Exponential Growth

The formula we wrote down recently for continuously compounded interest is an example of *exponential growth*.

Definition The number e is defined as follows:

$$e = \lim_{n \to \infty} \left(1 + \frac{1}{n}\right)^n \approx \_\_\_\_\_\_\_\_\_.$$

Interest Rates If we put \$$A$ in the bank at annual interest rate r (expressed as a decimal), continuously compounded, then after t years, we have \$_________.

e arises in all sorts of places. It comes up all over calculus. We'll see why it's *at least* as special as π over the next few lessons.

Example: Bacterial Growth In an experiment, a bacterial culture was placed in an environment that allows it unrestricted growth. The number of bacteria was measured every hour for 8 hours, resulting in the following data:

Time (hours)	0	1	2	3	4	5	6	7	8
# Bacteria	15	49	161	525	1719	5625	18406	60226	197059
Δ Bacteria	X	$49 - 15 = 34$							
Ratio # Bac		$49/15 \approx 3.27$							

- You can find this data in a spreadsheet here. Make a copy of it.
- In column B, compute the successive differences between the number of bacteria.
- In column C, compute the successive ratios between the number of bacteria (same as difference, but divide instead of subtract this time).

Questions

1. (a) Copy your results into the table above.
   (b) Explain why your results shows that bacterial growth is not linear.
   (c) Every hour, the number of bacteria increases by a ratio of approximately ______. Therefore, after t hours, the number of bacteria is given by

$$B(t) = \_\_\_\_\_\_\_\_\_.$$

(d) At what time does our model predict that the number of bacteria doubled from 15 to 30?

(e) At what time does the population reach 98, double the number of bacteria present at $t = 1$ hour?

(f) What do you conclude from the last two questions?

Definition The *doubling time* of an exponentially increasing function is the time required for the function to double in value.

2. What is the doubling time of the value of a bank account with interest rate 5%, compounded continuously? (Note: if the interest is 5%, then $r = 0.05$. Guess and check again.)

Example: Exponential Decay Radioactive elements decay with a constant *half-life*. In other words, if we start with n atoms of a radioactive substance with half-life T, we expect to be left with ______ atoms of the substance after time T has passed.

Example

3. Proctactinium-234, a product of the radioactive decay of uranium-238, has a half-life of 1.2 minutes. Suppose we have 10 moles of proctactinium-234 at time $t = 0$.

   (a) How much do we have after 1.2 minutes? 2.4 minutes?

   (b) Write down a formula of the form $P = P_0 a^t$ for the number of moles left after t minutes. (Hint: you already know three values of P. You really only need two...)

$P(t)$ is an *exponential function* of t if

$$P(t) = P_0 a^t,$$

where P_0 is the initial quantity and a is the factor by which $P(t)$ changes when t increases by 1.

If $a > 1$, we have *exponential growth*. If $0 < a < 1$, we have *exponential decay*.

Question What happens if $a = 1$?

Graphing Exponential Functions

Questions

4. (a) Using Geogebra (or otherwise), graph the functions $f(x) = a^x$ for $a = 1.5$, 2, 3, 5, and 10. Copy the graphs here:

   (b) Do the same for $a = 0.95$, 0.9, 0.8, 0.5 and 0.1:

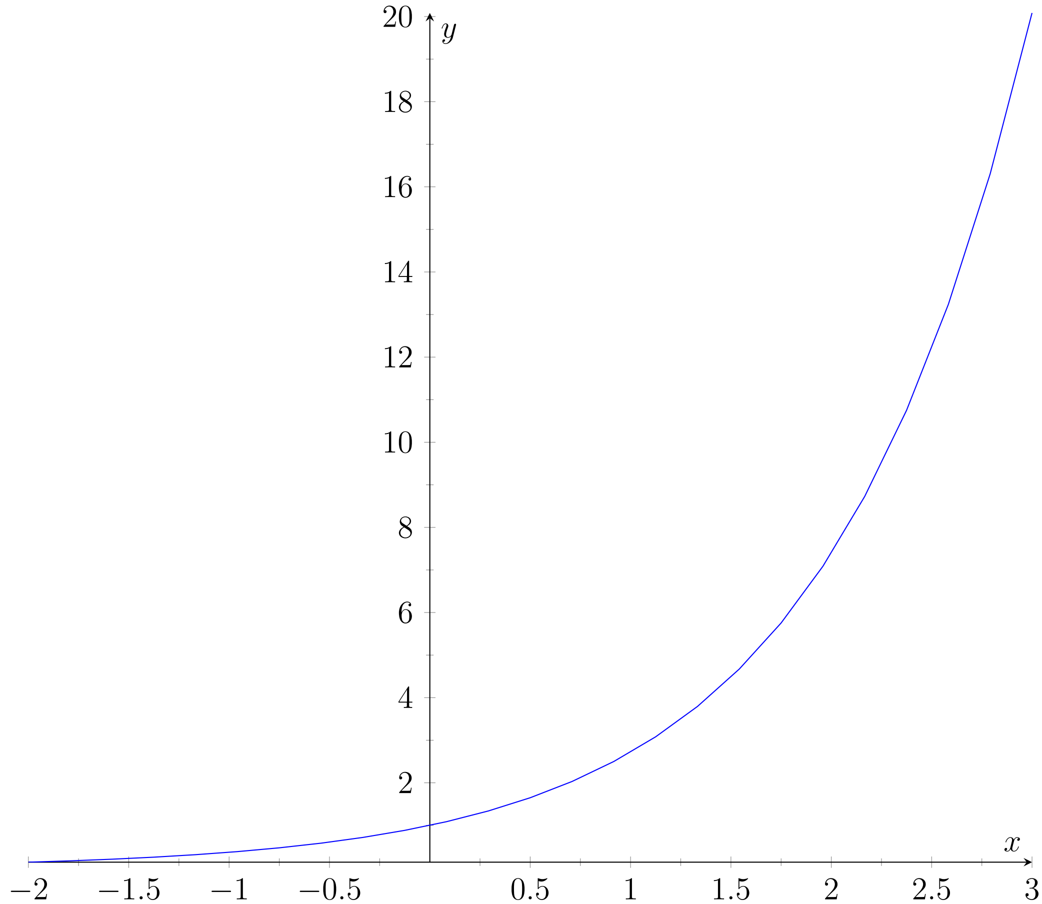
(c) All of the graphs you drew are concave ______

(d) When $a > 1$ is large, the graph of $f(x)$ increases ______ than when $a > 1$ is closer to 1.

(e) When $a < 1$ is close to 0, the graph of $f(x)$ decreases ______ than when $a < 1$ is closer to 1.

Previewing the Calculus

An exponential function is plotted on the following graph. Sketch the graph of its derivative on the same axes:



5. Fill in the following hypothesis: The derivative of an exponential function appears to be ________________________.