6x) - \log(2x - 1) = 2$
Answers to Extra Homework Problems

1. (a) Yes.
   (b) No.

2. Domain: all positive real numbers; Range: All real numbers.

3. (a) $-2$
   (b) 3
   (c) $-2.5$

4. (a) $\sqrt{x + y}$
   (b) $\frac{y}{x}$
   (c) $0.5x - 2y$
   (d) $x - y$

5. (a) $-\frac{99}{101}$
   (b) $-0.25, 0.25, 0.5$
   (c) 4
   (d) $\frac{50}{97}$