Continuity and Limits - Intuition

Previously, we said that a function is continuous at a point $x = c$ if both the following conditions are satisfied:

i. We know what number we expect $f(x)$ to take at $x = c$; and

ii. $f(x)$ is actually that number when $x = c$.

How This Can Go Wrong

A.

B.

C.

\[ \text{Definition} \quad \lim_{x \to c} f(x) = L \text{ if we can get } f(x) \text{ as close to } L \text{ as we want by taking } x \quad \quad \text{. (In words, we say} \]
\[ \text{“the limit as } x \text{ goes to } c \text{ of } f(x) \text{ equals } L.”] \]

- In this situation, as we get very close to $c$ from the horizontal direction, the graph of $f(x)$ gets very close to $L$ in the vertical direction.

- Also notice that $x \to c$ means that $x$ gets very close to $c$, but is not equal to $c$.

- We say $\lim_{x \to c} f(x)$ does not exist if there is no such number $L$. 
Definition (Left and Right Limits)

\[ \lim_{x \to c^+} f(x) \]

means that we only let \( x \) approach \( c \) from the \textit{positive side}. In terms of the graph, \( x \) is approaching \( c \) from the \textit{right}.

\[ \lim_{x \to c^-} f(x) \]

means that we only let \( x \) approach \( c \) from the \textit{negative side}. In terms of the graph, \( x \) is approaching \( c \) from the \textit{left}.

**Important point:** \( \lim_{x \to c} f(x) \) exists and is equal to \( L \) if and only if

\[ \lim_{x \to c^+} f(x) = L = \lim_{x \to c^-} f(x) \]

**Question**

1. By looking at \( \lim_{x \to 0^-} f(x) \) and \( \lim_{x \to 0^+} f(x) \), investigate \( \lim_{x \to 0} f(x) \) where

   (a) \( f(x) = \frac{|x|}{x} \)

   (b) \( f(x) = \frac{1}{x} \)

   (c) \( f(x) = \frac{1}{x^2} \)
**Limit Laws**

Here are properties that allow us to easily calculate limits. Note: To use them, \( \lim_{x \to c} f(x) \) and \( \lim_{x \to c} g(x) \) must exist.

- \( \lim_{x \to c} k = \) for any constant \( k \).
- \( \lim_{x \to c} x = \).
- \( \lim_{x \to c} [kf(x)] = \) for any constant \( k \).
- \( \lim_{x \to c} [f(x) + g(x)] = \).
- \( \lim_{x \to c} [f(x)g(x)] = \).
- \( \lim_{x \to c} \left[ \frac{f(x)}{g(x)} \right] = \) if \( \lim_{x \to c} g(x) \neq 0 \).

**Another Limit**

**Definition**  Limits at Infinity

We say \( \lim_{x \to \infty} f(x) = L \) if we can get \( f(x) \) as close to \( L \) as we please by taking \( x \).

**Questions**

2. Calculate the following limits (or show they don’t exist!):

   (a) \( \lim_{x \to 5} \frac{x^3 - x}{2x + 3} \)

   (b) \( \lim_{x \to \infty} \frac{3x^2 + 17x - 7}{2x^2 - 1} \)
(c) \[ \lim_{{h \to 0}} \frac{2(h + 2)^2 - 2h^2}{h} \] (Hint: simplify first. You may need to refer to the limit rules above and to question 1.)

(d) Let

\[ f(x) = \begin{cases} 
  x - 5 & \text{if } x \geq 1 \\
  -4x & \text{if } x < 1
\end{cases} \]

Evaluate \( \lim_{{x \to 1}} f(x) \).

---

**Definition (Continuity)** A function is said to be **continuous** on an interval \([a, b]\) if, intuitively, you can draw the graph of \( f \) over that interval without lifting your pencil from the paper. The mathematical definition is that a function \( f \) is continuous at the point \( x = c \) if \( \lim_{{x \to c}} f(x) = f(c) \).

3. Find \( k \) such that the function

\[ f(x) = \begin{cases} 
  kx & \text{if } x \geq 1 \\
  x^2 + 2 & \text{if } x < 1
\end{cases} \]

is continuous.