Continuity and Limits - Intuition

Previously, we said that a function is continuous at a point $x = c$ if both the following conditions are satisfied:

i We know what number we expect $f(x)$ to take at $x = c$; and

ii $f(x)$ is actually that number when $x = c$.

How This Can Go Wrong

A.

B.

C.

Definition $\lim_{x \to c} f(x) = L$ if we can get $f(x)$ as close to L as we want by taking x ____________________________________. (In words, we say "the limit as x goes to c of $f(x)$ equals L.")

- In this situation, as we get very close to c from the horizontal direction, the graph of $f(x)$ gets very close to L in the vertical direction.
- Also notice that $x \to c$ means that x gets very close to c, but is not equal to c.
- We say $\lim_{x \to c} f(x)$ *does not exist* if there is no such number L.

Definition (Left and Right Limits)

$$\lim_{x \to c^+} f(x)$$

means that we only let x approach c from the *positive side*. In terms of the graph, x is approaching c from the *right*.

$$\lim_{x \to c^-} f(x)$$

means that we only let x approach c from the *negative side*. In terms of the graph, x is approaching c from the *left*.

Important point: $\lim_{x \to c} f(x)$ exists and is equal to L if and only if

$$\lim_{x \to c^+} f(x) = L = \lim_{x \to c^-} f(x)$$

Question

1. By looking at $\lim_{x \to 0^-} f(x)$ and $\lim_{x \to 0^-} f(x)$, investigate $\lim_{x \to 0} f(x)$ where

   (a) $f(x) = \frac{|x|}{x}$.

   (b) $f(x) = \frac{1}{x}$

   (c) $f(x) = \frac{1}{x^2}$

Limit Laws

Here are properties that allow us to easily calculate limits. Note: To use them, $\lim_{x\to c} f(x)$ and $\lim_{x\to c} g(x)$ must exist.

- $\lim_{x\to c} k =$ ____________________________ for any constant k.
- $\lim_{x\to c} x =$ ____________________________ .
- $\lim_{x\to c} [kf(x)] =$ ____________________________ for any constant k.
- $\lim_{x\to c} [f(x) + g(x)] =$ ____________________________ .
- $\lim_{x\to c} [f(x)g(x)] =$ ____________________________ .
- $\lim_{x\to c} \left[\frac{f(x)}{g(x)}\right] =$ ____________________________ if $\lim_{x\to c} g(x) \neq 0$.

Another Limit

Definition Limits at Infinity

We say $\lim_{x\to\infty} f(x) = L$ if we can get $f(x)$ as close to L as we please by taking x __________________________.

Questions

2. Calculate the following limits (or show they don't exist!):

   (a) $\lim_{x\to 5} \frac{x^3 - x}{2x + 3}$

   (b) $\lim_{x\to\infty} \frac{3x^2 + 17x - 7}{2x^2 - 1}$

(c) $\lim_{h\to 0} \frac{2(h+2)^2 - 2h^2}{h}$ (Hint: simplify first. You may need to refer to the limit rules above and to question 1.)

(d) Let

$$f(x) = \begin{cases} x-5 & \text{if } x \geq 1 \\ -4x & \text{if } x < 1. \end{cases}.$$

Evaluate $\lim_{x\to 1} f(x)$.

> **Definition (Continuity)** A function is said to be **continuous** on an interval $[a, b]$ if, intuitively, you can draw the graph of f over that interval without lifting your pencil from the paper. The mathematical definition is that a function f is continuous at the point $x = c$ if $\lim_{x\to c} f(x) = f(c)$.

3. Find k such that the function

$$f(x) = \begin{cases} kx & \text{if } x \geq 1 \\ x^2+2 & \text{if } x < 1. \end{cases}.$$

is continuous.