Review - Exponential Functions

An exponential function is any function of the form $f(x) = P_0 a^x = P_0 e^{rt}$, where $a > 0$.

- If $P_0 a^t = P_0 e^{rt}$, then $a =$ _____. (Recall: $a^{bc} = (a^b)^c = (a^c)^b$.)
- P_0 is the _____ of the function. In other words, P_0 is the _____ of the graph of the function.
- If $a > 1$ (or $r > 0$), then the function is _____.
- If $0 < a < 1$ (or $r < 0$), then the function is _____.
- A bank account that pays interest rate r, compounded continuously, will yield \$_____ if we leave $\$A$ in it for t years.

Review - Inverses

- A function is *invertible* if it passes the _____.
- If $y = f(x)$, and $f(x)$ is invertible, then _____ $= x$

Preliminaries: Inverting Exponentials

Question Is every exponential function invertible? How do you know?

On the last worksheet, you were asked to find the doubling time of a bank account with interest rate 5%, compounded continuously. This required you to solve the equation

$$2A = Ae^{0.05t}$$

for t (where A is the initial deposit). To do so, you plugged in various numbers for t until you got pretty close to the answer. This is a little unsatisfactory.

Recasting this in terms of inverse functions: If we define $P(t) = Ae^{0.05t}$, we wish to compute _____.

If a quantity grows (or decays) exponentially, finding the time at which it reaches a particular value involves understanding the *inverse* of an exponential function.

For reasons we will see a bit later, we really only need to define the inverse of $f(x) = e^x$. For convenience, we will also define the inverse of $f(x) = 10^x$.

Logarithms

If $f(x) = 10^x$, then we define the *logarithm base 10* of x to be $f^{-1}(x)$. In other words

$$\log_{10} x = c \Leftrightarrow 10^c = x.$$

Often, we leave off the 10 and just write $\log x$.

If $f(x) = e^x$, then we define the *logarithm base e* of x to be $f^{-1}(x)$. In other words

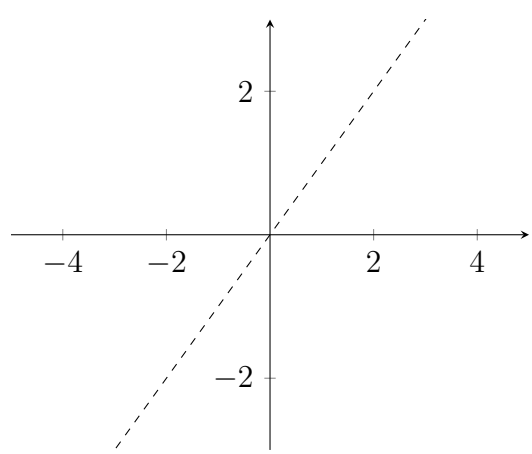
$$\log_e x = c \Leftrightarrow e^c = x.$$

For reasons we'll see next time, log base e is called the *natural logarithm* and is most often written $\log_e x = \ln x$.

- What are the range and domain of $\log x$ and $\ln x$?

- Why does it make no sense to find $\log 0$ or $\ln 0$? What about $\log x$ or $\ln x$ when x is negative?

- On the same axes, draw the graphs of 10^x and $\log x$, and of e^x and $\ln x$. (Hint: recall that the graph of the inverse of a function $f(x)$ is given by reflecting the graph of $f(x)$ in the line $y = x$, which is drawn for you).



- Do $\log(x)$ and $\ln x$ have vertical asymptotes? Where?

Properties of Logarithms

All the following properties can be deduced from the properties of exponents (eg $x^{a+b} = x^a x^b$):

$\log(AB) = \log A + \log B$	$\ln(AB) = \ln A + \ln B$
$\log\left(\frac{A}{B}\right) = \log A - \log B$	$\ln\left(\frac{A}{B}\right) = \ln A - \ln B$
$\log(A^p) = p\log(A)$	$\ln(A^p) = p\ln(A)$
$\log(10^x) = x$	$\ln(e^x) = x$
$10^{\log x} = x$	$e^{\ln x} = x$

Applications of Logs

Growth rates, Doubling times, and Half lives

1. (a) Find the doubling time of a bank account that has 5% interest rate compounded continuously.

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(b) If $P(x) = Ae^{rx}$, show that $P^{-1}(x) = \frac{1}{r}(\ln x - \ln A)$.

(c) Use part (b) to check your answer in part (a).

2. Suppose we start with 10 moles of radon-222, a radioactive element. After two days, ☐Regrade we find that there are 6.943 moles of radon-222 remaining.

(a) Without any computation: Is the half life more or less than two days? Explain.

(b) Let $N(t) = P_0 e^{rt}$ be the number of moles remaining after t days. Find P_0 and r.

(c) Find the half life of radon-222.

Continuous Growth and Decay Rates

If $f(x) = P_0 e^{rt}$, r is the *continuous growth rate*. It can be interpreted as the percentage rate at which function grows or decays, expressed as a decimal.

Examples The bank account in Question 1 above has a continuous growth rate of 5% per year, so $r = 0.05$. If you did Question 2 correctly, you should have found that $r \approx 0.1834$. So radon-222 decays at around 18.34% per day.

Inverses and Solving Equations

3. Find the inverses of the following functions.

☐Regrade (a) $f(t) = 10(7)^t$

(b) $g(t) = 2\ln(t) + 5$

4. Solve the following equations. Give exact answers in terms of ln or log, not decimals.

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(a) $7^x = 2$

(b) $10^{2x+9} = 6 \cdot 7^{x-4}$

(c) $9^{7-2x} + 6 = e$ (Careful with this one! Before charging ahead, think about $e - 6$.)

Converting Between Exponential Forms

5. Let $f(t) = 7(2^t)$. Write $f(t)$ in the form $P_0 e^{rt}$.

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Converting Between Exponential Forms

Any exponential function can be written in two forms:

$$P(t) = P_0 a^t = P_0 e^{rt}.$$

This gives $a = e^r$, or $r = \ln a$. We will mostly use the latter form from now on, but you should be able to convert between the two.

More Problems

6. Use log rules to evaluate the following expressions <u>without</u> using a calculator. Show all your work.
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(a) $\log\left(\frac{1}{100}\right)$

(b) $\log\left(200^2\right) - \log(40)$

(c) $\log\left(\frac{1}{10,000}\right) + \log\sqrt{1,000}$

7. Let $x = \log A$ and $y = \log B$. Rewrite the following expressions in terms of x and y.
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(a) $\sqrt{\log(AB)}$

(b) $\frac{\log B}{\log A}$

(c) $\log\left(\sqrt{A}B^{-2}\right)$

(d) $\log\left(\frac{A}{B}\right)$

8. If possible, use the properties of logarithms to find exact solutions of the the following ☐Regrade equations for x.

(a) $\log(1-x) - \log(1+x) = 2$

(b) $\log(10x-4) \cdot \log(16x^2) = 0$

(c) $\log(6x) + \log\left(2x - \frac{1}{3}\right) = 1$