And Once Again, The Past...

- The derivative of a function $f(x)$ is

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

- A function is continuous at a point $x = a$ if

$$\lim_{x \to a^+} f(x) = \lim_{x \to a^-} f(x) = f(a).$$

Preliminaries: Smooth and Sharp Graphs

- Find the tangent line to $f(x) = x^2$ at $x = 1$. Then draw the graph of x^2 and that tangent line in Geogebra. Zoom in a lot near $x = 1$. What do you observe?

- Now graph $h(x) = |x|$, and zoom in near 0? Do you see the same thing?

If we zoom in enough near any point of $y = x^2$ (or just about any function we've seen so far) the graph essentially coincides with its tangent line at that point. But this isn't the case for $y = |x|$ near $x = 0$.

We therefore conclude that $y = |x|$ ***does not have a tangent line at*** $x = 0$. We call it *non-differentiable* there.

Differentiability

A function f is *differentiable* at a point a if the derivative of $f(x)$ exists and is finite at $x = a$. In other words, $f(x)$ is differentiable at $x = a$ if

$$\lim_{h \to 0} \frac{f(a+h) - f(a)}{h}$$

exists and is finite.

We will show from this definition that the functions below are all non-differentiable at $x = 0$. In each case, draw the graph and write down a graphical description of what causes the function to be non-differentiable.

- $f(x) = |x|$

- $f(x) = x^{\frac{1}{3}}$

- $f(x) = \begin{cases} -1 & \text{if } x > 0 \\ 1 & \text{if } x \leq 0 \end{cases}$

Continuity and Differentiability

Theorem If $f(x)$ is differentiable at a point $x = a$, then $f(x)$ is continuous at $x = a$.

- Fill in the following blanks with the words "continuous" and "differentiable":
  - If $f(x)$ is not _____ at $x = a$, it is not _____ at $x = a$.
  - If $f(x)$ is _____ at $x = a$, it is _____ at $x = a$.
- If is possible for $f(x)$ to be _____ at $x = a$, but not _____ there. Give an example of such a function.

Examples

Question Find $f'(0)$ or explain why it does not exist if

$$f(x) = \begin{cases} 4 - x^2 & \text{if } x > 0 \\ x^2 - 4 & \text{if } x \leq 0 \end{cases}.$$

Question Find $f'(0)$ or explain why it does not exist if

$$f(x) = \begin{cases} 4 - x^2 & \text{if } x > 0 \\ x^2 + 4 & \text{if } x \leq 0 \end{cases}.$$

A Shortcut to Differentiability

Theorem Let $f(x)$ be a function. Then if $f'(x)$ is continuous at $x = a$, then $f(x)$ is differentiable at $x = a$.

This gives us a shortcut to check differentiability: take a derivative, and check if its continuous.

Important note: You must first check if the function itself is continuous! If it is not, it is also not differentiable.

1. Consider the last function on the previous page:

☐Regrade

$$f(x) = \begin{cases} 4 - x^2 & \text{if } x > 0 \\ x^2 + 4 & \text{if } x \leq 0 \end{cases}.$$

We know that it's continuous at $x = 0$.

(a) Differentiate it away from $x = 0$:

$$f'(x) = \begin{cases} \underline{\qquad\qquad} & \text{if } x > 0 \\ \underline{\qquad\qquad} & \text{if } x < 0 \end{cases}.$$

(b) Show that $f'(x)$ is continuous at $x = 0$. What does this tell you about $f(x)$ at $x = 0$?

2. Find $f'(0)$ or explain why it does not exist if $f(x) = (x + |x|)^2 + 1$.

☐Regrade (Hint: Write $|x|$ as a piecewise function, then differentiate as in Question 1.)

3. Find a and b such that the following function is differentiable everywhere:

☐Regrade

$$f(x) = \begin{cases} ax^3 & \text{if } x \leq 2 \\ x^2 + b & \text{if } x > 2 \end{cases}.$$

(Hint: First make $f(x)$ continuous, then differentiate...)