Preliminaries: Continuity and Limits - Intuition

Previously, we said that a function is continuous at a point $x = c$ if both the following conditions are satisfied:

- We know what number we expect $f(x)$ to take at $x = c$; and
- $f(x)$ is actually that number when $x = c$.

Three Ways This Can Go Wrong

Limits – A Better Definition

Definition $\lim_{x\to c} f(x) = L$ if we can get $f(x)$ as close to L as we want by taking x close to c. (In words, we say "the limit as x goes to c of $f(x)$ equals L.")

- In this situation, as we get very close to c from the horizontal direction, the graph of $f(x)$ gets very close to L in the vertical direction.
- Also notice that $x \to c$ means that x gets very close to c, but is not equal to c.
- We say $\lim_{x\to c} f(x)$ *does not exist* if there is no such number L.

Definition (Left and Right Limits)
$\lim_{x\to c^+} f(x)$ means that we only let x approach c from the *positive side*. In terms of the graph, x is approaching c from the *right*.

$\lim_{x\to c^-} f(x)$ means that we only let x approach c from the *negative side*. In terms of the graph, x is approaching c from the *left*.

Important point: $\lim_{x\to c} f(x)$ exists and is equal to L if and only if

$$\lim_{x\to c^+} f(x) = L = \lim_{x\to c^-} f(x)$$

1. By looking at $\lim_{x\to 0^-} f(x)$ and $\lim_{x\to 0^-} f(x)$, investigate $\lim_{x\to 0} f(x)$ where

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(a) $f(x) = \frac{|x|}{x}$.

(b) $f(x) = \frac{1}{x}$

(c) $f(x) = \frac{1}{x^2}$

Limit Laws

Here are properties that allow us to easily calculate limits. Note: To use them, $\lim_{x\to c} f(x)$ and $\lim_{x\to c} g(x)$ must exist.

- $\lim_{x\to c} k =$ _____ for any constant k.
- $\lim_{x\to c} x =$ _____.
- $\lim_{x\to c}[kf(x)] =$ _____ for any constant k.
- $\lim_{x\to c}[f(x)+g(x)] =$ _____.
- $\lim_{x\to c}[f(x)g(x)] =$ _____.
- $\lim_{x\to c}\left[\dfrac{f(x)}{g(x)}\right] =$ _____ if $\lim_{x\to c} g(x) \neq 0$.

Another Limit

Definition Limits at Infinity

We say $\lim_{x\to\infty} f(x) = L$ if we can get $f(x)$ as close to L as we please by taking x _____.

2. Calculate the following limits (or show they don't exist!):

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(a) $\lim_{x\to 5} \dfrac{x^3 - x}{2x+3}$

(b) $\lim_{x\to\infty} \dfrac{3x^2+17x-7}{2x^2-1}$

We already know this is $\frac{3}{2}$ from our time with rational functions, but let's do it from scratch using limit rules:

(c) $\lim_{h\to 0} \dfrac{2(h+2)^2-2h^2}{h}$ (Hint: simplify first. You may need to refer to the limit rules above and to question 1.)

(d) Let

$$f(x) = \begin{cases} x-5 & \text{if } x \geq 1 \\ -4x & \text{if } x < 1. \end{cases}.$$

Evaluate $\lim_{x\to 1} f(x)$.

Continuity and Limits

Definition (Continuity) A function is said to be **continuous** on an interval $[a, b]$ if, intuitively, you can draw the graph of f over that interval without lifting your pencil from the paper.

The mathematical definition is that a function f is continuous at the point $x = c$ if

$$\lim_{x \to c^+} f(x) = \lim_{x \to c^+} f(x) = f(c).$$

The first part of the equality says that the limit exists (that is, we know what value to expect at $x = c$). The second equality says that we also *get what we expect.*

3. Consider the following function:

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$$f(x) = \begin{cases} kx & \text{if } x \geq 1 \\ x^2 + 2 & \text{if } x < 1. \end{cases}.$$

(a) Find $\lim_{x \to 1^+} f(x)$ and $\lim_{x \to 1^-} f(x)$.

(b) What is $f(1)$?

(c) What must be true about your three answers in order for $f(x)$ to be continuous? What value of k makes it so?