Preliminaries: Rules from the Pas(t)

Relevant Limit Rules:

- Sums: If $f(x)$ and $g(x)$ are functions, then

$$\lim_{x\to c} f(x) + g(x) = \lim_{x\to c} f(x) + \lim_{x\to c} g(x).$$

- If b is a constant then

$$\lim_{x\to c} bf(x) = \lim_{x\to c} f(x)$$

  - Note: in this context a constant is anything that doesn't depend on x.

Relevant Derivative Rules:

- Constant multiple rule: If c is a constant, and $f(x)$ a differentiable function, then

$$\frac{d}{dx} cf(x) = cf'(x).$$

  - Note: in this context a constant is anything that doesn't depend on x.

- Sum rule: If $f(x)$ and $g(x)$ are differentiable functions, then

$$\frac{d}{dx}\left(f(x) + g(x)\right) = f'(x) + g'(x).$$

- Power rule:

$$\frac{d}{dx} ax^n = anx^{n-1}.$$

- Chain rule:

$$\frac{d}{dx} f(g(x)) = f'(g(x))g'(x)$$

The Product Rule

When the Obvious Doesn't Work...

In lab, we looked at finding derivatives of the sums and differences and constant multiples of functions. We do the next logical step today: products and quotients (i.e. multiplication and division). Naturally, we might want to try

$$\frac{d[(f(x)g(x)]}{dx} \stackrel{?}{=} f'(x)g'(x).$$

As the following example shows, though, this doesn't work:

Example Let $f(x) = \frac{1}{x}$ and $g(x) = x$. Then,

$$f(x)g(x) = \underline{\qquad\qquad}, \text{ so } \frac{d[f(x)g(x)]}{dx} = \underline{\qquad\qquad}.$$

However,

$$f'(x) = \underline{\quad}, \; g'(x) = \underline{\quad}, \text{ so } f'(x)g'(x) = \underline{\qquad\qquad}.$$

Hence

$$\frac{d[f(x)g(x)]}{dx} \neq f'(x)g'(x).$$

...We Have to Work Harder!

First, write down the derivative of the product $f(x)g(x)$ from the limit definition of a derivative:

$$\frac{d}{dx}f(x)g(x) = \underline{\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad}$$

Rabbit! We're going to subtract the term $f(x)g(x+h)$ from the numerator, and also add the same term:

$$= \lim_{h\to 0} \underline{\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad}$$

$$= \lim_{h\to 0} \underline{\qquad\qquad\qquad\qquad\qquad\qquad} + \lim_{h\to 0} \underline{\qquad\qquad\qquad\qquad\qquad\qquad}$$

$$= \lim_{h\to 0} \underline{\qquad\qquad} \cdot \lim_{h\to 0} \underline{\qquad\qquad\qquad} + \lim_{h\to 0} \underline{\qquad\qquad} \cdot \lim_{h\to 0} \underline{\qquad\qquad\qquad}$$

$$= \underline{\qquad\qquad} \cdot \underline{\qquad\qquad} + \underline{\qquad\qquad} \cdot \underline{\qquad\qquad}$$

The Product Rule
If $u = f(x)$ and $v = g(x)$ are differentiable, then

$$(fg)' = f'g + fg'$$

or, we can also write it as

$$\frac{d(uv)}{dx} = \frac{du}{dx} \cdot v + u \cdot \frac{dv}{dx}$$

In words, we say "The derivative of the product of two functions is the derivative of the first times the second plus the first times the derivative of the second."

The Quotient Rule

To differentiate a quotient, $\frac{f(x)}{g(x)}$, we can rewrite it as a product. Let $Q(x) = \frac{f(x)}{g(x)}$. Then

$$Q(x)g(x) = f(x).$$

We aim to compute $Q'(x)$.

- Use the product rule to differentiate $Q(x)g(x) = f(x)$:

- Solve this for $Q'(x)$:

- Replace $Q(x)$ with $\frac{f(x)}{g(x)}$, then multiply top and bottom by $g(x)$:

The Quotient Rule
If $u = f(x)$ and $v = g(x)$ are differentiable, then

$$\left(\frac{f}{g}\right)' = \frac{g(x)f'(x) - f(x)g'(x)}{g(x)^2}$$

or, if we refer to the numerator as 'hi' and the denominator as 'lo', and their derivatives as 'd(hi)' and 'd(lo)' we can write:

$$\frac{\text{lo d(hi)} - \text{hi d(lo)}}{\text{lo}^2}$$

Using the Product and Quotient Rules

Product Rule Example

Consider the function $(x^2+3)(x^3+4x^2+2x+1)$.

- FOIL this and collect like terms, then differentiate it.

- Now differentiate the function using the product rule instead.

- Lastly, FOIL your answer from the previous part and collect like terms to check we got the same answer.

Quotient Rule Example

Let $f(x) = \frac{x-1}{x+1}$.

- Find $f'(x)$ using the quotient rule.

- Fill in the blanks: Since $f'(x)$ is always _____, $f(x)$ is always _____.
- Draw the graph of $f(x)$. Briefly explain why it cannot cross its horizontal asymptote.

1. Carry out the same three part process as in the product rule example above on the
□Regrade function $(x^2 + \sqrt{x})\left(\frac{1}{x^2} + \frac{1}{\sqrt{x}}\right)$ to compute its derivative in two ways and check the answers are the same.

2. Suppose $h(3) = 1$, $h'(3) = 3$, $f(3) = 5$, and $f'(3) = 4$. If $g(x) = h(x)f(x)$, what is a
□Regrade formula for $g'(x)$? Use it to find $g'(3)$.

3. Let $h(3) = 1$, $h'(3) = 3$, $f(3) = 5$, and $f'(3) = 4$. Find:
□Regrade

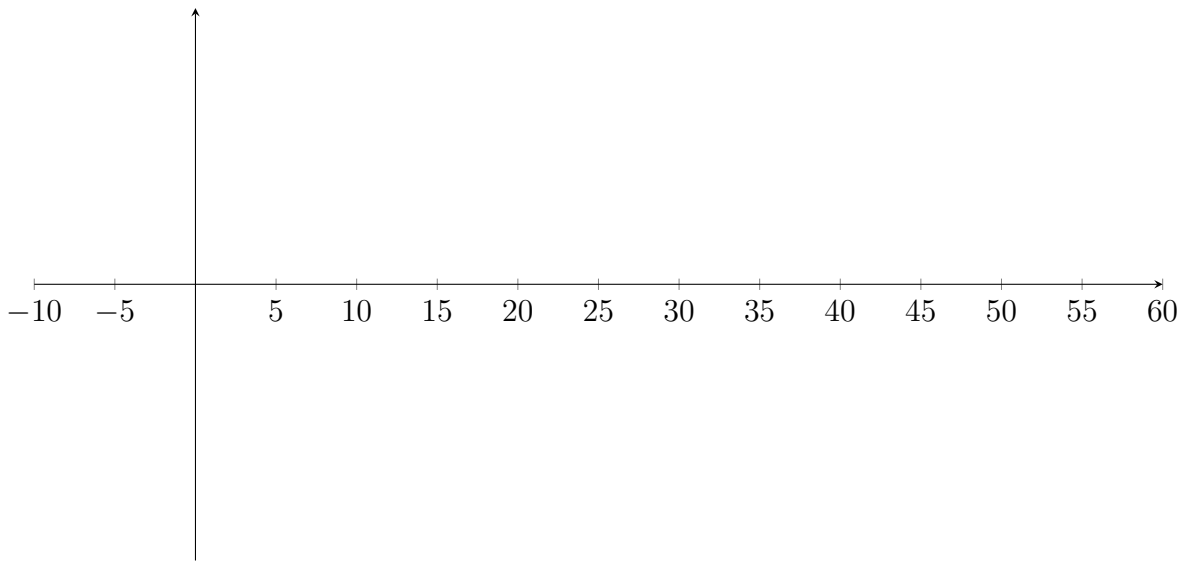
(a) $j'(3)$ if $j(w) = \frac{f(w)}{h(w)}$.

(b) $g'(3)$ if $g(w) = \frac{h(w)}{f(w)}$.

Questions 4 and 5 refer to the function $f(x) = \frac{2x^2+3x-2}{x^2+2x-15}$.

4. Draw the graph of this function in Geogebra or some other graphing software. Set
□Regrade your x axis to be from -10 to 60, and your y axis to be from -10 to 10. Copy the graph here. Find the roots, vertical asymptotes, and the horizontal asymptote of the function and add them to your graph.

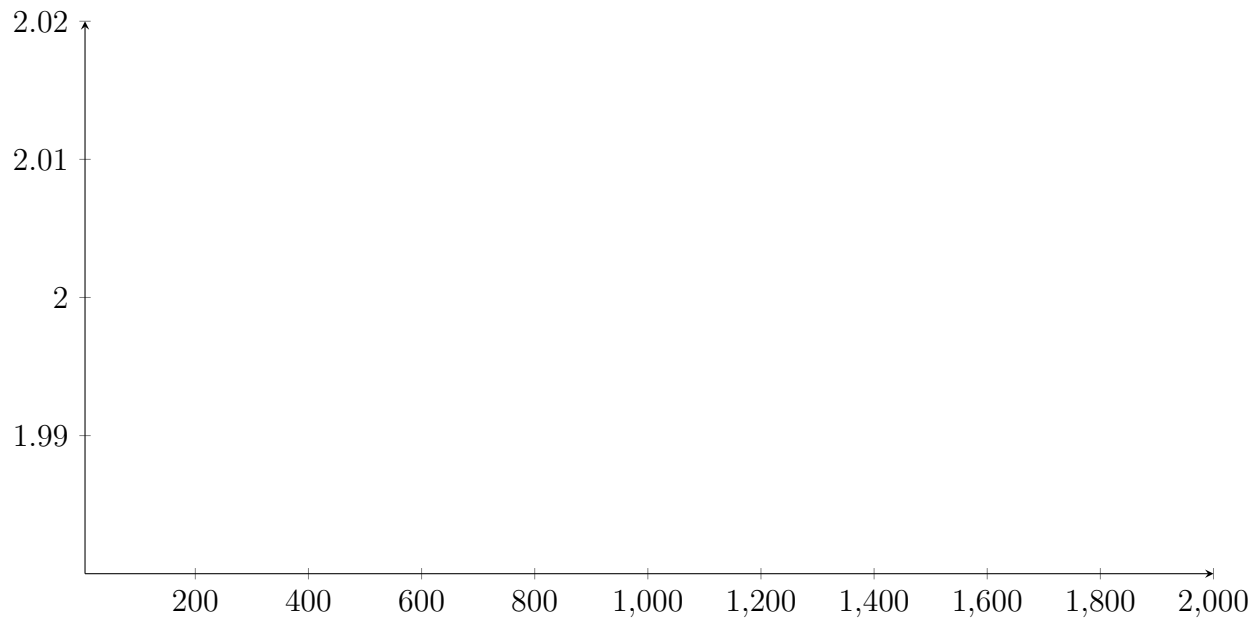
(Hint: see the rational functions worksheet.)



5. (a) Find $f'(x)$.
□Regrade

(b) Use your formula for $f'(x)$ to show that $f(x)$ is increasing for $x < -0.723$ and $x > 56.72$, and decreasing between those two numbers.

(Hint: If you did this correctly, you should have found that the denominator of $f(x)$ is always positive, so you really only need to consider the numerator. You'll need the quadratic formula.)

(c) If your graph in Question 4 is correct, it should look like the graph decreases toward its horizontal asymptote as $x \to \infty$. By looking at your answer in (b), explain why this cannot be the case.

(d) Going back to Geogebra (or other graphing software), set the x-axis to be from 5 to 2,000, and the y-axis to be from 1.98 to 2.02. You should see that the function drops below its horizontal asymptote, then turns around and approaches it from below. Copy your graph below and add in the asymptote, then compute the coordinates of the turning point to four decimal places and label it on the graph. (Hint: you already know the x-coordinate!)