Rules from the Pas(t)

Relevant Limit Rules:

- Sums: If $f(x)$ and $g(x)$ are functions, then

$$\lim_{x\to c} f(x) + g(x) = \underline{\hspace{6cm}}.$$

- If b is a constant then

$$\lim_{x\to c} bf(x) = \underline{\hspace{6cm}}$$

  – Note: in this context a constant is anything that doesn't depend on ______.

Relevant Derivative Rules:

- Constant multiple rule: If c is a constant, and $f(x)$ a differentiable function, then

$$\frac{d}{dx}cf(x) = \underline{\hspace{3cm}}.$$

  – Note: in this context a constant is anything that doesn't depend on ______.

- Sum rule: If $f(x)$ and $g(x)$ are differentiable functions, then

$$\frac{d}{dx}\left(f(x) + g(x)\right) = \underline{\hspace{6cm}}.$$

- Power rule:

$$\frac{d}{dx}ax^n = \underline{\hspace{3cm}}.$$

When the Obvious Doesn't Work...

In lab, we looked at finding derivatives of the sums and differences and constant multiples of functions. We do the next logical step today: products and quotients (i.e. multiplication and division). Naturally, we might want to try

$$\frac{d[(f(x)g(x)]}{dx} \stackrel{?}{=} f'(x)g'(x)$$

but, from the following example, we see why doesn't work.

Example Let $f(x) = \frac{1}{x}$ and $g(x) = x$. Then,

$$f(x)g(x) = \underline{\hspace{1cm}}, \text{ so } \frac{d[f(x)g(x)]}{dx} = \underline{\hspace{1cm}}.$$

However,

$$f'(x) = \underline{\hspace{1cm}},\ g'(x) = \underline{\hspace{1cm}}, \text{ so } f'(x)g'(x) = \underline{\hspace{1.5cm}}.$$

Hence

$$\frac{d[f(x)g(x)]}{dx} \neq f'(x)g'(x).$$

...We Have to Work Harder!

First, write down the derivative of the product $f(x)g(x)$ from the limit definition of a derivative:

$$\frac{d}{dx}f(x)g(x) = \underline{\hspace{8cm}}$$

Rabbit! We're going to subtract the term $f(x)g(x+h)$ from the numerator, and also add the same term:

$$= \lim_{h\to 0} \underline{\hspace{11cm}}$$

$$= \lim_{h\to 0} \underline{\hspace{6cm}} + \lim_{h\to 0} \underline{\hspace{6cm}}$$

$$= \lim_{h\to 0} \underline{\hspace{1.5cm}} \cdot \lim_{h\to 0} \underline{\hspace{3cm}} + \lim_{h\to 0} \underline{\hspace{1.5cm}} \cdot \lim_{h\to 0} \underline{\hspace{3cm}}$$

$$= \underline{\hspace{1.5cm}} \cdot \underline{\hspace{1.5cm}} + \underline{\hspace{1.5cm}} \cdot \underline{\hspace{1.5cm}}$$

We have just proved the Product Rule:

> If $u = f(x)$ and $v = g(x)$ are differentiable, then
>
> $$(fg)' = f'g + fg'$$
>
> or, we can also write it as
>
> $$\frac{d(uv)}{dx} = \frac{du}{dx} \cdot v + u \cdot \frac{dv}{dx}$$

In words, we say "The derivative of the product of two functions is the derivative of the first times the second plus the first times the derivative of the second."

Questions Now, we can compute all sorts of derivatives quickly. For example, differentiate the following in two ways. First by multiplying out, then by using the product rule. Are your answers the same?

1. $(x^2 + 3)(x^3 + 4x^2 + 2x + 1)$

2. $(x^2 + \sqrt{x})\left(\frac{1}{x^2} + \frac{1}{\sqrt{x}}\right)$

3. Suppose $h(3) = 1$, $h'(3) = 3$, $f(3) = 5$, and $f'(3) = 4$. If $g(x) = h(x) \cdot f(x)$, find $g'(3)$.

The Quotient Rule

Let $f(x)$ and $g(x)$ be differentiable functions with $g(x) \neq 0$. Suppose that $Q(x) = \frac{f(x)}{g(x)}$. Then,

$$f'(x) = \frac{d}{dx}\left(Q(x)g(x)\right) = \underline{\hspace{6cm}}.$$

Now solve for $Q'(x)$ and take common denominators to get:

$$Q'(x) = \frac{d}{dx}\left(\frac{f(x)}{g(x)}\right) = \qquad\qquad\qquad .$$

If $u = f(x)$ and $v = g(x)$ are differentiable and $g(x) \neq 0$, then

$$\left(\frac{f}{g}\right)' = \frac{f'g - g'f}{g^2}$$

Questions

4. Let $f(x) = \frac{x-1}{x+1}$.
   (a) Find $f'(x)$.

   (b) Fill in the blanks: Since $f'(x)$ is always __________________, $f(x)$ is always __________________.

5. Let $h(3) = 1$, $h'(3) = 3$, $f(3) = 5$, and $f'(3) = 4$. Find:
   - $j'(3)$ if $j(w) = \frac{f(w)}{h(w)}$.

   - $g'(3)$ if $g(w) = \frac{h(w)}{f(w)}$.

6. Let $f(x) = \frac{2x^2+3x-2}{x^2+2x-15}$.

   (a) Find $f'(x)$.

   (b) Where is $f(x)$ increasing? decreasing?

   (c) Using the techniques from earlier in the semester, plus the previous two parts of this question, sketch $f(x)$. How do calculus tools allow you to be more accurate in your sketching than you could back then? You should find that there is at least one feature in the graph that previous techniques missed completely!